1 Setting The Stage

Once upon a time, various philosophers (e.g., [1, 13]) defended the idea that the probability of an indicative conditional \( (P \rightarrow Q) \) is the conditional probability of its consequent \((Q)\), given its antecedent \((P)\). More precisely, the following principle has been proposed and defended by various authors:

**The Equation.** \( \Pr(P \rightarrow Q) = \Pr(Q \mid P) \).

David Lewis [8, 7] published several triviality results involving The Equation. Since then, several other authors have published similar triviality results (see, e.g., [5, 11]). In section two, I will explain the basic ideas behind all of these Lewisian triviality results. In section three, I will prove a new Lewisian triviality result. In fact, I will prove the strongest possible result of this kind. All other (published) Lewisian triviality results are strictly weaker than ours, and there can be no stronger result along these lines (in a sense to be made precise in section four).

2 Import-Export + The Equation \(\Rightarrow\) Lewisian Triviality

There’s nothing trivial about The Equation per se. But, if we combine The Equation with another (seemingly plausible) assumption about the probabilities of nested conditionals, then Lewisian trivialities ensue. That assumption is the so-called Import-Export Law, which (probabilistically) is expressed as follows.

**Import-Export.** \( \Pr(X \rightarrow (P \rightarrow Q)) = \Pr((P \& X) \rightarrow Q) \).

The conjunction of Import-Export and The Equation is equivalent to the following “resilient” equation:

**The Resilient Equation.** \( \Pr(P \rightarrow Q \mid X) = \Pr(Q \mid P \& X) \), provided \( \Pr(P \& X) > 0 \).

And, it is The Resilient Equation that is the true target of Lewisian triviality arguments. In the next section, I will present a new Lewisian triviality result, which subsumes all existing results of its kind.

3 The Strongest Lewisian Triviality Result

In this section, I will prove the following triviality result.

**Triviality.** The Resilient Equation \(\iff\) \( \Pr(P \& (Q \equiv (P \rightarrow Q))) = 1 \).

What follows is an algebraic proof of Triviality. The generic stochastic truth-table representation of the class of probability functions \( \Pr(\cdot) \) over the eight states determined by \( P, Q, P \rightarrow Q \) is as follows:

---

1See [5] for a nice survey article on The Equation and its history.
2I use the term “resilient” here, because it is reminiscent of the Skyrmian [12] notion of resiliency. More recently, Hannes Leitgeb has endorsed a resilient version of the Lockean Thesis [10], which also has various unintuitive consequences [11]. I think the ultimate source of Lewisian triviality is this requirement of resiliency (and not The Equation per se). Moreover, the Import-Export Law is implicated in various other “triviality” results for the indicative conditional [4, 10, 2]. As such, I’d be inclined to reject Import-Export here, rather than The Equation. But, I’ll have to leave the proper treatment of that question for another investigation.
3Strictly speaking, the \(\iff\) direction of Triviality also requires \( \Pr(P \& Q) > 0 \) and \( \Pr(P \& \neg Q) > 0 \). This is clarified in the proof.
4Here, I’m using the terminology and setup of [3], which provides a general technique for reasoning algebraically about the probability calculus. Moreover, I will be assuming (without loss of generality) that \( P, Q, \) and \( P \rightarrow Q \) are logically independent of each other. If there were logical dependencies between them, then this would only serve to strengthen our triviality result.
It turns out that one does not need the full strength of The Resilient Equation in order two show that it implies the right-hand side of Triviality. That is, one does not need to conditionalize on all X's such that Pr(P & X) > 0 in order to derive this (strongest) triviality result from The Resilient Equation. In fact, all we need are three instances of The Resilient Equation. I will now work my way up to Triviality, in three stages.

3.1 Stage 1: The $\sim Q$–instance of The Resilient Equation

Consider the following instance of The Resilient Equation, where $X := \sim Q$.

\[
\text{The Resilient Equation } \sim Q. \quad \Pr(P \rightarrow Q | \sim Q) = \Pr(Q | P & \sim Q), \quad \text{provided } \Pr(P & \sim Q) > 0.
\]

Algebraically, The Resilient Equation $\sim Q$ is equivalent to the following, provided $\Pr(P & \sim Q) > 0$.

\[
\Pr(P \rightarrow Q | \sim Q) = \frac{\Pr((P \rightarrow Q) & \sim Q)}{\Pr(\sim Q)} = \frac{c + g}{c + d + g + h} = 0 = \Pr(Q | P & \sim Q)
\]

This equation will be true iff $c + g = 0$, which implies that $c$ and $g$ must both be equal to zero. The effect of The Resilient Equation $\sim Q$ is therefore reflected in the following revised stochastic truth-table.

<table>
<thead>
<tr>
<th>P</th>
<th>Q</th>
<th>P $\rightarrow$ Q</th>
<th>Pr(·)</th>
</tr>
</thead>
<tbody>
<tr>
<td>T</td>
<td>T</td>
<td>T</td>
<td>$a$</td>
</tr>
<tr>
<td>T</td>
<td>T</td>
<td>F</td>
<td>$b$</td>
</tr>
<tr>
<td>T</td>
<td>F</td>
<td>T</td>
<td>$c$</td>
</tr>
<tr>
<td>T</td>
<td>F</td>
<td>F</td>
<td>$d$</td>
</tr>
<tr>
<td>F</td>
<td>T</td>
<td>T</td>
<td>$e$</td>
</tr>
<tr>
<td>F</td>
<td>T</td>
<td>F</td>
<td>$f$</td>
</tr>
<tr>
<td>F</td>
<td>F</td>
<td>T</td>
<td>$g$</td>
</tr>
<tr>
<td>F</td>
<td>F</td>
<td>F</td>
<td>$h$</td>
</tr>
</tbody>
</table>

3.2 Stage 2: The $P \supset Q$–instance of The Resilient Equation

Consider the following instance of The Resilient Equation, where $X := P \supset Q$.

\[
\text{The Resilient Equation } P \supset Q. \quad \Pr(P \rightarrow Q | P \supset Q) = \Pr(Q | P & (P \supset Q)), \quad \text{provided } \Pr(P & (P \supset Q)) > 0.
\]

Algebraically, The Resilient Equation $P \supset Q$ is equivalent to the following, provided $\Pr(P & Q) > 0$.

\[
\Pr(P \rightarrow Q | P \supset Q) = \frac{\Pr((P \rightarrow Q) & (P \supset Q))}{\Pr(P \supset Q)} = \frac{a + e}{a + b + e + f + h} = 1 = \Pr(Q | P & (P \supset Q))
\]

---

5This was one of the instances used by Lewis to derive his original triviality results. The other instance he used was $X := Q$. It can be shown that Lewis’s pair of constraints is strictly weaker than our (maximally strong) set of three constraints. For instance, Lewis’s pair of instances do not jointly entail $\Pr(P) = 1$. See the companion Mathematica notebook (fn.7) for a proof of this.

6This is the instance used by Milne to derive his triviality result. Milne’s instance is strictly weaker than our (maximally strong) set of three constraints. For instance, Milne’s instance does not entail either $\Pr(P) = 1$ or $\Pr(Q) = \Pr(P \rightarrow Q)$. See the companion Mathematica notebook (fn.7) for a proof of this.
Cross-multiplying (and expanding and simplifying) this equation yields

\[ 0 = b + f + h \]

This equation will be true iff \( b, f \) and \( h \) are all equal to zero. The effects of The Resilient Equation \( p \rightarrow q \) + The Resilient Equation \( p 
\equiv \neg q \) are reflected in the following revised stochastic truth-table.

\[
\begin{array}{ccc|c}
P & Q & P \rightarrow Q & Pr(\cdot) \\
T & T & T & a \\
T & T & F & 0 \\
T & F & T & 0 \\
T & F & F & d \\
F & T & T & e \\
F & T & F & 0 \\
F & F & T & 0 \\
F & F & F & 0 \\
\end{array}
\]

3.3 Stage 3: The \( \top \)-instance of The Resilient Equation — i.e., The Equation Itself

Consider the following instance of The Resilient Equation, where \( X := \top \).

The Resilient Equation \( \top \). \( \Pr(P \rightarrow Q | \top) = \Pr(Q | P \& \top) \), provided \( \Pr(P \& \top) > 0 \).

Of course, The Resilient Equation \( \top \) is just The Equation itself. Algebraically, The Equation is now

\[
\Pr(P \rightarrow Q) = a + e = \frac{a}{a + d} = \Pr(Q | P)
\]

Cross-multiplying (and expanding and simplifying) this equation yields the following quadratic equation

\[ a^2 + ad + ae + de - a = 0 \]

Recall, we are assuming (from Stage 1) that \( \Pr(P \& \neg Q) > 0 \). That is, we are assuming that \( d > 0 \). As it happens, when \( d > 0 \) (and the background probabilistic constraints on \( a, d, e \) hold [3]), the quadratic equation above is satisfied iff \( e = 0 \), \( d = 1 - a \), and \( a, d \in (0, 1) \).

The effects of The Resilient Equation \( \neg q \) + The Resilient Equation \( p \rightarrow q \) + The Equation are reflected in the following (final) single-parameter stochastic truth-table, where \( a \in (0, 1) \).

\[
\begin{array}{ccc|c}
P & Q & P \rightarrow Q & Pr(\cdot) \\
T & T & T & a \\
T & T & F & 0 \\
T & F & T & 0 \\
T & F & F & 1 - a \\
F & T & T & 0 \\
F & T & F & 0 \\
F & F & T & 0 \\
F & F & F & 0 \\
\end{array}
\]

In other words, The Resilient Equation \( \neg q \) + The Resilient Equation \( p \rightarrow q \) + The Equation jointly entail that the only two states which can be assigned non-zero probability are \( P \& Q \) and \( P \& \neg Q \& \neg (P \rightarrow Q) \). This is equivalent to saying that the proposition \( P \& (Q \equiv (P \rightarrow Q)) \) must receive maximal probability. QED

Triviality is very strong. It implies that, for every \( P \) and \( Q \) that feature as the antecedent and consequent of some indicative conditional \( P \rightarrow Q \), both \( P \) and the material biconditional \( Q \equiv (P \rightarrow Q) \) must receive maximal probability (and, as a result, we must also have \( \Pr(Q) = \Pr(P \rightarrow Q) \)). All of the existing Lewisian triviality results are strictly weaker than this one. In fact, there can be no stronger Lewisian triviality result.
4 Epilogue: Why Triviality is The Strongest (Lewisian) Triviality Result

**Triviality** is *the strongest* triviality result of its kind. Here’s what I mean. If one assumes *all* of the instances of **The Resilient Equation**, then this *still (only)* implies **Triviality**. That is, adding further instances of **The Resilient Equation** to the three we used above *does not add any additional constraints to* $\Pr(\cdot)$. This can be shown algebraically by proving that the conjunction of *all* (191) instances of **The Resilient Equation** (where $X$ ranges over the 256 propositions in the Boolean algebra generated by $P, Q, P \rightarrow Q$) is *equivalent to* the conjunction of the *three* instances of **The Resilient Equation** that we used above (and this also secures the $\Leftarrow$ direction of **Triviality**).

References


---

\[7\]This can easily be verified using *Mathematica*. I have created a *Mathematica* (version 10) notebook which verifies that **Triviality** is *the strongest* (Lewisian) triviality result for the indicative conditional. It also shows that the results of Lewis and Milne are strictly weaker than ours, and that one needs *at least three instances* of **The Resilient Equation** to establish **Triviality**. This *Mathematica* notebook can be downloaded from the following URL: [http://fitelson.org/triviality.nb](http://fitelson.org/triviality.nb)