Answers to Some Open Questions of Ulrich & Meredith

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1 The Open Questions

Ulrich [11] poses twenty six open questions concerning the axiomatization of various sentential logics. In this note, we report solutions to three of these open questions involving axiomatizations of classical sentential logic. We adopt Ulrich's notation (*viz.*, traditional, Polish notation) throughout. And, the sole rule of inference we will be assuming throughout is condensed detachment [2]: from Cpq and p infer q (where C is the implication operator, and where most general substitution instances are used in each *modus ponens* inference).

(Q_1) Is Meredith's [9] 21-symbol single axiom

(M_1) $CCCCCpqCNrNsrtCCtpCsp$

for classical sentential logic the shortest possible (where N is the negation operator)?

(Q_2) Are Meredith's [9] two 19-symbol single axioms

(M_4) $CCCCCpqCrfstCCtpCrp$

(M_5) $CCCpqCCfrsCCspCtCup$

for classical sentential logic the shortest possible (where f is the constant *falsum* operator, which is equivalent to an arbitrary classical contradiction)?

(Q_3) Are Meredith's [10] two 19-symbol single axioms

(M_2) $CCCpqCrCosCCspCrCtp$

(M_3) $CCCpqCorCsCCrpCtCup$

for classical sentential logic the shortest possible (where o is the constant *verum* operator, which is equivalent to an arbitrary classical tautology)?

In the next section, we report affirmative answers to each of these three questions, and we also report several new single axioms (of shortest length) for each of these three systems of classical sentential logic (thus answering three open questions posed by Meredith himself). Along the way, we will also pose several new open questions pertaining to these axiomatic systems.

†This paper is dedicated to the memory of Ted Ulrich (1940–2020) and Larry Wos (1930–2020). We thank Steve Kuhn for helpful comments on an earlier version of this paper.

2 Answering the Open Questions

2.1 Definitive Affirmative Answer to Open Question Q_1

The answer to Q_1 is: *Yes, M_1 is among the shortest possible single axioms for the C/N fragment of classical sentential logic* (assuming condensed detachment as the sole rule of inference).

Meredith [9] showed that M_1 is a single axiom for classical sentential logic, by deriving the following three axioms of Łukasiewicz (which are a well-known C/N basis).

(Ł$_1$) $CCpqCCqrCpr$

(Ł$_2$) $CpCNpq$

(Ł$_3$) $CCNppp$

We have shown (by exhaustive search and elimination — see Appendix A for details) that no C/N-tautology shorter than 21-symbols can derive Ł$_1$–Ł$_3$ using only condensed detachment.

Moreover, we have identified the following six (hitherto unknown) additional 21-symbol single axioms (thus answering a related open question due to Meredith himself).[1]

(A_1) $CCpCCNpqrCsCCNtCrtCpt$

(A_2) $CpCCqCprCCNrCCNstqCsr$

(A_3) $CpCCNqCCNrsCptCCtqCrq$

(A_4) $CpCCNqCCNrsCtqCCrtCrq$

(A_5) $CCpqCCCrCstCqCNsNpCps$

(A_6) $CCCpqCCCNrNsrtCCtpCsp$

We include a proof of Ł$_1$–Ł$_3$ from A_1 in Appendix B (proofs for other new axioms omitted).[2]

2.2 Definitive Affirmative Answer to Open Question Q_2

The answer to Q_2 is: *Yes, M_4 and M_5 are among the shortest possible single axioms for the C/f fragment of classical sentential logic* (assuming condensed detachment as the sole rule of inference).

Meredith [9] showed that M_4 is a single axiom for classical sentential logic, by deriving the following four axioms of Tarski-Bernays (which are a well-known C/f basis).

(T_1) $CCpqCCqrCpr$

(T_2) $CpCqp$

(T_3) $CCCpqpp$

(T_4) Cfp

[1] These six 21-symbol C/N-formulas are the only ones we've been able to formally rule-in as (new) single axioms. Some other 21-symbol C/N-formulas may yet turn out to be single axioms (*i.e.*, there are some remaining 21-symbol C/N-formulas that we have not been able to definitively rule-in our rule-out). For instance, *strictly speaking*, it remains an open question whether the following 21-symbol C/N-formula is a single axiom.

(O_1) $CCCpqCCrNsCtNtCCtpCrp$

But, based on extensive proof searches, we are confident that O_1 is not a single axiom.

[2] We were able to find condensed detachment proofs for all of these axioms, except for axiom A_2. We used binary resolution (rather than hyper-resolution) to verify that A_2 is a single axiom. The problem of finding a condensed detachment (*viz.*, hyper-resolution) proof of Ł$_1$–Ł$_3$ from A_2 remains as an open challenge problem for automated reasoning.

We have shown (by exhaustive search and elimination — see APPENDIX A for details) that no C/f-tautology shorter than 19-symbols can derive T_1–T_4 using only condensed detachment. Moreover, we have identified the following six (hitherto unknown) additional 19-symbol single axioms (thus answering a related open question due to Meredith himself).[3]

(A_7) $CCCCpCCqrfstCCtqCpq$

(A_8) $CCCCpfqrCCCrsCtpCtr$

(A_9) $CCCpqCrsCCsCCftpCrp$

(A_{10}) $CCCpqCrsCCsCrCrfCrp$

(A_{11}) $CCpqCCCrCstCqCpfCps$

(A_{12}) $CCpqCCCCCqrfsCtpCtq$

We include a proof of T_1–T_4 from A_7 in APPENDIX C (proofs for other new axioms omitted).[4]

2.3 Confident Affirmative Answer to Open Question Q_3

The answer to Q_3 *probably*[5] is: *Yes, M_4 and M_5 are among the shortest possible single axioms for the C/o fragment of classical sentential logic* (assuming condensed detachment as the sole rule of inference).

Meredith [10] showed that M_2 is a single axiom for classical sentential logic, by deriving the following two axioms of Łukasiewicz (which are a well-known C/o basis).

(Ł$_4$) $CCCpqrCCrpCsp$

(Ł$_5$) o

We have obtained strong evidence (by exhaustive search and nearly exhaustive elimination — see APPENDIX A for details) that no C/o-tautology shorter than 19-symbols can derive Ł$_4$–Ł$_5$ using only condensed deatchment.

Moreover, we have identified the following seven (hitherto unknown) additional 19-symbol single axioms (thus answering a related open question due to Meredith himself).[6]

(A_{13}) $CCpqCCCrCstCqCosCps$

(A_{14}) $CCCCCpqCorrsCCspCtp$

(A_{15}) $CCCpqCrsCCsCopCtCrp$

[3] These six 19-symbol C/f-formulas are the only ones we've been able to formally rule-in as (new) single axioms. Some other 19-symbol C/f-formulas may yet turn out to be single axioms (*i.e.*, there are some remaining 19-symbol C/f-formulas that we have not been able to definitively rule-in our rule-out). For instance, *strictly speaking*, it remains an open question whether the following 19-symbol C/f-formula is a single axiom.

(O_2) $CCCpqrCCCCsftCrpCsp$

But, based on extensive proof searches, we are confident that O_2 is not a single axiom.

[4] We were able to find condensed detachment proofs for all of these axioms, except for axiom A_{12}. We used binary resolution (rather than hyper-resolution) to verify that A_{12} is a single axiom. The problem of finding a condensed detachment (*viz.*, hyper-resolution) proof of T_1–T_4 from A_{12} remains as an open challenge problem for automated reasoning.

[5] As we explain in Appendix A, we have definitively ruled out all but three 17-symbol C/o single axiom candidates. And, we are confident (based on extensive searches) that none of these three remaining candidates is a single axiom.

[6] These six 19-symbol C/o-formulas are the only ones we've been able to formally rule-in as (new) single axioms. Some other 19-symbol C/o-formulas may yet turn out to be single axioms (*i.e.*, there are some remaining 19-symbol C/o-formulas that we have not been able to definitively rule-in our rule-out). For instance, *strictly speaking*, it remains an open question whether the following 19-symbol C/o-formula is a single axiom.

(O_3) $CCpCqrCCCrsCoqCtCpr$

But, based on extensive proof searches, we are confident that O_3 is not a single axiom.

(A_{16}) $CCCpqrCCCCrspCosCts$

(A_{17}) $CCpCCqrCosCCsqCtCpq$

(A_{18}) $CCpqCCCCoqrCCstpCtq$

(A_{19}) $CCpqCCCCoprCCstpCtq$

We include a proof of Ł$_4$–Ł$_5$ from A_{13} in Appendix D (proofs for other new axioms omitted).[7]

Appendix

A Methodology

Our methodology for solving these open questions can be broken down to the following five steps.[8]

1. Generate the set $\mathbf{S}_0$ of all well-formed formulas (*viz.*, all the wffs in the relevant language, depending on the open question at hand) that are *just shorter than* the shortest known single axiom. In the case of Q_1, this involves generating all the 19-symbol and 20-symbol wffs in the C/N language. In the case of Q_2, this involves generating all the 17-symbol wffs in the C/f language. And, in the case of Q_3, this involves generating all the 17-symbol wffs in the C/o language.

   · The reason why we only need to check the 19- & 20-symbol C/N formulas, and the 17-symbol C/f and C/o formulas is explained by the following simple result of Łukasiewicz [4].

     Theorem. If there is a single axiom for classical logic that contains n symbols, then there is also a single axiom for classical logic that contains $n + 2$ symbols.

     Proof. Suppose α is a single axiom for classical logic containing n symbols. Then, the sentence $Cz\alpha$ (where the variable z does not occur in α) is also a single axiom for classical logic. To see why, substitute '$Cz\alpha$' for z in $Cz\alpha$, which yields the sentence $CCz\alpha\alpha$. Since α is a tautology, so are both $Cz\alpha$ and $CCz\alpha\alpha$, which means we can apply condensed detachment to $CCz\alpha\alpha$ and $Cz\alpha$ yielding α, which is a single axiom by assumption. □

     Thus, by checking the 20-symbol C/N formulas, we are also implicitly checking all the C/N formulas of even length which are shorter than the shortest known single axioms. And, by checking the 19-symbol C/N formulas, we are also implicitly checking all the C/N formulas of odd length which are shorter than the shortest known single axioms. Because the C/f and C/o formulas can only be of odd length, we only need to check the 17-symbol formulas in those languages.

     Finally, it is worth noting that, in order to efficiently generate $\mathbf{S}_0$, we used *resonator templates* [13], which capture the structure of a class of formulas (where variables are treated indistinguishably).

2. Filter the set $\mathbf{S}_0$ down to the set $\mathbf{S}_1$ of *strongest tautological wffs* of the relevant length. At this stage, we converted each template to conjunctive normal form [1], which greatly sped-up the tautological filtering. We also applied subsumption as we went along, so as to only keep the strongest/most general tautological instances of each template.

[7] We were able to find condensed detachment proofs for all of these axioms, except for axiom A_{14}. We used binary resolution (rather than hyper-resolution) to verify that A_{14} is a single axiom. The problem of finding a condensed detachment (*viz.*, hyper-resolution) proof of Ł$_4$–Ł$_5$ from A_{14} remains as an open challenge problem for automated reasoning.

[8] All of the (python) code that we used to generate and filter the sets of formulas discussed below can be downloaded from `http://fitelson.org/walsh/code/`. The subfolder `http://fitelson.org/walsh/code/Results/` includes complete outputs for each of the three problems. To be more specific, for each of the candidates that was eliminated *via* a Vampire counter-model, we include that counter-model as a witness to the insufficiency of the candidate in question. A total of around 300 distinct counter-models were needed to rule-out all of the non-axioms.

3. Filter $\mathbf{S}_1$ down to the set $\mathbf{S}_2$ of formulas that survive a short satisfiability test for insufficiency of the candidate axiom (using VAMPIRE, version 4.5.1 [3]). Here, we use VAMPIRE to search (for a short period of time) for models in which *modus ponens* (*viz.*, condensed detachment) and the candidate are both satisfied, but a known single axiom fails to be satisfied. If such a counter-model is found, then the candidate is not included in $\mathbf{S}_2$, since the model in question shows that the candidate in question is insufficient to derive all theorems in the relevant language (*via* condensed detachment).

   - For questions, Q_1 and Q_2, *no formulas survived* this final filter (*viz.*, for those two questions, $\mathbf{S}_2 = \varnothing$). That is, for questions Q_1 and Q_2 Vampire was able to *definitively eliminate all* of the candidates that were just shorter than the shortest known single axiom. This settles those questions definitively.

4. In the case of Q_3, a fourth step was required. We needed to look more closely at the formulas remaining in $\mathbf{S}_2$. Specifically, *exactly three* 17-symbol C/o-formulas survived to $\mathbf{S}_2$. To wit:

   (O_4) $CCCpqCorCsCCrpCtp$

   (O_5) $CCCpqrCCrCoCrpCsp$

   (O_6) $CCpqCrCCCqsCopCtq$

   Here, we used a combination of PROVER9 [6], VAMPIRE, and OTTER to try to find (condensed detachment) proofs of sufficiency (or models showing insufficiency) for each of the remaining candidates (using our reference basis Ł$_4$–Ł$_5$ as target). After extensive proof (and model) searches (and careful studying of the kinds of formulas that can be generated using condensed detachment from these candidates), we are very confident that none of these three remaining candidates (O_4)–(O_6) are single axioms.[9] But, strictly speaking, their status remains open.

5. After convincing ourselves that no single axioms shorter than the shortest known single axioms exist (for any of the three languages in question), we then turned to the question of whether *other* (hitherto unknown) shortest single axioms exist. We applied the same four steps above to formulas of the shortest known length. This led to relatively small sets $\mathbf{S}_2$ of remaining candidates (at step 3) for each of the three languages. And, after studying the remaining candidates more carefully using PROVER9, VAMPIRE, and OTTER, we were able to find eight new single axioms for each of the three languages (as well as several other candidates which could not be settled one way or the other — see footnotes 1, 2, and 4 for examples). We report condensed detachment proofs for three of these new shortest single axioms in the remaining sections of the APPENDIX (one for each of the three languages).

[9] Of these three remaining 17-symbol C/o-candidates, only O_4 seems to be able to prove anything remotely interesting *via* condensed detachment (*e.g.*, O_4 *is* able to prove that the *verum* constant o is a theorem). O_5 and O_6, on the other hand, seem only to generate (increasingly) complex formulas *via* condensed detachment.

B Proof that A_1 is a Single Axiom

The following 45-step condensed detachment proof was discovered using OTTER.[10] For a description of some the techniques we used to discover the most elegant condensed detachment proofs we could find, see [12, 5].

1	$CCpCCpNqrCsCCtNCrtCpt$	A_1
2	$CpCCNqCCCNrCsrCtrqCCtCCNtusq$	1,1
3	$CpCCNqCCCrCCNrstuqCuq$	3,1
4	$CpCCNqCCrsqCsq$	3,1
5	$CCNpCCqrpCrp$	4,4
6	$CCCpqrCqr$	4,5
7	$CCCCNpCqpCrpsCCrCCNrtqs$	3,5
8	$CpCqp$	6,6
9	$CCCNpqrCsCCNtCrtCpt$	1,6
10	$CCpCCNpqrCCrsCps$	6,7
11	$CpCCNqCrqCrq$	8,1
12	$CpCCNqCCrCNstqCsq$	8,9
13	$CpCqCCNrCprCsr$	9,6
14	$CCCNpqrCCrsCps$	10,6
15	$CCNpCqpCqp$	11,11
16	$CCCpCNqrsCqs$	12,5
17	$CCCCNpCqpCrpsCqs$	13,10
18	$CpCCpqCrq$	14,6
⋆19	$CpCNpq$	15,16
20	$CpCNCqpr$	19,6
21	$CCCpCNCqprsCts$	20,18
22	$CNCpNqCrCsq$	21,17
23	$CCCpqrCNCsNqr$	22,10
24	$CNCpNqCCqrCsr$	14,23
25	$CCpNpCqNp$	24,15
26	$CCNpqCCpNpq$	25,10
27	$CCCCpNpqrCpr$	26,14
28	$CCpNpCpq$	27,27
29	$CNpCpq$	28,6
30	$CCCNpCpqrCsr$	29,18
31	$CCpCCNpqrCsCpr$	30,7
32	$CpCqCCNrCprr$	31,17
33	$CpCqCCNrCqrr$	32,31
34	$CCCCNpCqpprCqr$	32,10
35	$CpCCNqCpqq$	33,33
36	$CpCCpqq$	6,34
⋆37	$CCNppp$	35,5
38	$CpCqCCqrr$	36,8
39	$CCCpCqCCqrrss$	38,36
40	$CpCqCCprr$	39,17
41	$CCCCpqqrCpr$	40,10
42	$CCNpCqpCqCrp$	17,41
43	$CCpqCpCrq$	42,6

[10]For the purposes of presentation, we use the standard notational conventions of (a) labeling the line number of each of the three goals Ł_1–Ł_3 with a star (*e.g.*, line ⋆19 is Ł_2), and (b) stating the major and minor premises of each condensed deatchment inference in the right hand column (*e.g.*, line ⋆19 is derived *via* condensed deatchment with line 15 as major premise and line 16 as minor premise). One can use OTTER to explicitly generate all the relevant substitution instances for each step; and, one can use IVY to verify the correctness of OTTER proofs [7, 8]. An OTTER input file which allows for the generation and verification of this proof can be downloaded from the following URL: `http://fitelson.org/walsh/A1_proof.in`.

44	$CCpqCrCpCsq$	43,43
45	$CCCpCqrsCCprs$	44,10
*46	$CCpqCCqrCpr$	10,45

C Proof that A_7 is a Single Axiom

The following 24-step condensed detachment proof was discovered using OTTER.[11]

1	$CCCCpCCqrfstCCtqCpq$	A_7
2	$CCCCpqCrqsCCrCCqtfs$	1,1
3	$CCpCCqrfCCCpqsCts$	1,2
4	$CCCCCpqCCfrfspCCtfp$	2,1
5	$CCCCCCpqCrqfrsCts$	2,3
6	$CCCCpfqrCCCqsCCftfr$	4,1
7	$CCpCCqrfCsCpq$	5,2
8	$CCCpqCCfrfCsp$	5,6
9	$CCCpqCCfrfCCpsCts$	1,6
10	$CCCpCCqCCrsftrCqr$	7,1
11	$CpCfq$	8,8
12	$CCCpqrCqr$	8,1
13	$CCCCpqCrqsCps$	9,1
14	$CCCpqpCrp$	10,1
*15	Cfp	11,11
*16	$CpCqp$	12,12
17	$CpCCCCCqrsCtsfq$	13,7
18	$CCCCCpqrCsrfp$	17,17
19	$CCpqCCCprfq$	18,1
20	$CCCCpqrfCCCpsfq$	19,19
21	$CCCCCpqprfCsp$	14,19
22	$CCCCCpqfrsCCprs$	20,1
23	$CpCCCqrqq$	21,14
*24	$CCpqCCqrCpr$	1,22
*25	$CCCpqpp$	23,23

D Proof that A_{13} is a Single Axiom

The following 21-step condensed detachment proof was discovered using OTTER.[12]

1	$CCpqCCCrCstCqCosCps$	A_{13}
2	$CCCpCqrCCCCsCtuCvCotCwtCoqCCwvq$	1,1
3	$CCCpqpp$	1,2
4	$CCCpCqrCCCstuCoqCCCvCuwCCCCxCyzCtCoyCsyCouq$	2,1
5	$CCCpCCCqrCoqsCCCCtCuvCCqwCouCxuCoCCqrCoqq$	1,4
6	$CCCCCpCqrCsCoqCtquCCtsu$	4,2
*7	o	4,5

[11] An OTTER input file which allows for the generation and verification of this proof can be downloaded from the following URL: `http://fitelson.org/walsh/A7_proof.in`.

[12] An OTTER input file which allows for the generation and verification of this proof can be downloaded from the following URL: `http://fitelson.org/walsh/A13_proof.in`.

8	$CCpqCCqrCpr$	6,6
9	$CCCCpqCrqsCCrps$	8,8
10	$CCCpCqrCCCstCutCoqCCusq$	8,1
11	$CCCCpqrsCCCCqtCptrs$	9,8
12	$CCCCCpqrCsrCCCtuCvuCopCCvtp$	10,11
13	$CCCCCpqrCCsqrtCCpst$	9,11
14	$CCCpqpCCpqq$	9,12
15	$CCCCpqrqCpq$	12,3
16	$CCCpqpCCrsp$	12,15
17	$CCCCpqrsCCCrtrs$	16,8
18	$CCCpqpCrp$	15,17
19	$CCCpqpCCprr$	14,17
20	$CpCCpqq$	19,15
21	$CCCCCpqpCrpss$	18,20
*22	$CCCpqrCCrpCsp$	21,13

References

[1] T.B. de la Tour. "An optimality result for clause form translation." *Journal of Symbolic Computation* 14.4 (1992): 283-301.

[2] J.A. Kalman. "Condensed detachment as a rule of inference." *Studia Logica* 42.4 (1983): 443–451.

[3] L. Kovács, and A. Voronkov. "First-order theorem proving and Vampire." International Conference on Computer Aided Verification. Springer, Berlin, Heidelberg, 2013. Version 4.5.1 of VAMPIRE can be downloaded from the following URL: `https://vprover.github.io/index.html`.

[4] J. Łukasiewicz. "The shortest axiom of the implicational calculus of propositions." *Proceedings of the Royal Irish Academy. Section A: Mathematical and Physical Sciences.* Vol. 52. Royal Irish Academy, 1948.

[5] W. McCune, and L. Wos. "Experiments in automated deduction with condensed detachment." *International Conference on Automated Deduction* (pp. 209–223). Springer, Berlin, Heidelberg, 1992.

[6] W. McCune, "Prover9 and Mace4." Available at the following URL: `https://www.cs.unm.edu/~mccune/prover9/`, 2009.

[7] ______. "Otter 3.3 reference manual." arXiv preprint `cs/0310056`, 2003.

[8] W. McCune, and O. Shumsky. "IVY: A preprocessor and proof checker for first-order logic." In *Computer-Aided Reasoning* (pp. 265-281). Springer, Boston, MA.

[9] C. Meredith. "Single axioms for the systems (C, N), (C, O) and (A, N) of the two-valued propositional calculus." *Journal of Computing Systems* 1 (1953), 155–164.

[10] C. Meredith, and A.N. Prior. "Notes on the axiomatics of the propositional calculus." *Notre Dame Journal of Formal Logic* 4.3 (1963): 171–187.

[11] D. Ulrich. "A legacy recalled and a tradition continued." *Journal of Automated Reasoning* 27.2 (2001): 97–122.

[12] L. Wos. "Automating the search for elegant proofs." *Journal of Automated Reasoning* 21.2 (1998): 135–175.

[13] ______. "The resonance strategy." *Computers & Mathematics with Applications* 29.2 (1995): 133-178.