Comments on Some Completeness Theorems of Urquhart and Méndez & Salto

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Abstract. Urquhart [8] and Méndez & Salto [5, 6] claim to establish completeness theorems for the system **C** and two of its negation extensions. In this note, we do the following three things: (1) provide a counterexample to all of these alleged completeness theorems, (2) attempt to diagnose the mistakes in the reported completeness proofs, and (3) provide complete axiomatizations of the desired systems.

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1. Introduction

In [8], Urquhart presents an axiomatization of a positive logic **C** and claims that this system is complete with respect to the class of model structures over an ordered commutative monoid [8, Theorem 3.3, p. 103]. This claim is false. More recently, Méndez & Salto [5, 6] introduce two negation extensions of **C**, and claim that these extensions are complete with respect to certain relational semantics. These, too, are false claims. Below, we provide a counterexample to all of these claims. In closing, we will diagnose the mistakes in the reported completeness proofs and provide complete axiomatizations of the desired systems.

2. The Axiomatic System C and Two Negation Extensions

The positive logic **C** is given (in [8]) by the following ten axiom schemata:

$$\phi \to (\psi \to \phi) \tag{1}$$
$$(\phi \to \psi) \to ((\theta \to \phi) \to (\theta \to \psi)) \tag{2}$$
$$(\phi \to (\theta \to \psi)) \to (\theta \to (\phi \to \psi)) \tag{3}$$
$$(\phi \wedge \psi) \to \phi \tag{4}$$
$$(\phi \wedge \psi) \to \psi \tag{5}$$

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$$\phi \to (\psi \to (\phi \wedge \psi)) \tag{6}$$
$$\phi \to (\phi \vee \psi) \tag{7}$$
$$\psi \to (\phi \vee \psi) \tag{8}$$
$$((\phi \to \psi) \wedge (\theta \to \psi)) \to ((\phi \vee \theta) \to \psi) \tag{9}$$
$$(\phi \to \psi) \vee (\psi \to \phi) \tag{10}$$

together with the rule of *modus ponens*.[1]

Méndez & Salto [5, 6] consider two negation extensions of **C**. The first such extension is **CI′**, which is obtained by adding the following two negation axioms to **C**:

$$(\phi \to \neg\psi) \to (\psi \to \neg\phi) \tag{11}$$
$$\neg\phi \to (\phi \to \psi) \tag{12}$$

The second negation extension discussed by Méndez & Salto is **CIr**, which is obtained by adding the following negation axiom to **C**:

$$(\phi \to \neg\phi) \to \neg\phi \tag{13}$$

3. Counterexample to the Completeness of C, CI′, and CIr

The following formula is valid in Urquhart's class of model structures over ordered commutative monoids, and in each of the relational semantical systems considered by Méndez & Salto:

$$((\psi \to \phi) \wedge (\psi \to \theta)) \to (\psi \to (\phi \wedge \theta)) \tag{14}$$

However, (14) is not derivable in any of **C**, **CI′**, or **CIr**. Using John Slaney's powerful computer program MaGIC [7], we found the following four-valued matrices (in which the only *designated* value is 3, which is indicated by *) that establish the independence of (14) from each of the systems **C**, **CI′**, and **CIr**, simultaneously.[2]

[1] Axiom (1) is dependent. We note the strong resemblance of Urquhart's **C** with Dummett's system **LC** [2]. Dummett's **LC** is obtained by adding (10) to the axioms of intuitionistic logic. **C** can be obtained from **LC** by dropping the two **LC** negation axioms, and replacing $(\phi \to \psi) \to ((\phi \to (\psi \to \theta)) \to (\phi \to \theta))$ in **LC** by (2) and (3). In **LC** (unlike **C**), (1) is not dependent, and (14) is derivable.

[2] All matrices reported have been verified mechanically using Bill McCune's computer program Mace [4]. It is interesting to note that these are the *smallest possible* matrices which satisfy (1)–(13), but which violate (14). We know this because MaGIC performed an *exhaustive* search of all matrices up to this size.

$\to$	0	1	2	3
0	3	3	3	3
1	0	3	3	3
2	0	2	3	3
*3	0	1	2	3

$\wedge$	0	1	2	3
0	0	0	0	0
1	0	1	1	1
2	0	1	1	2
*3	0	1	2	3

$\vee$	0	1	2	3
0	0	1	2	3
1	1	1	2	3
2	2	2	2	3
*3	3	3	3	3

ϕ	0	1	2	*3
$\neg\phi$	3	0	0	0

To see why (14) fails in the above matrices, note that:

$$((2 \to 2) \wedge (2 \to 2)) = 3 \wedge 3 = 3,$$

but

$$2 \to (2 \wedge 2) = 2 \to 1 = 2.$$

Therefore,

$$((2 \to 2) \wedge (2 \to 2)) \to (2 \to (2 \wedge 2)) = 3 \to 2 = 2 \neq 3.$$

4. Diagnoses and Fixes of Reported Completeness Proofs

4.1. Diagnosis and Fix of Méndez & Salto's Proof(s)

All of the completeness proofs of Méndez & Salto make use of the fact that $\wedge$ distributes over $\vee$ in **C**. That is, their proofs presuppose that the following formula is a theorem of their negation extensions of **C**:

$$(\phi \wedge (\psi \vee \theta)) \to ((\phi \wedge \psi) \vee (\phi \wedge \theta)) \tag{15}$$

In lemmas 2, 3, and 4 of [6], Méndez & Salto appeal to distributivity to establish the existence of prime consistent theories.[3] Unfortunately, as the following (MaGICally discovered) matrices show, the distributive law (15) does *not* follow from (1)–(13).

$\to$	0	1	2	3	4	5	6
0	6	6	6	6	6	6	6
1	0	6	6	6	6	6	6
2	0	5	6	5	6	6	6
3	0	4	4	6	4	6	6
4	0	3	5	3	6	5	6
5	0	2	4	5	4	6	6
*6	0	1	2	3	4	5	6

$\wedge$	0	1	2	3	4	5	6
0	0	0	0	0	0	0	0
1	0	1	1	1	1	1	1
2	0	1	1	1	2	2	2
3	0	1	1	3	1	3	3
4	0	1	2	1	4	2	4
5	0	1	1	3	2	3	5
*6	0	1	2	3	4	5	6

$\vee$	0	1	2	3	4	5	6
0	0	1	2	3	4	5	6
1	1	1	2	3	4	5	6
2	2	2	2	5	4	5	6
3	3	3	5	3	6	5	6
4	4	4	4	6	4	6	6
5	5	5	5	5	6	5	6
*6	6	6	6	6	6	6	6

[3] Compare with the Pair Extension Lemma in [1, page 124].

ϕ	0	1	2	3	4	5	*6
$\neg\phi$	6	0	0	0	0	0	0

To see why (15) fails in the above matrices, note that:

$$2 \wedge (3 \vee 2) = 2 \wedge 5 = 2,$$

but

$$(2 \wedge 3) \vee (2 \wedge 2) = 1 \vee 1 = 1.$$

Therefore,

$$(2 \wedge (3 \vee 2)) \to ((2 \wedge 3) \vee (2 \wedge 2)) = 2 \to 1 = 5 \neq 6.$$

Happily, (15) *does* follow from (1)–(14).[4] In fact, if (14) is added as an axiom to any of Méndez & Salto's systems, then their reported completeness proofs go through, as stated. So, complete axiomatizations are obtained for their systems, simply by adding (14) as an axiom.

4.2. Diagnosis and Fix of Urquhart's Proof

The diagnosis of Urquhart's proof is a bit more subtle. Like the proofs of Méndez & Salto, Urquhart's proof in [8] makes implicit use of (14) in order to establish the existence of prime theories with certain requisite properties.[5] Unlike Méndez & Salto's proofs, Urquhart's proof [8, Lemma 3.1] also appeals to the following axiom scheme, which is a generalization of axiom (9):

$$((\phi^k \to \psi) \wedge (\theta^k \to \psi)) \to ((\phi \vee \theta)^k \to \psi) \tag{16}$$

We have been unable to verify Urquhart's claim in [8, Lemma 3.1] that all instances of (16) follow from his axioms (1)–(10).[6] So, following the suggestion of an anonymous referee, we suggest that both (14) and (16)

[4] In [3], we report an axiomatic proof of (15) in a system closely related to **C** + (14). Our proof is easily modified to produce the desired distributivity result.

[5] The property $\theta, \pi \in \Sigma \Rightarrow \theta \wedge \pi \in \Sigma$ of prime theories Σ presupposes (14). Urquhart makes this clear in a more recent completeness proof [9, Theorem 1.3 (d)].

[6] Thanks to an anonymous referee for pointing us to this scheme in Urquhart's [8] proof, and for giving us a more recent manuscript of Urquhart's [9] in which the problems reported here have been fixed. Although we have not been able to find a concrete counterexample to the claim that (for all k) (16) is derivable from Urquhart's [8] axioms (1)–(10), we doubt that (16) follows. This is why we suggest (along the lines of [9]) that *both* (14) *and* (16) be added to urquhart's original axiomatization of **C**. It is interesting to note that some instances of (16) are provable from (1)–(10). In particular, we have been able to prove the $k = 2$ instance of (16) from (1)–(10). However, the status of the $k > 2$ instances of (16) remains open.

be added to Urquhart's original axiomatization of **C**, in order to ensure that his published completeness proof is repaired. Indeed, this is what Urquhart has done in a more recent (unpublished) manuscript [9].

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