- Gibbard [5] argues that if the indicative conditional ($\rightsquigarrow$) satisfies *import-export* (and a few other assumptions), then it is logically equivalent to the material conditional ($\supset$).
- I will begin by rehearsing Gibbard's informal argument. Then, I will provide a rigorous, axiomatic proof of a more general "collapse theorem" for the indicative.

- Suppose the indicative satisfies *import-export*.

(IE) $A \rightsquigarrow (B \rightsquigarrow C)$ is *logically equivalent* to $(A \,\&\, B) \rightsquigarrow C$.

- If $\rightsquigarrow$ satisfies (IE), then (i) is equivalent to (ii).

(i) $(A \supset C) \rightsquigarrow (A \rightsquigarrow C)$.

(ii) $((A \supset C) \,\&\, A) \rightsquigarrow C$.

- Substitutivity of logical equivalents (in antecedents of indicatives) implies that (ii) [and ∴ (i)] is equivalent to (iii).

(iii) $(A \,\&\, C) \rightsquigarrow C$.

- So, if (iii) is a logical truth (as Gibbard supposes), then (i) and (ii) are too. Finally, suppose the indicative is at least as strong as the material conditional. That is, suppose (generally) that $p \rightsquigarrow q$ *entails* $p \supset q$. Then, (i) entails (iv).

(iv) $(A \supset C) \supset (A \rightsquigarrow C)$.

- Hence, (iv) is (also) a logical truth. So, $A \supset C$ entails $A \rightsquigarrow C$. Therefore, in general, $p \rightsquigarrow q$ entails $p \supset q$ and $p \supset q$ entails $p \rightsquigarrow q$. That is, in general, $\rightsquigarrow$ and $\supset$ are logically equivalent.

- Let $\mathscr{L}$ be a sentential (object) language containing atoms 'A', 'B', ..., and two *logical* connectives '&' and '$\rightarrow$'.
- $\mathscr{L}$ also contains another binary connective '$\rightsquigarrow$', which is meant to be interpreted as the English indicative.
- $\mathscr{L}$'s metalanguage contains metavariables $p, q, \ldots$ and two meta-linguistic relations: $\Vdash$ and $\vdash$. '$\Vdash$' is interpreted as *single premise deducibility* (or *entailment*). '$\vdash$' is interpreted as the property of *theoremhood* (or *logical truth*).

- Here's our weak background theory (independent axioms).

(1) $\vdash (p \,\&\, q) \rightarrow p$.

(2) $\vdash (p \,\&\, q) \rightsquigarrow q$.

(3) If $p \Vdash q$ and $p \Vdash r$, then $p \Vdash q \,\&\, r$.

(4) If $p \Vdash q$ and $q \Vdash p$, then $\vdash p \rightsquigarrow r$ if and only if $\vdash q \rightsquigarrow r$.

(5) If $\vdash p \rightarrow q$, then $p \Vdash q$.

(6) If $\vdash p \rightsquigarrow q$, then $\vdash p \rightarrow q$.

(7) $\vdash p \rightarrow (q \rightarrow r)$ if and only if $\vdash (p \,\&\, q) \rightarrow r$.

- The $\rightsquigarrow$ fragment of this background theory is *very* weak. (1)–(7) do *not* imply *any* of the following three principles.
  - If $\vdash p$ and $\vdash p \rightsquigarrow q$, then $\vdash q$.
  - $\vdash p \rightsquigarrow (q \rightsquigarrow p)$.
  - $\vdash (p \rightsquigarrow (q \rightsquigarrow r)) \rightsquigarrow ((p \rightsquigarrow q) \rightsquigarrow (p \rightsquigarrow r))$.

☞ *Modus ponens* for $\rightarrow$ does *not* follow from (1)–(7) either! So, *modus ponens* is *irrelevant* to Gibbardian collapse!

- Finally, consider this $\rightsquigarrow$*-import-export* axiom schema.

(8) $\vdash p \rightsquigarrow (q \rightsquigarrow r)$ if and only if $\vdash (p \,\&\, q) \rightsquigarrow r$.

Theorem 1. The schemata (1)–(8) are independent; and, given the background theory (1)–(7), (8) holds ***if and only if***

(9) $p \rightarrow q \Vdash p \rightsquigarrow q$ and $p \rightsquigarrow q \Vdash p \rightarrow q$.

Theorem 2. Axioms (1)–(8) do *not* entail collapse of $\rightsquigarrow$ to $\supset$. Even if we add *modus ponens* (MP) to (1)–(8), we *do **not*** get

(10) $\vdash ((p \rightsquigarrow q) \rightsquigarrow p) \rightsquigarrow p$.

That is, *Peirce's Law* is *not* implied by (1)–(8) + (MP). So, *classicality is inessential* to Gibbardian collapse.

Theorem 3. (1)–(8) + (MP) *do* imply that the indicative conditional collapses to a conditional that is *at least as strong as the intuitionistic conditional*: (1)–(8) + (MP) imply

(MP) If $\vdash p$ and $\vdash p \rightsquigarrow q$, then $\vdash q$.

(11) $\vdash p \rightsquigarrow (q \rightsquigarrow p)$.

(12) $\vdash (p \rightsquigarrow (q \rightsquigarrow r)) \rightsquigarrow ((p \rightsquigarrow q) \rightsquigarrow (p \rightsquigarrow r))$.

- While Gibbard's [5] argument was *logical* in nature, Lewis's triviality arguments [9, 8] were *probabilistic* in nature.
- I will derive these Lewisian results in a novel way. Normally, these results are derived *axiomatically* and in a way that obscures the crucial role of (probabilistic) import-export.
- I will adopt an *algebraic* approach. This will also allow us to derive *the strongest possible* Lewisian triviality result.
- Moreover, I will explain why these Lewisian triviality results all depend (implicitly) on (probabilistic) import-export.
- My presentation will mirror the way in which I presented my generalization of Gibbard's "collapse theorem."
- I will begin with a very weak probabilistic background theory for $\rightsquigarrow$. Then, I will show that, relative to this background theory, probabilistic import-export is *equivalent* to the condition that leads to Lewisian triviality.

- Let $\Pr(\cdot)$ be a probability function over a Boolean algebra of statements expressed *via* truth-functional connectives, *plus* a (possibly non-truth-functional) indicative connective $\rightsquigarrow$.
- Our background theory is the following single equational axiom schema [1, 12], sometimes called "The Equation."

  (I) $\Pr(p \rightsquigarrow q) = \Pr(q \mid p) \overset{\text{def}}{=} \frac{\Pr(p \& q)}{\Pr(p)}$, provided $\Pr(p) > 0$.
- The background theory (I) is *very* weak. That is, (I) *alone* does not entail any Lewisian trivialities for $\Pr(\cdot)$ and $\rightsquigarrow$.
- It is only when we combine (I) with the following *import-export* schema that we are led to Lewisian trivialities.

  (II) $\Pr(p \rightsquigarrow (q \rightsquigarrow r)) = \Pr((p \& q) \rightsquigarrow r)$, provided $\Pr(p \& q) > 0$.
- Given (I), (II) is *equivalent* to the following (*very* strong) equational axiom schema (see Extras #16 for a proof of this equivalence). I will call (III) "The *Resilient* Equation" [11].

  (III) $\Pr(p \rightsquigarrow q \mid x) = \Pr(q \mid p \& x)$, provided $\Pr(p \& x) > 0$.

- Lewis's original triviality results [9, 8] and all subsequent results of this kind [6, 10] are derived *via* x-instances of (III).
  - Lewis used the instances $x := q$ and $x := \neg q$ of (III) to derive his original triviality result [9]. Milne [10] used the instance $x := p \supset q$. More on these below. And, see [6] for a survey.
- A natural question is: What is *the strongest* triviality result that can be derived from (III), *via* instantiations of x?
- Using my decision procedure for Pr-calculus [4], I was able to determine the algebraic content of the conjunction of *all* (in a sense to be made precise shortly) x-instances of (III).
- Then, I was able to show that one only needs *three* x-instances of (III) to derive this *strongest* triviality result.
- Let's get more precise. Without loss of generality, consider the algebra $\mathcal{B}$ generated by the three ("atomic") statements $\{P, Q, P \rightsquigarrow Q\}$. We can visualize the family of probability functions $\Pr(\cdot)$ over $\mathcal{B}$ *via* a *stochastic truth-table* (STT).

P	Q	$P \rightsquigarrow Q$	$\Pr(\cdot)$
T	**T**	**T**	a
T	**T**	**F**	b
T	**F**	**T**	c
T	**F**	**F**	d
F	**T**	**T**	e
F	**T**	**F**	f
F	**F**	**T**	g
F	**F**	**F**	h

- So as to maximize generality, we assume $\{P, Q, P \rightsquigarrow Q\}$ are *logically independent*. In this way, *we assume nothing* about the *logical* relationship(s) between $P \rightsquigarrow Q$, P, and Q.
- Each of the $2^8 = 256$ propositions $x \in \mathcal{B}$ is then assigned a probability by $\Pr(\cdot)$ in the usual way — by adding up the probability masses of the states which feature in x's DNF.
- In this way, we can write down *algebraic* equations (in terms of $a, \ldots, h$) for each of the x-instances of (III). This allows us to determine the precise algebraic content of (III).

- It helps to re-state (III), so that it involves the "atomic" propositions $\{P, Q, P \rightsquigarrow Q\}$ from our algebra $\mathcal{B}$, above.

$(\mathrm{III}^{\mathcal{B}})$ $\Pr(P \rightsquigarrow Q \mid x) = \Pr(Q \mid P \,\&\, x)$, provided $\Pr(P \,\&\, x) > 0$.

- This rendition $(\mathrm{III}^{\mathcal{B}})$ makes it clear that (the universally quantified) x ranges over the 256 propositions in $\mathcal{B}$. As it happens, there are 191 instances of $(\mathrm{III}^{\mathcal{B}})$ which do not (by probability theory alone) violate $\Pr(P \,\&\, x) > 0$.
- The following theorem was verified [2, 4] by determining necessary and sufficient algebraic conditions for the joint satisfaction of all 191 of these equational constraints $(\mathrm{III}^{\mathcal{B}})$.

Theorem 4 ([2]). Provided that $\Pr(P \,\&\, Q) > 0$ and $\Pr(P \,\&\, \neg Q) > 0$, $(\mathrm{III}^{\mathcal{B}}) \iff \Pr(P \,\&\, (Q \equiv (P \rightsquigarrow Q))) = 1$.

- Luckily, the same result can be reached using *only three* of the 191 instances of $(\mathrm{III}^{\mathcal{B}})$. I will now go through that simpler proof of Theorem 4 (⇒). We proceed in three stages.

- **Stage 1.** The $\neg Q$–instance of $(\mathrm{III}^{\mathcal{B}})$.

$(\mathrm{III}^{\mathcal{B}}_{\neg Q})$ $\Pr(P \rightsquigarrow Q \mid \neg Q) = \Pr(Q \mid P \,\&\, \neg Q)$, provided $\Pr(P \,\&\, \neg Q) > 0$.

- Algebraically, $(\mathrm{III}^{\mathcal{B}}_{\neg Q})$ is equivalent to

$$\Pr(P \rightsquigarrow Q | \neg Q) = \frac{\Pr((P \rightsquigarrow Q) \,\&\, \neg Q)}{\Pr(\neg Q)} = \frac{c+g}{c+d+g+h} = 0 = \Pr(Q | P \& \neg Q)$$

- Assuming $\Pr(P \,\&\, \neg Q) > 0$, this equation will be true *iff* $c + g = 0$. Thus, $c = g = 0$, which yields this *revised* STT:

P	Q	$P \rightsquigarrow Q$	$\Pr(\cdot)$
T	**T**	**T**	a
T	**T**	**F**	b
T	**F**	**T**	0
T	**F**	**F**	d
F	**T**	**T**	e
F	**T**	**F**	f
F	**F**	**T**	0
F	**F**	**F**	h

- **Stage 2.** The $P \supset Q$–instance of $(\mathrm{III}^{\mathcal{B}})$.

$(\mathrm{III}^{\mathcal{B}}_{P \supset Q})$ $\Pr(P \rightsquigarrow Q \mid P \supset Q) = \Pr(Q \mid P \,\&\, (P \supset Q))$, if $\Pr(P \,\&\, Q) > 0$.

- Algebraically, $(\mathrm{III}^{\mathcal{B}}_{P \supset Q})$ is equivalent to

$$\Pr(P \rightsquigarrow Q | P \supset Q) = \frac{\Pr((P \rightsquigarrow Q) \,\&\, (P \supset Q))}{\Pr(P \supset Q)} = \frac{a+e}{a+b+e+f+h} = 1 = \Pr(Q | P \& (P \supset Q))$$

- Assuming $\Pr(P \,\&\, Q) > 0$, this equation will be true *iff* $b + f + h = 0$. So, $b = f = h = 0$, and our STT becomes:

P	Q	$P \rightsquigarrow Q$	$\Pr(\cdot)$
T	**T**	**T**	a
T	**T**	**F**	0
T	**F**	**T**	0
T	**F**	**F**	d
F	**T**	**T**	e
F	**T**	**F**	0
F	**F**	**T**	0
F	**F**	**F**	0

- **Stage 3.** The $\top$–instance of $(\mathrm{III}^{\mathcal{B}})$.

$(\mathrm{III}^{\mathcal{B}}_{\top})$ $\Pr(P \rightsquigarrow Q \mid \top) = \Pr(Q \mid P \,\&\, \top)$, provided $\Pr(P \,\&\, \top) > 0$.

- Algebraically, $(\mathrm{III}^{\mathcal{B}}_{\top})$ — which is just $(\mathrm{I}^{\mathcal{B}})$ — is equivalent to

$$\Pr(P \rightsquigarrow Q) = a + e = \frac{a}{a+d} = \frac{\Pr(P \,\&\, Q)}{\Pr(P)} = \Pr(Q \mid P)$$

- Since $\Pr(P) > 0$, this holds *iff* $a^2 + ad + ae + de - a = 0$. Since $\Pr(P \,\&\, \neg Q) = d > 0$, we have $e = 0$ and $a, d \in (0, 1)$. *Final* STT:

P	Q	$P \rightsquigarrow Q$	$\Pr(\cdot)$
T	**T**	**T**	$a \in (0,1)$
T	**T**	**F**	0
T	**F**	**T**	0
T	**F**	**F**	$1 - a$
F	**T**	**T**	0
F	**T**	**F**	0
F	**F**	**T**	0
F	**F**	**F**	0

- So, assuming $\Pr(P \,\&\, Q) > 0$ and $\Pr(P \,\&\, \neg Q) > 0$, ($\text{III}^{\mathcal{B}}$) implies that *exactly two states have non-zero probability*: $P \,\&\, Q \,\&\, (P \rightsquigarrow Q)$ and $P \,\&\, \neg Q \,\&\, \neg(P \rightsquigarrow Q)$. *QED*.
- *No stronger constraint* can be derived from ($\text{III}^{\mathcal{B}}$); and, *at least two instances* of ($\text{III}^{\mathcal{B}}$) are required for the result [2].
- Algebraically, it is easy to see exactly *how much* stronger our result is than previous results. Our result implies that *the six probability masses b, c, e, f, g and h are all zero*.
  - Lewis [9] relies on the two instances ($\text{III}^{\mathcal{B}}_{Q}$) and ($\text{III}^{\mathcal{B}}_{\neg Q}$), which only imply that the four masses b, c, f and g are zero. As a result, Lewis's results do not imply (*e.g.*) that $\Pr(P) = 1$.
  - Milne [10] relies on the single instance ($\text{III}^{\mathcal{B}}_{P \supset Q}$), which only implies that the three masses b, f and h are zero. As a result, he obtains *neither* $\Pr(P) = 1$ *nor* $\Pr(Q) = \Pr(P \rightsquigarrow Q)$.
- It's hard to think of *any* models that (generally) satisfy (III). Here's one. $\Pr(\cdot)$ is an *indicator function*, and $p \rightsquigarrow q \overset{\text{def}}{=} q$.

- The triviality results of Gibbard and Lewis seem to suggest that import-export is problematic. But, it is difficult to come up with intuitive counterexamples to either (8) or (II).
- Stephan Kaufmann [7] describes a possible counterexample to *both* (8) *and* (II). Here is (my rendition of) his example.

> Suppose that the probability that a given match ignites if struck is low, and consider a situation in which it is very likely that the match is *not* struck but instead is tossed into a camp fire, where it ignites without being struck. Now, consider the following two indicative conditionals.
>
> (a) If the match will ignite, then it'll ignite if struck.
>
> (b) If the match is struck and it'll ignite, then it'll ignite.

- According to Kaufmann, while (b) is clearly necessarily (even logically) true, (a) is not. Indeed, Kaufmann even claims that the probability of (a) should be less than 1.
- If he is right, we have a counterexample to *both* (8) *and* (II).

- Given our background theory (I), (II) $\Longleftrightarrow$ (III).
- Proof of $\Longrightarrow$. Assuming (I) and (II), prove (III).

①	$\Pr(x \rightsquigarrow (p \rightsquigarrow q)) = \Pr(p \rightsquigarrow q \mid x)$, if $\Pr(p \,\&\, x) > 0$	(I)
②	$\Pr(x \rightsquigarrow (p \rightsquigarrow q)) = \Pr((p \,\&\, x) \rightsquigarrow q)$, if $\Pr(p \,\&\, x) > 0$	(II)
③	$\Pr((p \,\&\, x) \rightsquigarrow q) = \Pr(q \mid p \,\&\, x)$, if $\Pr(p \,\&\, x) > 0$	(I)
(III)	$\therefore \Pr(p \rightsquigarrow q \mid x) = \Pr(q \mid p \,\&\, x)$, if $\Pr(p \,\&\, x) > 0$	①, ②, ③

- Proof of $\Longleftarrow$. Assuming (III) and (I), prove (II). [Note: (III) ⇒ (I).]

④	$\Pr(p \rightsquigarrow (q \rightsquigarrow r)) = \Pr(q \rightsquigarrow r \mid p)$, if $\Pr(p \,\&\, q) > 0$	(I)
⑤	$\Pr(q \rightsquigarrow r \mid p) = \Pr(r \mid p \,\&\, q)$, if $\Pr(p \,\&\, q) > 0$	(III)
⑥	$\Pr(r \mid p \,\&\, q) = \Pr((p \,\&\, q) \rightsquigarrow r)$, if $\Pr(p \,\&\, q) > 0$	(I)
(II)	$\therefore \Pr(p \rightsquigarrow (q \rightsquigarrow r)) = \Pr((p \,\&\, q) \rightsquigarrow r)$, if $\Pr(p \,\&\, q) > 0$	④, ⑤, ⑥

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