Univocity of Intuitionistic and Classical Connectives

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1 Introduction

A sentential connective $\star$ is said to be **univocal**, relative to a formal system $\mathcal{F}$ for a sentential logic containing $\star$ iff any two connectives $\star_1$ and $\star_2$ which satisfy the same $\mathcal{F}$ rules (and axioms) as $\star$ are such that *similar formulas involving* $\star_1$ *and* $\star_2$ *are inter-derivable in* $\mathcal{F}$.[1] To be more precise, suppose $\star$ is a unary connective. Then $\star$ is univocal relative to $\mathcal{F}$ iff for any $\star_1$ and $\star_2$ satisfying the same principles as $\star$ in $\mathcal{F}$, we have $\star_1\alpha \vdash_{\mathcal{F}} \star_2\alpha$. And, if $\star$ is binary, then $\star$ is univocal relative to $\mathcal{F}$ iff for any $\star_1$ and $\star_2$ satisfying the same principles as $\star$ in $\mathcal{F}$, we have $\alpha \star_1 \beta \vdash_{\mathcal{F}} \alpha \star_2 \beta$. In order to illustrate this definition of univocity, it is helpful to begin with a simple historical example.

It is easy to see that the conditional $\to$ is univocal, relative to Gentzen's (intuitionistic and classical) natural deduction systems **LJ** and **LK** [5]. In both of these systems, if we suppose there are two conditionals ($\to_1$ and $\to_2$) satisfying Gentzen's introduction and elimination rules for $\to$, then we can quickly prove that $\alpha \to_1 \beta \vdash \alpha \to_2 \beta$, as follows:

$$\dfrac{\dfrac{\overset{1}{\alpha} \qquad \alpha \to_1 \beta}{\beta}\,(\to_1 \mathrm{E})}{\alpha \to_2 \beta}\,(\to_2 \mathrm{I})_1.$$

Similar straightforward derivations may be found in order to prove that the connectives $\wedge$, $\vee$, $\bot$ and $\neg$ are each univocal, relative to Gentzen's system **LJ** (proofs omitted).

Also, it is well known, as Gentzen himself observed in [5, §5], that **LJ** and **LK** have the same derivations as the Hilbert systems for intuitionistic and classical sentential logic (**LHJ** and **LHK**). That is $\Gamma \vdash_{\mathbf{LJ}} \alpha$ iff $\Gamma \vdash_{\mathbf{LHJ}} \alpha$; and, $\Gamma \vdash_{\mathbf{LK}} \alpha$ iff $\Gamma \vdash_{\mathbf{LHK}} \alpha$.

In this context, it is natural to ask whether the conditional and the other connectives are also univocal, relative to the Hilbert systems for intuitionistic and classical sentential logic (**LHJ** and **LHK**). This is what will be considered in what follows.

We begin by reminding the reader of the axiomatic systems **LHJ** and **LHK**. The system **LHJ** consists of *modus ponens*, plus the following ten axiom schemata:

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[1] Sometimes the term "uniqueness" is used instead of "univocity." See [1], [2] and [3] for discussion.

($\mathbf{LHJ}_1$) $\beta \to (\alpha \to \beta)$,

($\mathbf{LHJ}_2$) $(\gamma \to (\alpha \to \beta)) \to ((\gamma \to \alpha) \to (\gamma \to \beta))$,

($\mathbf{LHJ}_3$) $\alpha \to (\beta \to (\alpha \wedge \beta)$,

($\mathbf{LHJ}_4$) $(\alpha \wedge \beta) \to \alpha$,

($\mathbf{LHJ}_5$) $(\alpha \wedge \beta) \to \beta$,

($\mathbf{LHJ}_6$) $(\alpha \to \gamma) \to ((\beta \to \gamma) \to ((\alpha \vee \beta) \to \gamma))$,

($\mathbf{LHJ}_7$) $\alpha \to (\alpha \vee \beta)$,

($\mathbf{LHJ}_8$) $\beta \to (\alpha \vee \beta)$,

($\mathbf{LHJ}_9$) $(\alpha \to \beta) \to ((\alpha \to \neg\beta) \to \neg\alpha)$,

($\mathbf{LHJ}_{10}$) $\neg\alpha \to (\alpha \to \beta)$.

One nice feature of **LHJ** is that the fragments of **LHJ** can be picked out simply by selecting those axioms which contain the connectives in question. This is not so for the system **LHK**, which has all ten axioms of **LHJ** *plus* the following eleventh axiom:[2]

(LEM) $\alpha \vee \neg\alpha$.

Famously, this extension of **LHJ** is non-conservative. The implicational fragment of **LHK** contains theorems not derivable in **LHJ** — specifically, Peirce's Law, that is,

(P) $((\alpha \to \beta) \to \alpha) \to \alpha$.

One can easily obtain the implicational fragment of **LHK** by adding (P) to the implicational fragment of **LHJ**. But, it would be preferable for our purposes to adopt a Hilbert system for classical sentential logic that allows us to easily extract all of the fragments of classical logic (just by selecting the axioms containing those connectives), as can be done with **LHJ**. To that end, we will adopt the following alternative (equivalent) axiomatization as our classical, Hilbert system **LHK**. It differs in two ways from **LHJ** in the axioms containing implication and negation. Specifically, our **LHK** replaces ($\mathbf{LHJ}_9$) and ($\mathbf{LHJ}_{10}$) with the following two axioms:

($\mathbf{LHK}_9$) $(\neg\beta \to \neg\alpha) \to (\alpha \to \beta)$,

($\mathbf{LHK}_{10}$) $((\alpha \to \beta) \to \alpha) \to \alpha$.

Thus, our system **LHK** consists of *modus ponens*, plus the ten axioms $\mathbf{LHJ}_1$–$\mathbf{LHJ}_8$, *and* $\mathbf{LHK}_9$ and $\mathbf{LHK}_{10}$. Because our **LHJ** and **LHK** share these first eight axioms, we will refer to these axioms sometimes as $\mathbf{LHJ}_1$–$\mathbf{LHJ}_8$ and sometimes as $\mathbf{LHK}_1$–$\mathbf{LHK}_8$, depending on what formal system(s) we are discussing in the context. Moreover, although the resulting axiomatization is now redundant (as $\mathbf{LHK}_{10}$ follows from $\mathbf{LHK}_1$, $\mathbf{LHK}_2$,

[2]This is Gentzen's way of obtaining **LHK** from **LHJ**. Another equivalent option is to replace $\mathbf{LHJ}_{10}$ with the *double-negation* axiom $\neg\neg\alpha \to \alpha$. This alternative is also non-conservative, of course, since Pierce's Law is also not derivable in the implication-negation fragment of **LHJ**.

and $\mathbf{LHK}_9$), we can now simply speak about a fragment of **LHK** as involving only those axioms of **LHK** that contain the connectives appearing in the fragment in question.

In Section 2, we will discuss questions of univocity of connectives in (fragments of) **LHJ**. In Section 3, we will discuss questions of univocity of connectives in (fragments of) **LHK**. In Section 4, we will discuss questions of univocity of connectives in some alternative formal systems for classical sentential logic that use the Sheffer stroke.

2 Univocity in Hilbert Systems for Intuitionism

2.1 Non-Univocity of the Intuitionistic Conditional

Interestingly, the conditional is *not* univocal, relative to the usual Hilbert systems for the implicational fragment of intuitionistic logic. More precisely, we have the following.

Theorem 1. *There exist conditionals $\to_1$ and $\to_2$ such that (a) both conditionals satisfy the laws of the implicational fragment $\mathbf{LHJ}^{\to}$ of* **LHJ***; but, (b) $\alpha \to_1 \beta \nvdash \alpha \to_2 \beta$.*

This result was known since at least 1962 (see [15, *p.* 430]) where a four-element counter-model is presented (see below). We recently discovered the following three-element counter-model.[3]

$\to_1$	1	2	3
1	1	2	2
2	1	1	1
3	1	1	1

$\to_2$	1	2	3
1	1	2	3
2	1	1	3
3	1	2	1

So, in this model, the (sole) designated value is 1 (*i.e.*, all and only formulas assigned the value 1 by the model are theorems of the logic defined by these 3-valued characteristic matrices). Inspection of the model reveals that (a) both conditionals satisfy *modus ponens* and axioms $\mathbf{LHJ}_1$ and $\mathbf{LHJ}_2$; *but*, (b) $2 \to_1 3 \nvdash 2 \to_2 3$, since $2 \to_1 3$ is assigned the designated value 1, but $2 \to_2 3$ is assigned the non-designated value 3.

In fact, a stronger result was established by Smiley [15]. As it turns out, the conditional is not univocal, relative to **LHJ** *as a whole*. That is, even if two conditionals $\to_1$ and $\to_2$ satisfy *all* of the laws of **LHJ**, it will *still* not generally be the case that $\alpha \to_1 \beta \vdash \alpha \to_2 \beta$.

Theorem 2. *There exist two conditionals $\to_1$ and $\to_2$ such that (a) both conditionals satisfy* ***all*** *the laws of* **LHJ***; but, (b) $\alpha \to_1 \beta \nvdash \alpha \to_2 \beta$.*

[3] Following Smiley [15], the sole designated value in all of our matrices will be 1 (all the values greater than 1 will be non-designated). Moreover, all of the matrices reported in this paper will be *normal* in Smiley's sense. That is, the matrices do not make a distinction between values which has no eventual effect on the designation status of any compound in which subformulas with those values differ (in modern terminology, all of our matrices are *reduced*). We have made extensive use of automated theorem-proving and model-finding software (*e.g.*, `Prover9` & `Mace4` [9], `Otter` [10], and `Vampire` [8]) to find and verify the derivations and models reported in this paper. Specifically, we were able to perform exhaustive searches with `Vampire` and `Mace4`, which is how we know that the counter-models presented here are the smallest possible. Input files for each of the results reported in the paper are available upon request.

Proof. In this case, four-element counter-models are required. Here is the model Smiley reports in [15, *p.* 430].[4]

$\to_1$	1	2	3	4
1	1	2	3	4
2	1	1	1	4
3	1	1	1	4
4	1	1	1	1

$\to_2$	1	2	3	4
1	1	2	3	4
2	1	1	3	4
3	1	1	1	4
4	1	1	1	1

$\wedge$	1	2	3	4
1	1	2	3	4
2	2	2	3	4
3	3	3	3	4
4	4	4	4	4

$\vee$	1	2	3	4
1	1	1	1	1
2	1	2	2	2
3	1	2	3	3
4	1	2	3	4

	1	2	3	4
$\neg$	4	4	4	1

Inspection of the model reveals that (a) both conditionals satisfy *modus ponens* and axioms $\mathbf{LHJ}_1$–$\mathbf{LHJ}_{10}$; *but*, (b) $2 \to_1 3 \nvdash 3 \to_2 3$, since $2 \to_1 3$ is assigned the designated value 1, but $2 \to_2 3$ is assigned the non-designated value 3. □

Remark. Theorem 2 implies that the axiomatic presentations of all substructural logics (see [12, §2.3] for a survey) have non-univocal conditionals.

2.2 Univocity of the Other Intuitionistic Connectives

The other connectives *are* univocal, relative to **LHJ**. For each connective, we will prove a corresponding positive theorem to establish this.

Theorem 3. *If two conjunctions $\wedge_1$ and $\wedge_2$ each satisfy the laws of the implication-conjunction fragment* $\mathbf{LHJ}^{\to,\wedge}$ *of* **LHJ**, *then* $\alpha \wedge_1 \beta \vdash \alpha \wedge_2 \beta$.

Proof. Here is a derivation of $\alpha \wedge_1 \beta \vdash \alpha \wedge_2 \beta$, where MP is *modus ponens*.

1.	$\alpha \wedge_1 \beta$	Assumption
2.	α	1, $\mathbf{LHJ}_4$, MP
3.	β	1, $\mathbf{LHJ}_5$, MP
4.	$\beta \to (\alpha \wedge_2 \beta)$	2, $\mathbf{LHJ}_3$, MP
5.	$\alpha \wedge_2 \beta$	3, 4, MP

[4]Strictly speaking, Smiley was interested not in the question of whether the intuitionistic conditional is *univocal* (relative to **LHJ**); but, rather, whether any two conditionals satisfying the laws of **LHJ** must be *synonymous*, where p and q are synonymous just in case they are *inter-substitutable in all contexts* (*i.e.*, if, for *any* propositional function ϕ, $\phi(p) \vdash \phi(q)$). Of course, if p and q are synonymous, then $p \vdash q$ (just let $\phi(x) = x$). So, univocity (of, *e.g.*, $u \to_1 v$ and $u \to_2 v$) follows from synonymy. In general, the converse does not hold, *i.e.*, there are some sentential logics in which the converse fails. Such logics are said to have non-congruential consequence relations [6, §3.3]. But, in intuitionistic logic **LHJ** and classical logic **LHK**, univocity *does* entail synonymy, since both of these logics have congruential consequence relations (*i.e.*, they both support general replacement theorems [7, §26]).

The strategy here is to *eliminate* $\wedge_1$ and then *introduce* $\wedge_2$. A similar strategy will work for the other connectives as well. □

Theorem 4. *If two disjunctions* $\vee_1$ *and* $\vee_2$ *each satisfy the laws of the implication-disjunction fragment* $\mathbf{LHJ}^{\rightarrow,\vee}$ *of* **LHJ**, *then* $\alpha \vee_1 \beta \vdash \alpha \vee_2 \beta$.

Proof. Here is a derivation of $\alpha \vee_1 \beta \vdash \alpha \vee_2 \beta$.

1.	$\alpha \vee_1 \beta$	Assumption
2.	$\alpha \rightarrow (\alpha \vee_2 \beta)$	$\mathbf{LHJ}_7$
3.	$\beta \rightarrow (\alpha \vee_2 \beta)$	$\mathbf{LHJ}_8$
4.	$(\alpha \vee_1 \beta) \rightarrow (\alpha \vee_2 \beta)$	2, 3, $\mathbf{LHJ}_6$, MP($\times$2)
5.	$\alpha \vee_2 \beta$	1, 4, MP

Here, we first *introduce* $\vee_2$ and then *eliminate* $\vee_1$. □

Theorem 5. *If two negations* $\neg_1$ *and* $\neg_2$ *each satisfy the laws of the implication-negation fragment* $\mathbf{LHJ}^{\rightarrow,\neg}$ *of* **LHJ**, *then* $\neg_1\alpha \vdash \neg_2\alpha$.

Proof. Here is a derivation of $\neg_1\alpha \vdash \neg_2\alpha$.

1.	$\neg_1\alpha$	Assumption
2.	$\alpha \rightarrow \neg_2\alpha$	1, $\mathbf{LHJ}_{10}$, MP
3.	$\alpha \rightarrow \alpha$	Theorem of $\mathbf{LHJ}^{\rightarrow}$
4.	$\neg_2\alpha$	2, 3, $\mathbf{LHJ}_9$, MP($\times$2)

Here, we omit the derivation of step 3, which is a well-known theorem of the implicational fragment $\mathbf{LHJ}^{\rightarrow}$ of **LHJ**. □

3 Univocity in Hilbert Systems for Classical Logic

It follows from the results in Section 2.2 that the connectives $\wedge, \vee, \neg$ are univocal, relative to (the relevant fragments of) **LHK**. The only remaining questions involve the univocity of the conditional in (fragments of) **LHK**. Here, there are two positive results and two negative results. We begin with the negative results.

Theorem 6. *The conditional is non-univocal, relative to the implication-conjunction fragment* $\mathbf{LHK}^{\rightarrow,\wedge}$ *of* **LHK**.[5]

Proof. In this case, four-element counter-models are required. Here is one.

$\rightarrow_1$	1	2	3	4
1	1	2	2	2
2	1	1	1	1
3	1	1	1	1
4	1	1	1	1

$\rightarrow_2$	1	2	3	4
1	1	2	3	4
2	1	1	3	3
3	1	2	1	2
4	1	1	1	1

[5]It is a corollary of Theorem 6 that $\rightarrow$ is non-univocal, relative to the implicational fragment of $\mathbf{LHK}^{\rightarrow}$ of **LHK**. In that case, three-element counter-models exist. In fact, the model we reported above (Theorem 1) will suffice for this purpose (as $\rightarrow_1$ and $\rightarrow_2$ in that model also satisfy Peirce's law).

∧	1	2	3	4
1	1	2	3	4
2	2	2	4	4
3	3	4	3	4
4	4	4	4	4

Inspection of the model reveals that (a) both conditionals satisfy *modus ponens* and axioms $\mathbf{LHK}_1$–$\mathbf{LHK}_5$; *but*, (b) $2 \to_1 3 \nvdash 2 \to_2 3$, since $2 \to_1 3$ is assigned the designated value 1, but $2 \to_2 3$ is assigned the non-designated value 3. □

Theorem 7. *The conditional is non-univocal, relative to the implication-negation fragment* $\mathbf{LHK}^{\to,\neg}$ *of* **LHK**.

Proof. In this case, six-element counter-models are required. Here, we report an eight-element counter-model.[6]

$\to_1$	1	2	3	4	5	6	7	8
1	1	2	3	4	5	6	7	8
2	1	1	1	5	5	5	1	5
3	1	7	1	6	5	6	7	5
4	1	1	1	1	1	1	1	1
5	1	2	3	2	1	7	7	3
6	1	3	3	3	1	1	1	3
7	1	3	3	8	5	5	1	8
8	1	7	1	7	1	7	7	1

$\to_2$	1	2	3	4	5	6	7	8
1	1	2	3	4	5	6	7	8
2	1	1	3	5	5	7	7	3
3	1	2	1	6	7	6	7	2
4	1	1	1	1	1	1	1	1
5	1	2	1	2	1	2	1	2
6	1	1	3	3	3	1	1	3
7	1	2	3	8	3	2	1	8
8	1	1	1	7	7	7	7	1

	1	2	3	4	5	6	7	8
¬	4	5	6	1	2	3	8	7

Inspection of the model reveals that (a) both conditionals satisfy *modus ponens* and axioms $\mathbf{LHK}_1$, $\mathbf{LHK}_2$, and $\mathbf{LHK}_9$; *but*, (b) $2 \to_1 3 \nvdash 2 \to_2 3$, since $2 \to_1 3$ is assigned the designated value 1, but $2 \to_2 3$ is assigned the non-designated value 3. □

Remark. Theorem 7 implies that many axiomatic presentations of classical logic — dating all the way back to Frege's *Begriffsschrift* [4] — have non-univocal conditionals.

Theorem 8. *The conditional is univocal, relative to the implication-disjunction fragment* $\mathbf{LHK}^{\to,\vee}$ *of* **LHK**.

Proof. Here is a derivation of $\alpha \to_1 \beta \vdash \alpha \to_2 \beta$.

1.	$\alpha \to_1 \beta$	Assumption
2.	$\alpha \vee (\alpha \to_2 \beta)$	Theorem of $\mathbf{LHK}^{\to,\vee}$
3.	$\alpha \to_1 (\beta \vee ((\alpha \to_1 \beta) \to_1 (\alpha \to_2 \beta)))$	Theorem of $\mathbf{LHK}^{\to,\vee}$
4.	$(\alpha \to_2 \beta) \to_1 (\beta \vee ((\alpha \to_1 \beta) \to_1 (\alpha \to_2 \beta)))$	Theorem of $\mathbf{LHK}^{\to,\vee}$
5.	$\beta \vee ((\alpha \to_1 \beta) \to_1 (\alpha \to_2 \beta))$	2, 3, 4, $\mathbf{LHK}_6$, MP

[6] We prefer this counter-model to the six-element counter-models we initially discovered for the following reason. These eight-element models allow one of the conditionals ($\to_1$) to satisfy the laws of Boolean algebra. This is analogous to Smiley's 4-element matrices (from Theorem 2, above), in which one of the two conditionals ($\to_2$) satisfies the laws of Heyting algebra.

6.	$\beta \to_2 (\alpha \to_2 \beta)$	$\mathbf{LHK}_1$
7.	$(\alpha \to_2 \beta) \vee ((\alpha \to_1 \beta) \to_1 (\alpha \to_2 \beta))$	5, 6 $\mathbf{LHK}_6$, MP
8.	$(\alpha \to_2 \beta) \to_1 ((\alpha \to_1 \beta) \to_1 (\alpha \to_2 \beta))$	$\mathbf{LHK}_1$
9.	$(\alpha \to_1 \beta) \to_1 (\alpha \to_2 \beta)$	7, 8, $\mathbf{LHK}_6$, MP
10.	$\alpha \to_2 \beta$	1, 9, MP

We omit the derivations of steps 2, 3, 4, which are theorems of the implication-disjunction fragment $\mathbf{LHK}^{\to,\vee}$ of **LHK**. In particular, note that steps 2 and 3 hold by Peirce's Law. In order to derive steps 7 and 9, we have also made use of the fact that $\alpha \to \alpha$ is a theorem of $\mathbf{LHK}^{\to}$. □

Theorem 9. *The conditional is univocal, relative to the implication-conjunction-negation fragment* $\mathbf{LHK}^{\to,\wedge,\neg}$ *of* **LHK**.

Proof. This is a corollary of Theorem 8. Because we are in the implication-conjunction-negation fragment of classical logic, we can now define a disjunction connective (which will satisfy all of the **LHK** disjunction laws, relative to both $\to_1$ and $\to_2$) in terms of negation and conjunction, as $\alpha \vee \beta =_{df} \neg(\neg\alpha \wedge \neg\beta)$. Then, we can apply the same strategy that we used above to demonstrate $\alpha \to_1 \beta \vdash \alpha \to_2 \beta$ in our proof of Theorem 8. □

4 Univocity in Classical Systems with Sheffer's Stroke

4.1 Univocity in Hilbert Systems Using Sheffer's Stroke

Nicod [11] was the first to develop a formal system **N** for classical sentential logic using (only) Sheffer's stroke (|). He presented a Hilbert system, which uses the following single rule of inference:

(D) From $\alpha \mid (\beta \mid \gamma)$ and α, infer γ.

Various sets of axioms have been shown to be complete for classical sentential logic, in the presence of rule (D) [14]. We will adopt Nicod's original 23-symbol single axiom, that is,

$$(N) \quad (\alpha \mid (\beta \mid \gamma)) \mid ((\epsilon \mid (\epsilon \mid \epsilon)) \mid ((\delta \mid \beta) \mid ((\alpha \mid \delta) \mid (\alpha \mid \delta)))).$$

In other words, the system **N** consists of the single rule D and the single axiom N. The question of univocity is a simple one in this context. And, as it happens, we have the following negative result.

Theorem 10. *The Sheffer stroke is non-univocal, relative to Nicod's Hilbert system* **N**.

Proof. In this case, four-element counter-models are required. Here is one.

$\mid_1$	1	2	3	4
1	2	1	4	3
2	1	1	1	1
3	4	1	4	1
4	3	1	1	3

$\mid_2$	1	2	3	4
1	4	3	2	1
2	3	3	1	1
3	2	1	2	1
4	1	1	1	1

Inspection of the model reveals that (a) both strokes satisfy D and N; *but*, (b) $1|_1 2 \nvdash 1|_2 2$, since $1 \,|_1\, 2$ is assigned the designated value 1, but $1 \,|_2\, 2$ is assigned the non-designated value 3. □

4.2 Univocity in Gentzen Systems Using Sheffer's Stroke

We know of two Gentzen systems for classical sentential logic including Sheffer's stroke ($|$). One is due to Read [13], and the other is due to Zach [16]. In both systems, the Sheffer stroke is univocal, and this is our final theorem.

Theorem 11. *The Sheffer stroke is univocal, relative to both Read's Gentzen system* **SC***, and Zach's Gentzen system* **NS***.*

Proof. Here is a derivation of $\alpha \,|_1\, \beta \vdash_{\mathbf{SC}} \alpha \,|_2\, \beta$.

$$\dfrac{\dfrac{\dfrac{\alpha \,|_1\, \beta \qquad \overset{1}{\alpha} \qquad \overset{1}{\beta}}{\alpha \,|_2\, \beta}\,(|_1\mathrm{E})}{\alpha \,|_2\, \beta \vee \bot}\,(|_2\mathrm{I}_c)_1 \qquad \overset{2}{\alpha \,|_2\, \beta} \qquad \dfrac{\overset{2}{\bot}}{\alpha \,|_2\, \beta}\,(\bot\mathrm{E})}{\alpha \,|_2\, \beta}\,(\vee\mathrm{E})_2$$

And, here is a derivation of $\alpha \,|_1\, \beta \vdash_{\mathbf{NS}} \alpha \,|_2\, \beta$.

$$\dfrac{\dfrac{\alpha \,|_1\, \beta \qquad \overset{1}{\alpha} \qquad \overset{1}{\beta}}{\bot}\,(|_1\mathrm{E})}{\alpha \,|_2\, \beta}\,(|_2\mathrm{I})_1$$

Remark. It may seem surprising that the $\bot$-rule is not used in our derivation in Zach's system (of $\alpha|_1\beta \vdash_{\mathbf{NS}} \alpha|_2\beta$). But, something completely analogous happens when proving the univocity of negation, relative to Gentzen's original systems. To wit:

$$\dfrac{\dfrac{\overset{1}{\alpha} \qquad \neg_1\alpha}{\bot}\,(\neg_1\mathrm{E})}{\neg_2\alpha}\,(\neg_2\mathrm{I})_1$$

□

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