CHAPTER 1

Relations

1.1 Notation and Terminology

In this chapter, we present a mathematical topic, the theory of relations. The concepts and techniques presented here will be used throughout the rest of the volume. The notation we shall use is summarized in Table 1.1, fundamental properties of relations are defined in Table 1.2, and types of relations are defined in Table 1.3. Many readers will be familiar with most of the material of this chapter. The reader may simply want to glance at Tables 1.1, 1.2, and 1.3. He should, however, familiarize himself with Theorems 1.2, 1.3, and 1.4 of Section 1.5 and with the material of Section 1.8.

1.2 Definition of a Relation

Suppose A and B are sets. The *Cartesian product* of A with B, denoted $A \times B$, is the set of all *ordered* pairs (a, b) so that a is in A and b is in B. More generally, if $A_1, A_2, \ldots, A_n$ are sets, the *Cartesian product*

$$A_1 \times A_2 \times \ldots \times A_n$$

is the set of all ordered n-tuples $(a_1, a_2, \ldots, a_n)$ such that $a_1 \in A_1$, $a_2 \in A_2, \ldots, a_n \in A_n$. The notation A^n denotes the Cartesian product of A with itself n times.

A *binary relation* R on the set A is a subset of the Cartesian product $A \times A$, that is, a set of ordered pairs (a, b) such that a and b are in A. If A is the set $\{1, 2, 3, 4\}$, then examples of binary relations on A are given by

$$R = \{(1, 1), (1, 2), (2, 1), (3, 3), (3, 4), (4, 3)\}, \tag{1.1}$$

$$S = \{(1, 1), (1, 2), (2, 1), (2, 2), (3, 3), (3, 4), (4, 3), (4, 4)\}, \tag{1.2}$$

ENCYCLOPEDIA OF MATHEMATICS and Its Applications, Gian-Carlo Rota (ed.).
Vol. 7: Fred S. Roberts, Measurement Theory

Table 1.1. Notation

Set-Theoretic Notation	
$\cup$	union
$\cap$	intersection
$\subseteq$	subset (contained in)
$\subsetneqq$	proper subset
$\nsubseteq$	is not a subset
$\supseteq$	contains (superset)
$\in$	member of
$\notin$	not a member of
$\varnothing$	empty set
$\{\ldots\}$	the set ...
$\{\ldots : \ldots\}$	the set of all ... such that ...
A^c	complement of A
$A-B$	$A \cap B^c$
$\lvert A \rvert$	cardinality of A, the number of elements in A
Logical Notation	
$\sim$	not
$\Rightarrow$	implies
$\Leftrightarrow$	if and only if (equivalence)
$\forall$	for all
$\exists$	there exists
iff	if and only if
Sets of Numbers	
Re	the real numbers
Re^+	the positive real numbers
Q	the rational numbers
Q^+	the positive rational numbers
N	the positive integers
Z	the integers
Miscellaneous	
$f \circ g$	composition of the two functions f and g
$f(A)$	the image of the set A under the function f; i.e., $\{f(a): a \in A\}$.
$\approx$	approximately equal to
$\equiv$	congruent to
Π	product
Σ	sum
$\int$	integral sign

Table 1.2. Properties of Relations

A Binary Relation (A, R) Is:	Provided That:
Reflexive	aRa, all $a \in A$
Nonreflexive	it is not reflexive
Irreflexive	$\sim aRa$, all $a \in A$
Symmetric	$aRb \Rightarrow bRa$, all $a, b \in A$
Nonsymmetric	it is not symmetric
Asymmetric	$aRb \Rightarrow \sim bRa$, all $a, b \in A$
Antisymmetric	aRb & $bRa \Rightarrow a = b$, all $a, b \in A$
Transitive	aRb & $bRc \Rightarrow aRc$, all $a, b, c \in A$
Nontransitive	it is not transitive
Negatively transitive	$\sim aRb$ & $\sim bRc \Rightarrow \sim aRc$, all $a, b, c \in A$; equivalently: $xRy \Rightarrow xRz$ or zRy, all $x, y, z \in A$
Strongly complete	for all $a, b \in A$, aRb or bRa
Complete	for all $a \neq b \in A$, aRb or bRa
Equivalence relation	it is reflexive, symmetric and transitive

Table 1.3. Order Relations*

	Relation Type						
Property	Quasi Order	Weak Order	Simple Order	Strict Simple Order	Strict Weak Order	Partial Order	Strict Partial Order
Reflexive	√					√	
Symmetric							
Transitive	√	√	√	√		√	√
Asymmetric				√	√		√
Antisymmetric			√			√	
Negatively transitive					√		
Strongly complete		√	√				
Complete				√			

*A given type of relation can satisfy more of these properties than those indicated. Only the defining properties are indicated.

and

$$T = \{(1, 2), (1, 3), (1, 4), (2, 3), (2, 4), (3, 4)\}. \quad (1.3)$$

The binary relation T is the "less than" relation on A; an ordered pair (a, b) is in the binary relation T if and only if $a < b$. Similarly, "less than" defines a binary relation on the set A of all real numbers, as does "greater than," "equals," and so on. Of course, the set A does not have to be a set of numbers. If A is the set $\{SO_2, \text{DDT}, NO_x\}$, then examples of binary relations on A are given by

$$U = \{(SO_2, \text{DDT}), (\text{DDT}, NO_x), (SO_2, NO_x)\} \quad (1.4)$$

and

$$V = \{(SO_2, NO_x), (NO_x, SO_2)\}. \quad (1.5)$$

In the case of a binary relation R on a set A, we shall usually write aRb to denote the statement that $(a, b) \in R$. Thus, for example, if S is the relation defined by Eq. (1.2), then $3S3$ and $2S1$, but not $3S1$. If U is the relation defined by Eq. (1.4), then $SO_2 U DDT$.

Binary relations arise very frequently from everyday language. For example, if A is the set of people in the world, then the set

$$F = \{(a, b): a \in A \text{ and } b \in A \text{ and } a \text{ is the father of } b\} \quad (1.6)$$

defines a binary relation on A, which we may call, by a slight abuse of language, "father of." To give another example, suppose A is any collection of alternatives among which you are choosing, for example, a collection of designs for a regional transportation system, and suppose

$$P = \{(a, b) \in A \times A: \text{you (strictly) prefer } a \text{ to } b\}. \quad (1.7)$$

Then P may be called your relation of "strict preference."* Someone else's relation of preference might be quite different, and that of course is where problems arise. To give yet another example, suppose A is a set of airplane engines and L is the relation "sounds louder than when heard at a horizontal distance of 500 feet." The relation L on A can play a role in the design of airplanes. It is hoped that the relation L is related to some physical characteristics of the engine design and of the sounds engines emit. The study of the relationship between the physical properties of these sounds and the psychological ones, such as perceived loudness, is the subject matter of the field called psychophysics. (We return to this study in Chapter 4.)

The properties of a relation are not clearly defined without giving its underlying set. Thus, if A is the set of all people in the United States and B is the set of all males in the United States, then the relation

$$R = \{(a, b) \in A \times A: a \text{ is the brother of } b\} \quad (1.8)$$

is different from the relation

$$R' = \{(a, b) \in B \times B: a \text{ is the brother of } b\}. \quad (1.9)$$

For example, R' has certain symmetry properties; that is, if $aR'b$, then $bR'a$. These properties are not shared by R. Moreover, R' has different properties if it is thought of as a relation on the set B or as a relation on the set A (even though R' contains no ordered pairs with elements not in B). To make sure that the set A on which a relation R is defined is given explicitly, it is necessary to speak formally of a *relational system* (A, R)

*Strict preference is to be distinguished from weak preference: the former means "better than," and the latter "at least as good as." See Section 1.5.

rather than just a relation R. By an abuse of language, we shall simply call (A, R) a relation. We shall see more precisely below why specification of the underlying set is important. In general, if $B \subseteq A$ and R is a binary relation on A, we shall refer to

$$S = \{(a, b) \in B \times B: aRb\}$$

as the *restriction of R to B* or the *subrelation generated by B*. Thus the relation (B, R') defined by Eq. (1.9) is the restriction of the relation (A, R) of Eq. (1.8) to the set B of all males in the United States.

We may also speak of n-ary relations, where n is a positive integer. An *n-ary relation* R on a set A is a subset of the Cartesian product A^n. (We shall frequently speak of a relation, rather than an n-ary relation, if n is understood.) For example, if $A = \{1, 2, 6\}$, then a 3-ary or *ternary* relation R on A is given by

$$R = \{(1, 2, 6), (6, 2, 1), (6, 6, 6)\}. \tag{1.10}$$

If A is the set of all lines in the plane, we might define a ternary relation R on A as follows:

$$(a, b, c) \in R \Leftrightarrow a, b, \text{ and } c \text{ are parallel and } b \text{ is strictly between } a \text{ and } c. \tag{1.11}$$

A similar ternary relation R might be defined on the set A of all students in a given school. Then we would take

$$(a, b, c) \in R \Leftrightarrow b\text{'s grade point average is strictly between those of } a \text{ and } c. \tag{1.12}$$

If $A = \{1, 2, 6\}$, then a 4-ary or *quaternary* relation R on A is given by

$$R = \{(1, 2, 2, 6), (1, 2, 1, 6), (6, 6, 6, 6)\}. \tag{1.13}$$

If A is once again a set of alternatives such as designs for alternative regional transportation systems, you might make statements like $\mathbb{S}$: "I prefer a to b at least as much as I prefer c to d." A quaternary relation on A is given by the collection D of all ordered 4-tuples (a, b, c, d) such that a, b, c, d are in A and for which you assert such a statement $\mathbb{S}$ of comparative preference. In such a case, we shall use either the notation $D(a, b, c, d)$ or the notation $abDcd$ to mean that $(a, b, c, d) \in D$.

In what follows, we shall most frequently deal with binary relations, and shall often speak just of a *relation* when we mean a binary one.

Exercises

1. If (A, R) is a binary relation, the *converse* is the relation R^{-1} on A defined by

$$aR^{-1}b \quad \text{iff} \quad bRa.$$

(The notation $\breve{R}$ is sometimes used in place of R^{-1}.) For example, if A is the set of all males in the United States and (A, R) is "father of," then (A, R^{-1}) is "son of." Identify the converse of the following relations:

(a) "Sister of" on the set of all people in the United States.
(b) "Uncle of" on the set of all people in the Soviet Union.
(c) "Greater than" on the set Re.
(d) $(Re, =)$.
(e) "As tall as" on the set of all men in New Jersey.

2. If (A, R) and (A, S) are binary relations, the *intersection* $R \cap S$ on A is defined by

$$R \cap S = \{(a, b)\colon aRb \text{ and } aSb\}.$$

For example, if (A, R) is "brother of" and (A, S) is "sibling of," then $(A, R \cap S)$ is "brother of." Identify $(A, R \cap S)$ in the following cases:

(a) A = a set of people, R = "father of," S = "relative of."
(b) $A = Re$, $R = \geqq$, $S = \neq$.
(c) A = a set of people, R = "older than," S = "father of."
(d) A = a set of sets, $R = \subsetneqq$, S = "are disjoint."

3. If (A, R) and (A, S) are binary relations, the *union* $R \cup S$ on A is defined by

$$R \cup S = \{(a, b)\colon aRb \text{ or } aSb\}.$$

For example, if (A, R) is "brother of" and (A, S) is "sister of," then $(A, R \cup S)$ is "sibling of." Identify $(A, R \cup S)$ in the examples of Exer. 2.

4. If (A, R) and (A, S) are binary relations, then the *relative product* $R \circ S$ on A is defined by

$$R \circ S = \{(a, b)\colon \text{for some } c,\ aRc \text{ and } cSb\}.$$

For example, if (A, R) is "father of" and (A, S) is "parent of," then $a(R \circ S)b$ holds if and only if, for some c, a is father of c and c is parent of b—that is, if and only if a is grandfather of b. Identify $R \circ S$ in the following examples:

(a) A = a set of people, R = "father of," S = "mother of."
(b) A = a set of people, R = "older than," S = "older than."
(c) $A = Re$, $R = >$, $S = >$.
(d) $A = Re$, $R = >$, $S = <$.

5. Show the following:

(a) $(R \cup S) \circ T = (R \circ T) \cup (S \circ T)$.
(b) $(R \cap S) \circ T$ may not be $(R \circ T) \cap (S \circ T)$.
(c) $R \circ (S \circ T) = (R \circ S) \circ T$.
(d) $(R \circ S)^{-1}$ may not be $R^{-1} \circ S^{-1}$.

6. Show that $(A, S \circ R)$ can be different from $(A, R \circ S)$.

7. Suppose $A = \{1, 2, 6\}$ and (A, R) is the quaternary relation of Eq. (1.13). Then we may define a ternary relation S on A and a binary relation T on A by

$$(a, b, c) \in S \Leftrightarrow (\exists x)[R(a, b, c, x)]$$

and
$$aTb \Leftrightarrow (\exists x)(\exists y)[R(a, b, x, y)].$$

(a) Write out (A, S) and (A, T) as sets of ordered n-tuples.

(b) Let U be the quaternary relation defined by the restriction of R to $B = \{1, 6\}$. Write out (B, U).

1.3 Properties of Relations

There are certain properties that are common to many naturally occurring relations. We discuss some of these properties in this section, and they are summarized in Table 1.2 at the beginning of the chapter. Let us say that a binary relation (A, R) is *reflexive* if, for all $a \in A$, aRa. Thus, for example, if A is a set of real numbers and R is the relation "equality" on A, then (A, R) is reflexive because a number is always equal to itself. If $A = \{1, 2, 3, 4\}$, then the relation (A, R) defined by Eq. (1.1) is not reflexive, since $2R2$ does not hold. On the other hand, the restriction of this relation to the set $B = \{1, 3\}$ is reflexive. This observation demonstrates again why it is important to refer to the underlying set when speaking of a relation.

If A is the set of people in the world and F is the relation "father of" on A, then (A, F) is not reflexive, since a person is not his own father. This relation is not reflexive in a very strong way, since the condition of reflexivity is violated for every a in A. We shall say that a binary relation (A, R) is *irreflexive* if, for all $a \in A$, $\sim aRa$. Thus the relation "father of" is irreflexive. So is the relation (A, T) where $A = \{1, 2, 3, 4\}$ and T is defined by Eq. (1.3), and the relation (A, U) where $A = \{SO_2, \text{DDT}, NO_x\}$ and U is defined by Eq. (1.4). This terminology should be distinguished from the terminology *nonreflexive*, which means simply "not reflexive."

A binary relation (A, R) is called *symmetric* if, for all $a, b \in A$,
$$aRb \Rightarrow bRa.$$
That is, (A, R) is symmetric if, whenever $(a, b) \in R$, then $(b, a) \in R$. The relation "equality" on any set of numbers is symmetric. So are the relations (A, R) where $A = \{1, 2, 3, 4\}$ and R is defined by Eq. (1.1), and (A, V) where $A = \{SO_2, \text{DDT}, NO_x\}$ and V is defined by Eq. (1.5). The relation (A, T) where $A = \{1, 2, 3, 4\}$ and T is given by Eq. (1.3), is not symmetric, for $1T2$ but $\sim 2T1$. The relation "brother of" on the set of all people in the United States is not symmetric, for if a is the brother of b, it does not follow that b is the brother of a. However, the relation "brother of" on the set of all males is symmetric.

Other examples of *nonsymmetric* (not symmetric) relations are the relation "father of" on the set of people in the world, the relation P of strict preference on a set of alternatives, and the relation "sounds louder than" on a set of sounds. These three relations are highly nonsymmetric. They are *asymmetric*, i.e., they satisfy the rule
$$aRb \Rightarrow \sim bRa.$$

Other asymmetric relations are the relation "greater than," $>$, on the set of real numbers, the relation "strictly contained in," $\subsetneqq$, on any collection of sets, and the relation (A, U) where $A = \{SO_2, \text{DDT}, NO_x\}$ and U is given by Eq. (1.4).

Some relations (A, R) are not quite asymmetric, but are almost asymmetric in the sense that aRb and bRa holds only if $a = b$. Let us say that (A, R) is *antisymmetric* if, for all $a, b \in A$,

$$aRb \;\&\; bRa \Rightarrow a = b.$$

An example of an antisymmetric relation is the relation "greater than or equal to," $\geqq$, on the set of real numbers. Another example is "contained in," $\subseteqq$, on any collection of sets. Every asymmetric binary relation (A, R) is antisymmetric. But the converse is false: the relation $\geqq$ is antisymmetric but not asymmetric.

A relation (A, R) is called *transitive* if, for all $a, b, c \in A$, whenever aRb and bRc, then aRc. In symbols, (A, R) is transitive if

$$aRb \;\&\; bRc \Rightarrow aRc.$$

Examples of transitive relations are the relations "equality" and "greater than" on the set of real numbers and "implies" on a set of statements. It is left to the reader to verify that the relations (A, S) where $A = \{1, 2, 3, 4\}$ and S is defined by Eq. (1.2), and (A, U) where $A = \{SO_2, \text{DDT}, NO_x\}$ and U is defined by Eq. (1.4), are transitive. It seems reasonable to assume that the relation of strict preference among alternative designs of transportation systems is transitive, for if you prefer a to b and b to c, you should be expected to prefer a to c. We shall discuss this point further in later chapters. Similarly, it seems reasonable to assume that the relation L, "sounds louder than," on a set of airplane engines is transitive, though this must be left to empirical data to verify. If $A = \{SO_2, \text{DDT}, NO_x\}$ and V is defined by Eq. (1.5), then (A, V) is not transitive. For $SO_2 V NO_x$ and $NO_x V SO_2$, but not $SO_2 V SO_2$. Another relation that is not transitive is the relation "father of."

In studying binary relations (A, R), it will often be convenient to use the abbreviation $aRbRc$ for the statement aRb & bRc. Thus, (A, R) is transitive if $aRbRc$ implies aRc. Similarly, $aRbRcRd$ will abbreviate aRb & bRc & cRd. And so on. If transitivity holds, then $aRbRcRd$ implies aRd. More generally, $a_1Ra_2Ra_3 \ldots Ra_n$ implies a_1Ra_n. The proof is easily accomplished by mathematical induction.

A binary relation (A, R) is called *negatively transitive* if, for all $a, b, c \in A$, not aRb and not bRc imply not aRc. A binary relation (A, R) is negatively transitive if the relation "not in the relation R," defined on the set A, is transitive. To give an example, the relation $R =$ "greater than" on a set of real numbers is negatively transitive, for "not in R" is the relation "less than or equal to," which is transitive. It is easy to show that if $A = \{SO_2, \text{DDT}, NO_x\}$ and U is defined by Eq. (1.4), then (A, U) is negatively transitive. Similarly, strict preference on a set of alternatives and

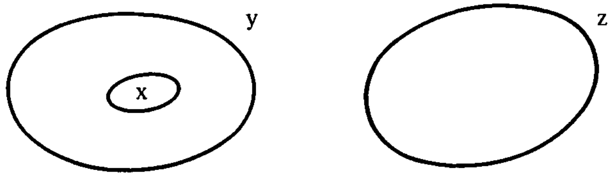

Figure 1.1. The relation "contained in" is not necessarily negatively transitive.

"sounds louder than" on a set of sounds are probably negatively transitive. Verifying negative transitivity can be annoyingly confusing; it is often easier to test the following equivalent condition: For all $x, y, z \in A$, if xRy, then xRz or zRy. To prove that these two conditions are equivalent, we observe that the equivalent version is just the contrapositive of the condition in negative transitivity.* Using this notion, we see easily that the relation "greater than" is negatively transitive, for if $x > y$, then for all z, either $x > z$ or $z > y$. Similarly, one sees that "contained in" is not negatively transitive, for if x is contained in y, there may very well be a z so that x is not contained in z and z is not contained in y. (See Fig. 1.1 for an example.) The relation "father of" on the set of all people in the world is not negatively transitive, nor is the relation (A, R) where $A = \{1, 2, 3, 4\}$ and R is given by Eq. (1.1). To see the latter, note that $1R2$, but not $1R3$ and not $3R2$.

Exercises

1. (a) Show that if (A, R) is a binary relation and (A, R^{-1}) is its converse (Exer. 1, Section 1.2), then (A, R^{-1}) is reflexive if and only if (A, R) is.
   (b) Is a similar statement true for the property irreflexive?
   (c) Symmetric?
   (d) Asymmetric?
   (e) Antisymmetric?
   (f) Transitive?
   (g) Negatively transitive?

2. Suppose (A, R) and (A, S) are binary relations.
   (a) If both relations (A, R) and (A, S) are reflexive, is the intersection $(A, R \cap S)$ reflexive?
   (b)–(g) Repeat for the properties in (b) through (g) of Exer. 1.

3. Repeat Exer. 2 for the union $(A, R \cup S)$.

4. Repeat Exer. 2 for the relative product $(A, R \circ S)$.

5. Which of the properties in (a) through (g) of Exer. 1 hold for the relational system $(A, \varnothing)$?

6. Which of the properties in (a) through (g) of Exer. 1 hold for the relational system $(A, A \times A)$?

*If $\mathbb{S}$ is the statement A implies B, then the *contrapositive* of $\mathbb{S}$ is the statement "not B" implies "not A."

7. (a) Show that it is not possible for a binary relation to be both symmetric and asymmetric.

(b) Show that it is possible for a binary relation to be both symmetric and antisymmetric.

8. Show that there are binary relations that are
   (a) transitive but not negatively transitive;
   (b) negatively transitive but not transitive;
   (c) neither negatively transitive nor transitive;
   (d) both negatively transitive and transitive.

9. (a) Consider the relation x divides y on the set of positive integers. Which of the properties in (a) through (g) of Exer. 1 does this relation have?
   (b) Repeat part (a) for the relation "uncle of" on a set of people.
   (c) For the relation of "having the same weight as" on a set of mice.
   (d) For the relation "feels smoother than" on a set of objects.
   (e) For the relation "admires" on a set of people.

10. Suppose $A = Re$ and

$$aRb \Leftrightarrow a > b + 1.$$

This relation will arise in our study of preference in Chapter 6. Which of the properties in (a) through (g) of Exer. 1 hold for (A,R)?

11. Consider the binary relation (A,S) where $A = Re$ and

$$aSb \Leftrightarrow |a - b| \leqq 1.$$

This relation is closely related to the binary relation (A, R) of Exer. 10, and will arise in our study of indifference in Chapter 6. Which of the properties in (a) through (g) of Exer. 1 hold for (A, S)?

12. If (A, R) is a binary relation, the *symmetric complement* is the binary relation S on A defined by

$$aSb \Leftrightarrow (\sim aRb \ \& \sim bRa).$$

Note that if R is strict preference, then S is indifference: you are indifferent between two alternatives if and only if you prefer neither.

(a) Show that the symmetric complement is always symmetric.

(b) Show that if (A,R) is negatively transitive, then the symmetric complement is transitive.

(c) Show that the converse of (b) is false.

(d) Show that if $A = Re$ and R is as defined in Exer. 10, then S as defined in Exer. 11 is the symmetric complement.

(e) Identify the symmetric complement of the following relations:
   (i) $(Re, >)$.
   (ii) $(Re, =)$.
   (iii) (N, R), where xRy means that x does not divide y.

13. The data of Table 1.4 shows the (consensus) preferences among composers of the members of an orchestra. The i,j entry is 1 if and only if composer i is (strictly) preferred to composer j. Which of the properties in

(a) through (g) of Exer. 1 hold for the orchestra members' relation of preference on the set

$$A = \{\text{Beethoven, Brahms, Mozart, Wagner}\}?$$

Table 1.4. Preferences of Orchestra Members Among Composers*
(The i,j entry is 1 iff composer i is (strictly) preferred to composer j.)

	Beethoven	Brahms	Mozart	Wagner
Beethoven	0	1	1	1
Brahms	0	0	1	1
Mozart	0	0	0	1
Wagner	0	0	0	0

*Data based on an experiment of Folgmann [1933], with strict preference for the orchestra taken to mean a majority of the orchestra members have that preference.

14. The data of Table 1.5 represent taste preferences for vanilla puddings by a group of judges, with entry i, j equal to 1 if and only if the group (strictly) prefers pudding i to pudding j. Which of the properties in (a) through (g) of Exer. 1 hold for this relation of preference on the set $\{1, 2, 3, 4, 5\}$?

Table 1.5. Taste Preference for Vanilla Puddings*
(Entry i,j is 1 iff pudding i is (strictly) preferred to pudding j by a group of judges.)

	1	2	3	4	5
1	0	0	1	1	0
2	1	0	0	1	0
3	0	1	0	0	0
4	0	0	1	0	0
5	1	1	0	1	0

*Data obtained from an experiment of Davidson and Bradley [1969].

15. The data of Table 1.6 state judgments of relative loudness of different sounds, with entry i, j taken to be 1 if and only if sound i is judged (definitely) louder than sound j. Which of the properties in (a) through (g) of Exer. 1 hold for the relation "louder than" on the set of sounds $\{1, 2, 3, 4, 5\}$?

Table 1.6. Judgments of Relative Loudness.
(Entry i,j is 1 iff sound i is judged (definitely) louder than sound j.)

	1	2	3	4
1	0	1	0	1
2	0	0	1	1
3	1	0	0	0
4	0	0	0	0

16. The data of Table 1.7 present the judgments of "sameness" among different sounds by individuals in a group, with entry i, j taken to be 1 if and only if sound i is judged to be the same as sound j a sufficiently large percentage of the time. Which of the properties in (a) through (g) of Exer. 1 hold for this relation of sameness on the set {B, C, F, J}?

Table 1.7. Judgments of Sameness of Sounds*
(Entry i,j is taken to be 1 iff sound i is judged to be the same as sound j at least 25% of the time.)

	B	C	F	J
B	1	1	1	0
C	1	1	1	1
F	1	1	1	1
J	0	1	1	1

*Data from an experiment of Rothkopf [1957].

17. The data of Table 1.8 present judgments of relative importance of different objectives for a library system in Dallas, with entry i, j equal to 1 if and only if objective i is judged more important than objective j. Which of the properties in (a) through (g) of Exer. 1 hold for this relation of relative importance on the set {a, b, c, d, e, f}?

Table 1.8. Judgments of Relative Importance of Objectives for a Library System in Dallas*
(Entry i,j is 1 iff i is judged more important than j.)

	a	b	c	d	e	f
a	0	1	1	1	1	1
b	0	0	0	0	0	0
c	0	0	0	0	0	1
d	0	1	0	0	1	0
e	0	0	0	0	0	0
f	0	0	0	0	0	0

*Data from Farris [1975].

Key

a Convenient, accessible library facilities
b Convenient operating hours
c Efficient inter-library system
d Good local libraries
e Good reproduction facilities
f Rapid inter-library response time

18. The data of Table 1.9 present judgments of relative importance of different objectives for a state environmental agency in Ohio, with entry

Table 1.9. Judgments of Relative Importance of Goals for a State Environmental Agency in Ohio*
(Entry i,j is 1 iff i is judged more important than j.)

	a	b	c	d	e	f
a	0	1	1	1	1	1
b	0	0	0	0	0	0
c	0	1	0	0	1	0
d	0	1	0	0	1	0
e	0	1	0	0	0	0
f	0	1	0	0	1	0

*Data from Hart [1975].

Key

a Enhance and protect State's environment
b Improve and insure the quality of the air
c Develop a comprehensive program of environmental quality planning
d Protect and promote the State's natural attractions
e Prevent the future occurrence of pollution emergencies
f Promote an environment that is beneficial to human health and welfare

i, j taken to be 1 if and only if objective i is judged more important than objective j. Which of the properties in (a) through (g) of Exer. 1 hold for this relation of relative importance on the set {a, b, c, d, e, f}?

1.4 Equivalence Relations

Many binary relations satisfy the three properties reflexivity, symmetry, and transitivity. Such relations are called *equivalence relations*. The relation equality on any set of numbers is an equivalence relation. So is the relation (A, S), where A is $\{1, 2, 3, 4\}$ and S is defined by Eq. (1.2). If A is a set of lines and aRb holds if and only if a and b are parallel, then (A, R) is an equivalence relation provided that we say a line is parallel to itself. Other examples are the following: A is the set of integers $\{0, 1, 2, \ldots, 26\}$, and aRb iff $a \equiv b(\text{mod } 3)$; A is a set of people who have been blood-typed, and aRb iff a and b have the same blood type; A is a set of people, and aRb iff a has the same height as b; A is a set of children, and aRb iff a and b have the same IQ; A is a set of animals, and aRb iff a and b are in the same species.

If (A, R) is an equivalence relation and $a \in A$, let a^* denote

$$\{b \in A : aRb\}.$$

This set will be called the *equivalence class containing a*. The element a will be called a *representative* of the equivalence class a^*. To give an example, if $A = \{1, 2, 3, 4\}$ and S is defined by Eq. (1.2), then

$$1^* = \{1, 2\},$$
$$2^* = \{1, 2\},$$
$$3^* = \{3, 4\},$$
$$4^* = \{3, 4\}.$$

Here, there are two different equivalence classes, $\{1, 2\}$ and $\{3, 4\}$. If $A = \{0, 1, 2, \ldots, 26\}$ and aRb iff $a \equiv b \pmod 3$, then

$$0^* = \{0, 3, 6, 9, 12, 15, 18, 21, 24\},$$
$$1^* = \{1, 4, 7, 10, 13, 16, 19, 22, 25\},$$
$$2^* = \{2, 5, 8, 11, 14, 17, 20, 23, 26\},$$
$$3^* = \{0, 3, 6, 9, 12, 15, 18, 21, 24\},$$
$$\ldots$$

Here, there are three different equivalence classes, 0^*, 1^*, and 2^*.

The most important properties of equivalence relations are summarized in the following theorem.

THEOREM 1.1. *Suppose (A, R) is an equivalence relation. Then:*

(a) Any two equivalence classes are either disjoint or identical.

(b) The collection of (distinct) equivalence classes partitions A; that is, every element of A is in one and only one (distinct) equivalence class.

Proof. To prove (a), we shall show that for all $a, b \in A$, either $a^* = b^*$ or $a^* \cap b^* = \varnothing$. In particular, we shall show that

$$aRb \Rightarrow a^* = b^* \tag{1.14}$$

and

$$\sim aRb \Rightarrow a^* \cap b^* = \varnothing. \tag{1.15}$$

To demonstrate (1.14), we assume aRb and show that aRc holds if and only if bRc holds. Thus, suppose aRc holds. Then cRa follows, since (A, R) is symmetric. Now we have cRa and aRb, so we conclude cRb from transitivity of (A, R). Finally, cRb implies bRc, by symmetry. A similar proof, left to the reader, shows that bRc implies aRc. Thus, we have established (1.14). To prove (1.15), suppose $a^* \cap b^* \neq \varnothing$ and let $c \in a^* \cap b^*$. Then $c \in a^*$ implies aRc, and $c \in b^*$ implies bRc. By symmetry, we have cRb. Finally, by transitivity, aRc and cRb imply aRb. This proves (1.15), and completes the proof of part (a).

To prove part (b), note that every element a in A is in some equivalence class, namely a^*, and in no more than one, by part (a). ■

In general, whenever (A, R) is an equivalence relation, we can gain much in the way of economy by dealing with the equivalence classes rather than the objects of A themselves. There are in general many fewer such. We shall see this clearly in the next section, when we discuss the process of reduction.

Exercises

1. Suppose A is the set of sequences of 0's and 1's of length 10, and aRb holds if and only if sequences a and b have the same number of 1's.

(a) Show that (A,R) is an equivalence relation.

(b) Identify the equivalence classes.

2. (a) Show that if A is a set of sounds and aRb holds if and only if a and b *sound* equally loud, then (A,R) may not be an equivalence relation.

(b) If A is a set of sounds, and aRb holds if and only if a and b are measured to have the same decibel level (a measure of sound intensity), is (A,R) an equivalence relation?

(c) If A is a set of people and aRb holds if and only if a and b *look* equally tall, is (A,R) an equivalence relation?

(d) If A is a set of people and aRb holds if and only if a and b are measured to have the same height, is (A,R) an equivalence relation?

3. Suppose (A,S) is the binary relation of Exer. 11, Section 1.3. Is (A,S) an equivalence relation?

4. Show that all the properties in the definition of an equivalence relation are needed. In particular, show that there are binary relations that are

(a) reflexive, symmetric, and not transitive;

(b) reflexive, transitive, and not symmetric;

(c) symmetric, transitive, and not reflexive.

5. Suppose (A, R) and (A, S) are equivalence relations.

(a) Show that $(A, R \cap S)$ is an equivalence relation.

(b) Show that $(A, R \cup S)$ does not have to be an equivalence relation.

(c) What about $(A, R \circ S)$?

6. Show that if (A, R) is an equivalence relation, then

$$a^* = b^* \quad \text{iff} \quad a \in b^*.$$

7. If (A,R) is an equivalence relation, we may define a binary relation R^* on the set A^* of equivalence classes as follows: if α and β are equivalence classes, and a is in α and b is in β, then

$$\alpha R^* \beta \quad \text{iff} \quad aRb.$$

(a) Show that R^* is well-defined in the sense that if a' is in α and b' is in β, then

$$aRb \quad \text{iff} \quad a'Rb'.$$

(b) Moreover, show that (A^*,R^*) is an equivalence relation.

8. Suppose A is finite and R is a binary relation on A. Let R^2 be the binary relation $R \circ R$ on A, and define R^n to be $R^{n-1} \circ R$.

(a) Show that if $n \geqq 1$, then aR^nb if and only if there are $a_1, a_2, \ldots, a_{n-1}$ so that $aRa_1, a_1Ra_2, \ldots, a_{n-2}Ra_{n-1}, a_{n-1}Rb$.

(b) Define R^0 on A to be $\{(a, a): a \in A\}$. If aR^nb for some $n \geqq 0$, we say that there is a (finite) *path* from a to b of *length n*. If A is finite, show that there is a number k so that if there is a path from a to b, there is one of length at most k.

(c) Let S be defined on A as follows: aSb if and only if there is a path from a to b. Show that (A, S) is transitive.

(d) Let T be defined on A as follows: $T = S \cap S^{-1}$. Show that aTb if and only if there is a path from a to b and a path from b to a.

(e) Show that (A, T) is an equivalence relation. (The equivalence classes are called *strong components*.)

(f) (A, R) is called *strongly connected* if there is just one equivalence class under T. Is the relation $<$ on a finite set of numbers strongly connected?

(g) If A is finite, the *transitive closure* of (A, R) is the transitive relation on A containing all ordered pairs in R and the smallest possible number of ordered pairs. Show that the notion of transitive closure is well-defined.

(h) How is transitive closure related to the relation S defined in (c)?

(i) Identify R^2, R^3, S, T, the strong components, and the transitive closure in the following examples. Determine which of these examples is strongly connected.

(i) $A = Re, R = \geqq$.
(ii) $A = \{1, 2, 3, 4\}, R = \{(1, 2), (2, 3), (3, 4)\}$.
(iii) $A = \{1, 2, 3\}, R = \{(1, 2), (2, 1), (1, 3)\}$.
(iv) $A = \{+, *, \#, \$\}, R = \{(+, \#), (*, *), (+, *)\}$.

9 (a) Does the data of Table 1.4 define an equivalence relation?
(b) What about the data of Table 1.5?
(c) Table 1.6?
(d) Table 1.7?
(e) Table 1.8?
(f) Table 1.9?

1.5 Weak Orders and Simple Orders

In this section, we define various order relations. The results are summarized at the beginning of the chapter in Table 1.3, which the reader is urged to consult for reference.

Suppose (A,P) is the binary relation of (strict) preference defined by Eq. (1.7), where A is a set of alternatives among which you are choosing—for example, alternative designs for a regional transportation system. In general, we can suppose that for each pair of alternatives a and b, you do one of three things: You prefer a to b, you prefer b to a, or you are indifferent between a and b. Let us say you *weakly prefer a* to b if either you (strictly) prefer a to b or you are indifferent between a and b. We denote the binary relation of weak preference on the set A by W, and the binary relation of indifference on A by I. Then we have

$$aWb \Leftrightarrow (aPb \text{ or } aIb).$$

It is reasonable to assume that the relation (A, W) is both reflexive and transitive, though later we shall question transitivity. A binary relation that satisfies these two properties is called a *quasi order* or pre-order. The relation $\geqq$ on the set of real numbers is another example of a quasi order; so is the relation $\subseteqq$ on any collection of sets; so is the relation "at least as tall as"; and so is any equivalence relation.

The relation (A, W) presumably also has the property that for every a and b in A, including $a = b$, either aWb or bWa. A binary relation with this property is called *strongly complete* (sometimes the terms connected or strongly connected are used). A binary relation is called a *weak order* if it is transitive and strongly complete. Thus, weak preference is a weak order. So is the relation $\geqq$ on the set Re of real numbers. The relation $>$ on Re is not a weak order, for it is not strongly complete; it is not the case that $1 > 1$. Similarly, $\subseteqq$ is not weak, because it is not strongly complete. Any weak order (A, R) is a quasi order. It suffices to prove reflexivity, which follows by strong completeness. A quasi order is not necessarily a weak order. An example of a quasi order that is not weak is given by any equivalence relation with more than one equivalence class, as, for example, the relation (A, S), where $A = \{1, 2, 3, 4\}$ and S is defined by Eq. (1.2). One of the most helpful examples of a weak order is the following, which we shall use as an example frequently. Let $A = \{0, 1, 2, \ldots, 26\}$. Every number a in A is congruent (mod 3) to one of the numbers 0, 1, or 2. Let us call this number a mod 3. Thus, 8 mod 3 is 2, 10 mod 3 is 1, etc. We define R on A as follows:

$$aRb \Leftrightarrow a \bmod 3 \geqq b \bmod 3. \tag{1.16}$$

Thus, $2R1$, $8R10$, etc. It is left to the reader to prove that (A,R) is a weak order. We may think of (A,R) as follows: Elements of A are listed in vertical columns above the number 0, 1, or 2 to which they are congruent. Then aRb if and only if a is at least as far to the right as b. (See Fig. 1.2.) It will follow from Theorem 1.2 below that essentially every weak order on a finite set can be thought of as an ordering "weakly to the right of" on a comparable array of points arranged in vertical columns. A sample array is shown in Fig. 1.3.

A weak order (A,R) that is also antisymmetric is called a *simple order*. (The terms *linear order* and *total order* are also used.) The prototype of simple orders is the relation $\geqq$. In a simple order R on a finite set A, the elements of A may be laid out on the line with aRb holding if and only if a is to the right of b or equal with b, as shown in Fig. 1.4. (We prove this formally in Section 3.1.)

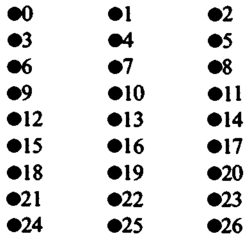

Figure 1.2. The binary relation $\geqq$ (mod 3) on $A = \{0, 1, 2, \ldots, 26\}$.

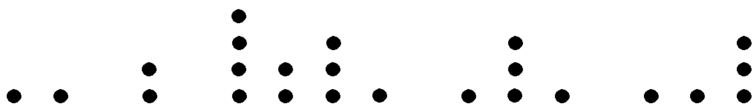

Figure 1.3. A weak order (A, R); aRb iff a is to the right of b or in the same vertical column as b.

Figure 1.4. A simple order.

If (A, R) is a binary relation, suppose we define a binary relation E on A by

$$aEb \Leftrightarrow aRb \;\&\; bRa. \tag{1.17}$$

If we think of an array like that of Fig. 1.3, the relation E can be interpreted as "being in the same vertical column." If (A, R) is a simple order, then antisymmetry implies that E is equality. More generally, if (A, R) is a quasi order, then E is an equivalence relation. To see this, suppose (A, R) is a quasi order. We verify first that (A, E) is symmetric. If aEb holds, then aRb and bRa hold; hence bRa and aRb hold; hence bEa holds. (Notice that this proof does not use any of the properties of a quasi order.) Proof that (A, E) is reflexive and transitive is left to the reader.

If (A, R) is a weak order, then it is a quasi order, and so E is an equivalence relation. The relation R tells how to simply order the equivalence classes under E. To be precise, let A^* be the collection of equivalence classes, and define R^* on A^* as follows: If α and β are equivalence classes, pick $a \in \alpha$ and $b \in \beta$, and let $\alpha R^* \beta$ hold if and only if aRb holds. Of course, this process may lead to ambiguities, because whether we take $\alpha R^* \beta$ may depend on which a and b we chose. We shall show below that this is not the case, that is, that R^* is well-defined. Assuming this for now, we can summarize the definition of R^* as follows:

$$a^*R^*b^* \Leftrightarrow aRb. \tag{1.18}$$

Then (A^*, R^*) is called the *reduction* or *quotient* of (A, R).†

To give an example, suppose $A = \{0, 1, 2, \ldots, 26\}$ and aRb holds iff $a \bmod 3 \geqq b \bmod 3$. Then aEb holds iff $a \equiv b \pmod 3$. There are three equivalence classes: 0^*, 1^*, and 2^*. We have $2^*R^*1^*$, since $2R1$; and similarly $2^*R^*0^*$, $2^*R^*2^*$, etc. We see that here the reduction (A^*, R^*) is a simple order on the set A^* of equivalence classes; it is like the usual simple order $\geqq$ on $\{0, 1, 2\}$. That this example is not a special case is summarized in the following theorem.

THEOREM 1.2. *Suppose (A,R) is a weak order. Then the reduction (A^*, R^*) is well-defined, and it is a simple order.*

Proof. To show that R^* is well-defined, we need to show that its definition does not depend on the particular choice of elements $a\in\alpha$ and $b\in\beta$ of the equivalence classes α and β. Put in other words, if we choose $c\in\alpha$ and $d\in\beta$, then we need to show that aRb iff cRd. To show this, suppose aRb. Since a and c are in α, we conclude that aEc. Similarly, bEd. Now aEc implies aRc and cRa, and bEd implies bRd and dRb. Now, if aRb, then using cRa, we conclude cRb by transitivity. From cRb and bRd we conclude cRd, again by transitivity. The proof that cRd implies aRb is analogous. Thus, R^* is well-defined.

To prove that (A^*, R^*) is a simple order, one must verify that it is transitive, strongly complete, and antisymmetric. To show that it is transitive, suppose that α, β, and γ are in A^* and that $\alpha R^*\beta R^*\gamma$. To show $\alpha R^*\gamma$, pick a in α and c in γ and show aRc. Now given $b\in\beta$, we have aRb, since $\alpha R^*\beta$, and we have bRc, since $\beta R^*\gamma$. Since (A,R) is transitive, aRc follows. This implies $\alpha R^*\gamma$. The rest of the proof is left to the reader. ■

COROLLARY *The reduction is well-defined even if (A,R) is only a quasi order.*

Proof. The proof of well-definedness uses only this fact. ■

If we accept the fact that every simple order on a finite set can be realized as the ordering $\geqq$ on a set of real numbers, then we may intepret Theorem 1.2 as follows: Every weak order (A,R) on a finite set A may be

†(A^*, R^*) is sometimes denoted $(A/E, R/E)$, and the process of reduction is sometimes called "canceling out the equivalence relation." This is similar to what we do in group theory when we pass from a group to its cosets.

realized on the real line with equivalent elements in the same vertical column and so that aRb holds if and only if a is to the right of b or equal with b (cf. Fig. 1.3).

The relation $>$ on the real numbers is not a simple order. It is not strongly complete, because it is not reflexive. But it is *complete* in the sense that for all $a \neq b$, aRb or bRa. The relation $(Re, >)$ has many properties, among them transitivity, completeness, asymmetry, antisymmetry, and negative transitivity. We would like to characterize this ordering $>$ just as we characterized the ordering $\geqq$ by listing properties or axioms that essentially determined it—namely, the axioms for a simple order. In axiom-building, one tries to be as frugal as possible, and list only those properties that are needed. Of the above list, some are superfluous. For asymmetry implies antisymmetry. Similarly, transitivity and completeness imply negative transitivity. (The proofs are left to the reader.) These observations lead us to adopt the following definition: A binary relation (A,R) is called a *strict simple order* if it is asymmetric, transitive, and complete. (It is left to the reader to show that none of the conditions in this definition is superfluous.) It is often assumed that strict preference is a strict simple order, though it probably violates at least completeness: We can be indifferent between two alternatives. The relation of strict containment $\subsetneqq$ is not a strict simple order, because it violates completeness. The relation "beats" in a round-robin tournament usually does not determine a strict simple order, for transitivity is violated. Naturally, $>$ on the set of reals is a strict simple order. We shall prove in Section 3.1 that in every strict simple order R on a finite set A, the elements of A may be laid out on the line with aRb holding if and only if a is strictly to the right of b. Thus, $>$ is indeed the prototype of strict simple orders. It is also easy to show, and we shall ask the reader to show it, that the strict simple orders and simple orders are related to each other in the same way that $>$ is related to $\geqq$. Namely, suppose (A, R) is an irreflexive relation and S is defined on A by

$$aSb \Leftrightarrow aRb \text{ or } a = b.$$

Then (A,R) is a strict simple order if and only if (A,S) is a simple order.

To complete our discussion of order relations, we would like to define a type of order relation, called *strict weak*, which corresponds to strict simple orders in the same way that weak orders correspond to simple orders. The paradigm example will again be an ordering of an array of vertical columns, but now an element a is in the relation R to an element b if and only if a is strictly to the right of b. An example can be defined as follows: Let $A = \{0, 1, 2, \ldots, 26\}$, and let aRb hold if and only if $a \bmod 3 > b \bmod 3$. Such a relation will clearly be asymmetric and transitive, but no longer complete: of two elements a and b in the same vertical column, neither aRb nor bRa. Two elements a and b are in the same vertical column if and only if $\sim aRb$ and $\sim bRa$, and this suggests that we should study the following binary relation:

$$aEb \Leftrightarrow \sim aRb \text{ and } \sim bRa.^* \tag{1.19}$$

If (A, R) is strict simple, then (A, E) is equality. In general, we would like (A, E) to be an equivalence relation. We could define (A, R) to be a strict weak order if it is asymmetric and transitive and (A, E) is an equivalence relation. (The reader might wish to check that our relation $>$ (mod 3) on $\{0, 1, 2, \ldots, 26\}$ satisfies these properties.) However, there is something unsatisfactory about this definition. Specifically, all our definitions of order relations (A, R) so far have been stated in terms of properties of (A, R), and not in terms of properties of relations like (A, E) which are defined from (A, R). Any definition of strict weak order should turn out to be equivalent to this potential definition, and that will be the case with the definition we adopt. We shall say that (A, R) is by definition a *strict weak order* if (A, R) is asymmetric and negatively transitive. To justify this definition, we prove the following theorem.

THEOREM 1.3. *(A, R) is a strict weak order if and only if*

(a) *(A, R) is asymmetric,*

and

(b) *(A, R) is transitive,*

and

(c) *(A, E) is an equivalence relation, where E is defined by Eq.* (1.19).

Proof. Assume (A,R) is strict weak. Then it is asymmetric by definition. To show that it is transitive, suppose aRb and bRc. To show aRc, suppose by way of contradiction that $\sim aRc$. By asymmetry, bRc implies $\sim cRb$. Now by negative transitivity, $\sim aRc$ and $\sim cRb$ imply $\sim aRb$, which is a contradiction. It is left to the reader to prove that if (A,R) is strict weak, then (A,E) is an equivalence relation.

To complete the proof of Theorem 1.3, let us assume that (A,R) satisfies conditions (a), (b), and (c). To show that (A,R) is strict weak, it is sufficient to show that (A,R) is negatively transitive. To demonstrate negative transitivity, we assume that $\sim aRb$ and $\sim bRc$, and show $\sim aRc$. We argue by cases.

CASE 1: bRa. In this case, if aRc, we conclude bRc by transitivity. Thus, $\sim aRc$.

CASE 2: $\sim bRa$. Here, there are two subcases.

CASE 2a: cRb. In this case, if aRc, we conclude aRb by transitivity. Thus, $\sim aRc$.

*In Exer. 12 of Section 1.3, (A,E) is called the symmetric complement of (A,R).

CASE 2b: $\sim cRb$. Here, we have aEb and bEc, since $\sim aRb$, $\sim bRa$, $\sim bRc$, and $\sim cRb$. Since (A, E) is an equivalence relation, we conclude aEc, from which $\sim aRc$ follows. ∎

Even if indifference is allowed, it is often assumed that the relation of strict preference is an example of a strict weak order, though later we shall question this assumption. Another example is the binary relation "weighs more than" on a set of people, if weight is measured on a precise scale. A third example is the binary relation "warmer than" on a set of objects, if warmer than is based on temperature and temperature is measured on a precise scale. We shall return to these examples in Chapter 3.

The process of reduction is the same for strict weak orders as it is for weak orders. If (A, R) is a strict weak order, let the equivalence relation E be defined by Eq. (1.19) and let A^* be the collection of equivalence classes under E. Then define the *reduction* R^* on A^* as before, by

$$a^* R^* b^* \Leftrightarrow aRb.$$

THEOREM 1.4 *Suppose (A, R) is a strict weak order. Then the reduction (A^*, R^*) is well-defined and it is a strict simple order.*

Proof. The proof is left to the reader. ∎

Exercises

1. (a) Is the converse R^{-1} of a strict weak order R necessarily strict weak?
   (b) Is the converse of a weak order necessarily weak?
   (c) Is the converse of a strict simple order necessarily strict simple?
   (d) Is the converse of a simple order necessarily simple?
2. Is every quasi order a simple order?
3. Is every strict weak order strict simple?
4. Let $A = \{(a, b): a, b \in \{1, 2, 3, 4\}\}$, and suppose

$$(a, b)R(c, d) \quad \text{iff} \quad a > c.$$

Show that (A, R) is a strict weak order and calculate the reduction (A^*, R^*).

5. Suppose $A = \{a, b, x, y, \alpha, \beta, \gamma\}$, and R consists of the following ordered pairs:

$$\begin{aligned}&(a, a), (b, b), (x, x), (y, y), (\alpha, \alpha), (\beta, \beta), (\gamma, \gamma),\\ &(y, x), (x, y), (x, a), (x, b), (y, a), (y, b), (\alpha, \beta),\\ &(\alpha, \gamma), (\beta, \alpha), (\beta, \gamma), (\gamma, \alpha), (\gamma, \beta), (\alpha, x), (\alpha, y),\\ &(\alpha, a), (\alpha, b), (\beta, x), (\beta, y), (\beta, a), (\beta, b), (\gamma, x),\\ &(\gamma, y), (\gamma, a), (\gamma, b).\end{aligned}$$

Note that (A, R) is a weak order and calculate the reduction.

6. Suppose $A = Re \times Re$, and suppose P is defined on A by

$$(a, b)P(s, t) \quad \text{iff} \quad a \geqq s \;\&\; b \geqq t \;\&\; (a > s \text{ or } b > t).$$

Show that (A,P) is not a strict weak order or a weak order.

7. Suppose $A = Re \times Re$, and suppose P is defined on A by

$$(a,b)P(s,t) \quad \text{iff} \quad a > s \text{ or } (a = s \text{ and } b > t).$$

(a) Show that (A,P) is a strict weak order—it is called the *lexicographic ordering of the plane*.
(b) Is (A,P) strict simple?
(c) Is (A,P) weak?

8. Suppose (A,R) is the binary relation of Exer. 10, Section 1.3.
(a) Is (A,R) a strict weak order?
(b) A weak order?

9. Suppose A is a set of students and aRb holds if and only if a has a higher grade point average than b, or a and b have the same grade point average and a has a smaller number of absences.
(a) Is (A,R) strict weak?
(b) If so, what is its reduction?

10. Suppose (A, R) and (A, S) are weak orders.
(a) Is $(A, R \cap S)$ necessarily weak?
(b) What about $(A, R \cup S)$?
(c) What about $(A, R \circ S)$?

11. Show that the axioms for a simple order are all needed by giving examples of binary relations that are
(a) transitive, antisymmetric, and not strongly complete;
(b) transitive, strongly complete, and not antisymmetric;
(c) antisymmetric, strongly complete, and not transitive.

12. Show that the axioms for a strict simple order are all needed by giving examples of binary relations that are
(a) asymmetric, transitive, and not complete;
(b) asymmetric, complete, and not transitive;
(c) transitive, complete, and not asymmetric.

13. Prove that transitivity and completeness imply negative transitivity.

14. Prove that asymmetry implies antisymmetry.

15. If (A, R) is a simple order, prove that (A^*, R^*) is
(a) strongly complete;
(b) antisymmetric.

16. Show that if (A,R) is a strict weak order and E is defined on A by Eq. (1.19), then (A,E) is an equivalence relation.

17. If (A,R) is a quasi order and E is defined on A by Eq. (1.17), show that (A,E) is
 (a) reflexive;
 (b) transitive.

18. Prove Theorem 1.4.

19. Suppose (A,R) is irreflexive. Define S on A by

$$aSb \quad \text{iff} \quad (aRb \text{ or } a=b).$$

Show that (A,R) is strict simple if and only if (A,S) is simple.

20. Suppose (A,R) is strict weak and E is defined on A by Eq. (1.19). Define S on A by

$$aSb \quad \text{iff} \quad (aRb \text{ or } aEb).$$

Show that (A,S) is weak.

21. Suppose (A,R) is weak. Define S on A by

$$aSb \quad \text{iff} \quad (aRb \;\&\; \sim bRa).$$

Show that (A, S) is strict weak.

22. (a) Does the data of Table 1.4 define a strict weak order?
 (b) A weak order?
 (c) A strict simple order?
 (d) A simple order?

23. Repeat Exer. 22 for the data of Table 1.5.

24. Repeat Exer. 22 for the data of Table 1.6.

25. Repeat Exer. 22 for the data of Table 1.7.

26. Repeat Exer. 22 for the data of Table 1.8.

27. Repeat Exer. 22 for the data of Table 1.9.

1.6 Partial Orders

Very often an ordering relation violates completeness, as we have seen. For example, the relation $>(\text{mod}\,3)$ does; the relation "father of" does; most probably, the relation "louder than" on a set of airplane engines does, since very possibly two different engines will sound equally loud; and the relation $\subseteq$ of containment on a collection of sets does. The latter relation has the properties of reflexivity and transitivity and also antisymmetry: $X \subseteq Y$ and $Y \subseteq X$ implies $X = Y$. A binary relation satisfying these three properties is called a *partial order*. Thus, a partial order is an antisymmetric quasi order. Each simple order is a partial order, but not conversely. The relations "father of" and "louder than" are not partial orders, since they are not reflexive.

Partial orders often arise if we are stating preferences among alternatives that have several aspects or dimensions. Thus, suppose A is a set of alternative designs for a transportation system. Suppose we judge these designs on the basis of two aspects: cost and number of people served. Let us suppose we (weakly) prefer design a to design b if and only if a costs no more than b and a serves at least as many people as b. Let us denote this weak preference relation as W. Then (A, W) is certainly not complete: if a costs less than b and serves fewer people, then $\sim aWb$ and $\sim bWa$. On the other hand, (A, W) is transitive and, if we assume that no two designs both cost the same and serve the same number of people, it is also antisymmetric. Thus, (A, W) is a partial order.

More generally, suppose A is a collection of alternatives each of which has n dimensions or aspects, and suppose the real number $f_i(x)$ measures the "worth" of x on the ith aspect. It is not unreasonable to define a (weak) preference relation W on A by

$$aWb \quad \text{iff} \quad [f_i(a) \geqq f_i(b) \quad \text{for each } i].$$

The binary relation (A, W) defines a partial order.

To give yet another example of a partial order, let A be a set of points in the plane, some of which are joined by straight lines, as in the diagram of Fig. 1.5. If $a, b \in A$, let aRb hold if and only if $a = b$ or there is a continually descending path from a to b, following lines of the diagram. For example, in Fig. 1.5, we have $1R2$, $1R3$, $1R4$, $1R5$, $2R4$, $3R5$, and aRa for every a. It is fairly easy to prove that (A, R) is a partial order. (Proof is left to the reader.) What is not so obvious is that every partial order R on a finite set A arises from such a diagram, called a *Hasse diagram* of the partial order. To give an example, let us consider the partial order $\subseteqq$ on A = the set of subsets of $\{1, 2, 3\}$. A Hasse diagram corresponding to $(A, \subseteqq)$ is shown in Fig. 1.6. The reader will note that the Hasse diagram of a simple order is a "chain," as shown in Fig. 1.7. Usually it is convenient in Hasse diagrams to omit a line from point a to point b if there is a continuously descending path of more than one link from a to b. We shall follow this procedure in our diagrams.

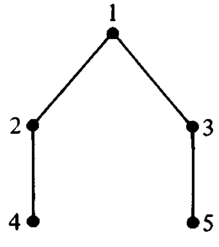

Figure 1.5. Hasse diagram of the partial order (A, R) defined by $A = \{1, 2, 3, 4, 5\}$, $R = \{(1, 1), (2, 2), (3, 3), (4, 4), (5, 5), (1, 2), (1, 3), (1, 4), (1, 5), (2, 4), (3, 5)\}$.

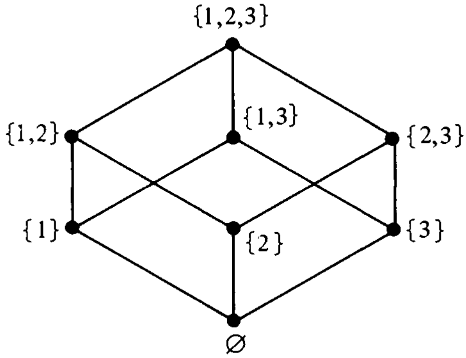

Figure 1.6. Hasse diagram of the partial order of inclusion $\subseteq$ on the set of subsets of $\{1, 2, 3\}$.

Figure 1.7. Hasse diagram of a simple order.

Analogously to strict simple orders, we can speak of strict partial orders. A binary relation is a *strict partial order* if it is asymmetric and transitive. Partial orders and strict partial orders are related in the same way that simple orders and strict simple orders are. Namely, suppose (A, R) is an irreflexive binary relation and we define

$$aSb \quad \text{iff} \quad (aRb \text{ or } a = b).$$

Then (A, R) is a strict partial order if and only if (A, S) is a partial order. Again, a strict partial order R corresponds to a Hasse diagram, except that now an element is not in the relation R to itself.

Exercises

1. Suppose $A = \{1, 2, 3, 4\}$ and

$$R = \{(1, 1), (2, 2), (3, 3), (4, 4), (1, 3), (1, 4), (2, 3), (2, 4), (3, 4)\}.$$

Show that

(a) (A, R) is a partial order.

(b) Draw the Hasse diagram.

2. Which of the following are strict partial orders?
(a) $\subsetneqq$ on the collection of subsets of $\{1,2,3,4\}$.
(b) (A,P) where $A = Re \times Re$ and

$$(a,b)P(s,t) \quad \text{iff} \quad (a > s \text{ and } b > t).$$

(c) (A, Q) where A is a set of n-dimensional alternatives, $f_1, f_2, \ldots, f_n$ are real-valued scales on A, and Q is defined by

$$aQb \Leftrightarrow [f_i(a) > f_i(b) \quad \text{for each } i].$$

(d) The relation (A, Q) where A is as in part (c) and

$$aQb \Leftrightarrow [f_i(a) \geqq f_i(b) \text{ for each } i \text{ and } f_i(a) > f_i(b) \text{ for some } i].$$

(e) (A,R) of Exer. 10, Section 1.3.

3. (a) Is the converse R^{-1} of a strict partial order necessarily a strict partial order?
(b) Is the converse of a partial order necessarily a partial order?

4. (a) Is every weak order a partial order?
(b) Is every partial order a weak order?

5. Is every quasi order a partial order?

6. Show that there are quasi orders that are not partial orders, not weak orders, and not equivalence relations.

7. Show that every strict weak order is a strict partial order.

8. Draw the Hasse diagram of the general strict weak order.

9. If (A,R) is strict weak, define S on A by

$$aSb \quad \text{iff} \quad (aRb \text{ or } a = b).$$

Show that (A,S) is a partial order.

10. Show that none of the axioms for a partial order is superfluous by giving examples of binary relations that are
(a) reflexive, transitive, and not antisymmetric;
(b) reflexive, antisymmetric, and not transitive;
(c) transitive, antisymmetric, and not reflexive.

11. Suppose (A, R) is irreflexive, and define S on A as in Exer. 9. Show that (A, R) is a strict partial order if and only if (A, S) is a partial order.

12. Exercises 12 through 18 introduce the notion of dimension of a strict partial order. For further reference on this subject, see Baker, Fishburn, and Roberts [1971] or Trotter [1978]. Also, see Exer. 2 of Section 5.1 and Exers. 21 and 31 of Section 6.1. Suppose (A, P) is a strict partial order. A strict simple order R on A such that $P \subseteqq R$ is called a *strict simple extension* of (A, P). For example, the strict partial order defined from the Hasse

diagram of Fig. 1.5 has as one strict simple extension the ordering in which 1 comes first, 2 second, 3 third, 4 fourth, and 5 fifth. List all the strict simple extensions of this strict partial order.

13. Szpilrajn's Extension Theorem [1930] states that if (A,P) is a strict partial order, if $a \neq b$, and if $\sim aPb$ and $\sim bPa$, then there is a strict simple extension (A,R) of (A,P) so that aRb. Show from this that there is a family $\mathfrak{F}$ of strict simple extensions so that

$$P = \cap \{R: R \in \mathfrak{F}\}.$$

14. Dushnik and Miller [1941] define the *dimension* of a strict partial order (A,P) as the smallest cardinal number m so that (A,P) is the intersection of m strict simple extensions. (By Exer. 13, dimension is well-defined.) As an example, the strict partial order (A,P) defined from Fig. 1.5 has dimension 2. Show this by observing that (A,P) is not strict simple and writing (A,P) as the intersection of two strict simple extensions.

15. If A is the set of all subsets of $\{1, 2, 3\}$ and P is the strict partial order $\subsetneq$, show that (A, P) has dimension 3. (Komm [1948] proves that the strict partial order $\subsetneq$ on the set of subsets of a set S has dimension $|S|$.)

16. Show that every strict weak order has dimension at most 2.

17. Hiraguchi [1955] shows that if (A,P) is a strict partial order with $|A|$ finite and at least 4, then (A,P) has dimension at most $|A|/2$. For a simple proof of this result, see Trotter [1975]. Show that dimension can be less than $[|A|/2]$, where $[a]$ is the greatest integer less than or equal to a.

18. (a) Use Hiraguchi's Theorem (Exer. 17) to obtain upper bounds for dimension of the strict partial orders whose Hasse diagrams are shown in Fig. 1.8.

(b) Use Komm's Theorem (Exer. 15) in one case and a specific construction in the other case to determine the exact dimensions.

19. (a) Does the data of Table 1.4 define a partial order?

(b) A strict partial order?

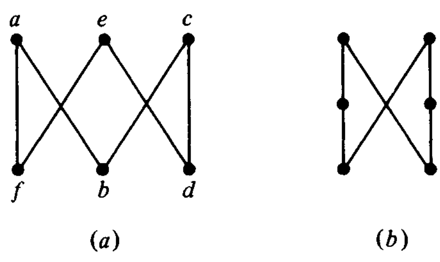

Figure 1.8. Hasse diagrams of two strict partial orders.

20. Repeat Exer. 19 for the data of Table 1.5.
21. Repeat Exer. 19 for the data of Table 1.6.
22. Repeat Exer. 19 for the data of Table 1.7.
23. Repeat Exer. 19 for the data of Table 1.8.
24. Repeat Exer. 19 for the data of Table 1.9.

1.7 Functions and Operations

Suppose A is a set. A function $f{:}A \to A$ can be thought of as a binary relation (A, R) with the following properties:

$$(\forall a \in A)(\exists b \in A)(aRb), \tag{1.20}$$

$$(\forall a, b, c \in A)(aRb \,\&\, aRc \Rightarrow b = c). \tag{1.21}$$

Conversely, any binary relation (A, R) satisfying (1.20) and (1.21) can be thought of as a function from A into A. More generally, a function $f{:}A^n \to A$ can be thought of as an $(n+1)$-ary relation (A, R) satisfying properties analogous to (1.20) and (1.21). For example, if $n=2$, these properties are as follows:

$$(\forall a, b \in A)(\exists c \in A)[(a, b, c) \in R], \tag{1.22}$$

$$(\forall a, b, c, d \in A)[(a, b, c) \in R \,\&\, (a, b, d) \in R \Rightarrow c = d]. \tag{1.23}$$

Such functions from $A \times A$ into A are sometimes called *binary operations*, or just *operations* for short. Abstractly, a ternary relation defines a binary operation if and only if it satisfies Eqs. (1.22) and (1.23).

Let us give some examples. Consider the operation + of addition of real numbers. Given a pair of real numbers a and b, + assigns a third real number c so that $c = a + b$. The corresponding ternary relation $\oplus$ on *Re* is defined as follows:

$$(a, b, c) \in \oplus \quad \text{iff} \quad c = a + b.$$

Thus, $(1, 2, 3) \in \oplus$ and $(1, 3, 4) \in \oplus$, but $(2, 5, 3) \notin \oplus$. The operation $\times$ of multiplication corresponds to the ternary relation $\otimes$ on *Re* defined as follows:

$$(a, b, c) \in \otimes \quad \text{iff} \quad c = a \times b.$$

To give yet another example, suppose $A = Re$ and

$$\mathbf{o}(a, b, c) \Leftrightarrow c = a/b.$$

Then the ternary relation $(A, \mathbf{o})$ is not an operation because there is no c

so that $\mathbf{o}(1,0,c)$. Next, suppose $A = Re$ and

$$\mathbf{o}(a, b, c) \Leftrightarrow c = \sqrt{ab}\ .$$

Then $(A, \mathbf{o})$ is not an operation because there is no c so that $\mathbf{o}(2, -2, c)$. If we again take $A = Re$ and now define

$$\mathbf{o}(a, b, c) \Leftrightarrow c = \sqrt{|ab|}\ ,$$

then $(A, \mathbf{o})$ is still not an operation, for $\mathbf{o}(2, 2, 2)$ and $\mathbf{o}(2, 2, -2)$. However, $(A, \mathbf{o})$ is an operation if we only allow the positive square root. Another operation $\mathbf{o}$ on $A = Re$ can be defined by taking

$$\mathbf{o}(a, b, c) \Leftrightarrow c = a + 2b.$$

If $(A, \mathbf{o})$ is an operation and $\mathbf{o}(a,b,c)$ holds, we usually write $c = a \mathbf{o} b$. Thus, in our present example, $5 = 1 \mathbf{o} 2$ and $8 = 2 \mathbf{o} 3$.

To give two further examples, suppose $A = \{1,2,3\}$ and we define

$$R = \{(1, 1, 1), (1, 2, 2), (1, 3, 1), (2, 1, 2), (2, 2, 1), (2, 3, 2), (3, 1, 1), (3, 2, 2), (3, 3, 1)\}$$

and

$$S = \{(1, 1, 1), (1, 2, 1), (1, 3, 1), (2, 1, 2), (2, 2, 2), (2, 3, 2), (3, 1, 3), (3, 2, 3), (1, 3, 2)\}.$$

Then (A, R) is a binary operation: it is the operation that assigns to a and b the number 1 if $a + b$ is even and the number 2 if $a + b$ is odd. (A, S) is not a binary operation, since both $S(1, 3, 1)$ and $S(1, 3, 2)$.

To give still another example, which we shall encounter later, suppose A is the set of aircraft engines discussed above, among which we are interested in comparing subjective loudness. If a and b are two engines, let us think of $a \mathbf{o} b$ as the object consisting of both engines placed next to each other. As far as loudness is concerned, the loudness of $a \mathbf{o} b$ is compared to other loudnesses by running both engines at once. We might call $\mathbf{o}$ an operation of "combination" and speak of comparing combined loudness. Unfortunately, $\mathbf{o}$ does not allow us to define an operation on the set A in the precise sense we have defined. We would think of defining the ternary relation $\mathbf{o}(a, b, c)$ which holds if and only if c is $a \mathbf{o} b$. Unfortunately, if a and b are in A, $a \mathbf{o} b$ is not necessarily in A, and so there is no c for which $\mathbf{o}(a, b, c)$ holds, violating condition (1.22). Moreover, the combination $a \mathbf{o} a$ does not make sense, and yet an operation must be defined on all pairs in $A \times A$ including pairs (a, a).

(To rectify this situation, we can think in terms of a hypothetical set B consisting of infinitely many copies of each element a of A, and to speak of a set C consisting of finite sets of elements from B. Two sets in C are considered equivalent if for each element a in A the two sets have the same number of copies of a. Then equivalence is indeed an equivalence relation. Let C^* be the collection of equivalence classes. If α and β are in C^*, pick disjoint representatives x in α and y in β and define $\alpha \circ \beta$ to be $x \cup y$. It is not hard to show that $\circ$ is well-defined and defines an operation on C^*.)

Exercises

1. Suppose A is the set of positive integers.
   (a) Show that $R(a, b, c)$ iff $a + b = c$ defines an operation on A.
   (b) Show that $S(a, b, c)$ iff $a - b = c$ does not define an operation on A.
   (c) Show that $T(a, b, c)$ iff $a + b + c = 0$ does not define an operation on A.
2. Suppose A is all of the integers.
   (a) Is the relation (A, R) of Exer. 1 an operation?
   (b) What about the relation (A, S) of Exer. 1?
   (c) What about the relation (A, T) of Exer. 1?
3. Which of the following relations (A, U) define operations?
   (a) $A = Re$, $U(a, b, c)$ iff $a + b + c = 0$.
   (b) $A = Re$, $U(a, b, c)$ iff $abc = 0$.
   (c) $A = Re$, $U(a, b, c)$ iff $a > b > c$.
4. If $A = \{0, 1, 2\}$ and

$$R = \{(0, 0, 0), (0, 1, 0), (0, 2, 0), (1, 0, 0), (1, 1, 1), (1, 2, 2), (2, 0, 0), (2, 1, 2), (2, 2, 1)\},$$

show that (A, R) is an operation. (What operation is it?)

5. (a) If $A = \{$Los Angeles, Chicago, New York$\}$, show that the following relation on A is not a binary operation:

$R = \{$(Los Angeles, Chicago), (Los Angeles, New York), (Chicago, New York)$\}$.

   (b) Which of the following relations on A is a binary operation?
   (i) S given by the following set of triples:
   (Los Angeles, Los Angeles, Chicago)
   (Los Angeles, Chicago, Chicago)
   (Los Angeles, New York, Chicago)
   (Chicago, Chicago, Chicago)
   (Chicago, New York, Chicago)
   (New York, New York, Chicago).

(ii) $T = \{(x, y, \text{NewYork}): x, y \in A\} \cup$
$\{(\text{Chicago, Chicago, Chicago})\}$.

6. (a) Show that the following binary relations (A, R) are functions:
   (i) $A = Re, R = \{(a, a + 1): a \in A\}$.
   (ii) $A = Re, aRb$ iff $b = a^2$.
   (iii) $A = Re, aRb$ iff $a = b$.
   (b) Show that the following binary relations are not functions:
   (i) $A = Re, aRb$ iff $a = b^2$.
   (ii) $A = Re, aRb$ iff $a > b$.
   (c) Which of the following binary relations are functions?
   (i) $A = Re, aRb$ iff $6a + 2b = 0$.
   (ii) $A = Re, aRb$ iff $a > b + 1$.
   (iii) $A = Re, aRb$ iff a divides b.

7. Which of the following quaternary relations (A, R) correspond to functions from $A \times A \times A$ to A?
   (a) $A = \{SO_2, \text{DDT}\}$
   $R = \{(SO_2, SO_2, SO_2, NO_x), (NO_x, NO_x, NO_x, SO_2)\}$.
   (b) $A = Re, R = \{(a, b, c, d): d = \sqrt{abc}\}$.
   (c) $A = Re, R = \{(a, b, c, d): d > a + b + c\}$.

8. Suppose R is a function on A and S is a function on A.
   (a) Is the relative product $R \circ S$ necessarily a function on A?
   (b) How does the notion of relative product compare to the usual notion of composition of two functions?

1.8 Relational Systems and the Notion of Reduction

Suppose $R_1, R_2, \ldots, R_p$ are (not necessarily binary) relations on the same set A and $o_1, o_2, \ldots, o_q$ are binary operations on A. We shall call the $(p + q + 1)$-tuple $\mathfrak{A} = (A, R_1, R_2, \ldots, R_p, o_1, o_2, \ldots, o_q)$ a *relational system*. Of course, we could treat binary operations as relations, and so simply speak of a relational system as a $(p + 1)$-tuple $(A, R_1, R_2, \ldots, R_p)$. However, in what follows, it will be convenient to single out the binary operations. On the other hand, we do not single out any other functions.

It will be useful to generalize the reduction or quotient procedure of Section 1.5 to relational systems $\mathfrak{A}$. The procedure is called by Scott and Suppes [1958] the *method of cosets*. Let us start with a relational system $(A, R_1, R_2, \ldots, R_p)$ having no operations. We define a relation of equivalence E on A by saying that aEb holds if and only if a and b are "perfect substitutes" for each other with respect to all the relations R_i. (Formally, a and b are *perfect substitutes* for each other with respect to an m-ary relation R_i if the following condition holds: given sequences $(a_1, a_2, \ldots, a_m)$ and $(b_1, b_2, \ldots, b_m)$ from A, if for $j = 1, 2, \ldots, m$, $a_j \neq b_j$ implies $\{a_j, b_j\} = \{a, b\}$, then

$$R_i(a_1, a_2, \ldots, a_m) \quad \text{iff} \quad R_i(b_1, b_2, \ldots, b_m).$$

This definition, and the notion of reduction defined below, appear in Scott and Suppes [1958].) It is not hard to show that if a^* is the equivalence class containing a, then the following relations R_i^* are well-defined on the collection A^* of equivalence classes.

$$R_i^*(a_1^*, a_2^*, \ldots, a_m^*) \quad \text{iff} \quad R_i(a_1, a_2, \ldots, a_m).$$

The relational system $\mathfrak{A}^* = (A^*, R_1^*, R_2^*, \ldots, R_p^*)$ is called the *reduction* or *quotient* of $\mathfrak{A}$, and is often denoted as $\mathfrak{A}/E$. The reader should verify that if (A, R) is a strict weak order, then the perfect substitutes relation E is the same as the tying relation E defined in Eq. (1.19) of Section 1.5.

Handling the general case, suppose we are given a relational system $\mathfrak{A} = (A, R_1, R_2, \ldots, R_p, \mathbf{o}_1, \mathbf{o}_2, \ldots, \mathbf{o}_q)$, and aEb holds for a, b in A if and only if a and b are perfect substitutes for each other with respect to all the relations R_i. Let us define binary operations $\mathbf{o}_i^*$ on A^* as follows:

$$a^* \mathbf{o}_i^* b^* = (a \mathbf{o}_i b)^*.$$

The operation $\mathbf{o}_i^*$ is well-defined provided the following condition holds:

$$(aEa' \ \& \ bEb') \Rightarrow (a \mathbf{o}_i b)E(a' \mathbf{o}_i b'). \tag{1.24}$$

If (1.24) holds for all i, we say the relational system $\mathfrak{A}$ is *shrinkable*, and we call the relational system $\mathfrak{A}^* = (A^*, R_1^*, R_2^*, \ldots, R_p^*, \mathbf{o}_1^*, \mathbf{o}_2^*, \ldots, \mathbf{o}_q^*)$ the *reduction* of $\mathfrak{A}$. We say that a relational system $\mathfrak{A}$ is *irreducible* if every equivalence class with respect to E has exactly one element.

We illustrate these ideas with an example. Let $A = \{0, 1, 2, \ldots, 26\}$, and define R and $\mathbf{o}$ on A by

$$aRb \quad \text{iff} \quad a \bmod 3 > b \bmod 3, \tag{1.25}$$

$$c = a \mathbf{o} b \quad \text{iff} \quad [c \equiv a + b (\bmod 3) \text{ and } c \in \{0, 1, 2\}]. \tag{1.26}$$

Thus, $2R1$, $8R10$, etc. Similarly, $2 \mathbf{o} 2 = 1$, $6 \mathbf{o} 3 = 0$. There are three equivalence classes under E, namely:

$$0^* = \{0, 3, 6, 9, 12, 15, 18, 21, 24\},$$

$$1^* = \{1, 4, 7, 10, 13, 16, 19, 22, 25\},$$

and

$$2^* = \{2, 5, 8, 11, 14, 17, 20, 23, 26\}.$$

We have $2^*R^*1^*$, since $2R1$. Equation (1.24) holds, and hence $\mathbf{o}^*$ is well-defined, and $\mathfrak{A} = (A, R, \mathbf{o})$ is shrinkable. We have, for example, $2^* \mathbf{o}^* 2^* = 1^*$, since $2 \mathbf{o} 2 = 1$. We shall return to the idea of reduction in Chapter 2.

Exercises

1. If $\mathfrak{A} = (A, E)$ is an equivalence relation, what is the reduction $\mathfrak{A}^*$?
2. Suppose $A = Re \times Re$,

$$(a, b)R(c, d) \Leftrightarrow a > c, \tag{1.27}$$

and

$$(a, b)S(c, d) \Leftrightarrow a = c. \tag{1.28}$$

(a) Identify the reduction $(A^*_;\ R^*)$.
(b) Identify the reduction $(A^*_;\ S^*)$.
(c) Identify the reduction $(A^*_;\ R^*_;\ S^*)$.

3. Suppose $A = Re \times Re$, R is defined on A by Eq. (1.27), and

$$(a, b)S(c, d) \Leftrightarrow b > d.$$

Identify the reduction $(A^*_;\ R^*_;\ S^*)$.

4. Suppose $A = Re$, $R = >$, and $\mathbf{o} = +$. Show that $(A, R, \mathbf{o})$ is shrinkable.

5. Are the following relational systems shrinkable? If so, find their reductions.

(a) $(A, R, \mathbf{o})$, where $A = Re^+$, $R = >$, $\mathbf{o} = \times$.
(b) $(A, R, \mathbf{o})$, where $A = Re \times Re$, R is defined by Eq. (1.27), and

$$(a, b)\,\mathbf{o}\,(c, d) = (a + c, b + d).$$

(c) $(A, R, \mathbf{o}, \mathbf{o}')$, where A, R, and $\mathbf{o}$ are as in (b) and

$$(a, b)\,\mathbf{o}'\,(c, d) = (ac, bd).$$

(d) $(A, R, S, \mathbf{o})$, where A, R, and $\mathbf{o}$ are as in (b) and S is defined by Eq. (1.28).
(e) $(A, R, S, \mathbf{o}, \mathbf{o}')$, where A, R, S, and $\mathbf{o}$ are as in (d) and $\mathbf{o}'$ is as in (c).
(f) $(A, R, \mathbf{o})$, where $A = \{0, 1, 2, \ldots, 26\}$, $R = >$, and $\mathbf{o}$ is defined by Eq. (1.26).

6. Show that if (A, R) is a strict weak order, then the perfect substitutes relation E is the same as the tying relation of Eq. (1.19) of Section 1.5.

7. Show that the reduction $\mathfrak{A}^*$ is always irreducible.

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