Chapter 3

Justifying Belief as Probability

In this chapter we shall present three arguments supporting the assumption that belief functions should be identified with probability functions, that is satisfy ($P1$–2), and that conditional belief should be identified with conditional probability (when defined).

First Justification

Returning yet again to our example $\mathbb{E}$ consider attempting to give a meaning to the doctor's statement that

Symptom D is rather uncommon

which, with the doctor's assent, has been formalised as

$$Bel_0(D) = 0.01.$$

We might argue as follows: The doctor was not born with his knowledge, the simplest explanation of where it came from is from the, presumably considerable, set M of patients x he has previously seen. Given this model of his knowledge as simply this set M the obvious explanation of his statement that symptom D is rather uncommon is that rather a small proportion of patients in M have complained of D and indeed, further, that the figure 0.01 represents an estimate of this proportion. In other words, ideally, $Bel_0(D)$ is to be identified with

$$\frac{\mid \{x \in M \mid x \text{ has } D\} \mid}{\mid M \mid}.$$

More generally $Bel_0(\theta)$ is to be identified with

$$\frac{\mid \{x \in M \mid x \text{ has } \theta\} \mid}{\mid M \mid}$$

for $\theta \in SL$, and by an exactly similar argument $Bel_0(\theta|\phi)$ is to be identified with

$$\frac{\mid \{x \in M \mid x \text{ has } \theta \text{ and } x \text{ has } \phi\} \mid}{\{x \in M \mid x \text{ has } \phi\} \mid}.$$

Then with this simple model of the doctor's knowledge, commonly referred to as the *urn model*, and its relationship to his knowledge statements, Bel_0 does indeed come out to be a probability function and $Bel_0(\ |\ )$ the corresponding conditional probability. To see this notice that for each $x \in M$ we can define a valuation V_x on the language L by

$V_x(p) = 1$ (i.e. true) if x has p,
$V_x(p) = 0$ (i.e. false) if x does not have p,

for $p \in L$ (i.e. p a propositional variable of L). Then for $\theta \in SL$, $x \in M$,

$$V_x(\theta) = 1 \Leftrightarrow x \text{ has } \theta$$

so

$$\{x \in M \mid x \text{ has } \theta\} = \{x \in M \mid V_x(\theta) = 1\}.$$

Now if $\models \theta$ then $V(\theta) = 1$ for any valuation V so $\{x \in M \mid V_x(\theta) = 1\} = M$ and $\mathrm{Bel}_0(\theta) = 1$, giving (P1). Also if $\models \neg(\theta \wedge \phi)$ then for no valuation V does $V(\theta) = V(\phi) = 1$ hold (otherwise $V(\theta \wedge \phi) = 1$) and so, since

$$V(\theta \vee \phi) = 1 \Leftrightarrow V(\theta) = 1 \text{ or } V(\phi) = 1,$$

$\{x \in M \mid V_x(\theta \vee \phi) = 1\}$ is the union of the disjoint sets $\{x \in M \mid V_x(\theta) = 1\}$ and $\{x \in M \mid V_x(\phi) = 1\}$ from which (P2) follows easily.

An obvious criticism of this model is that it is not clear how it is supposed to model beliefs obtained indirectly, such as from books or theoretical considerations. Furthermore the model also assumes that, in this case, the doctor has complete knowledge of all the previous patients $x \in M$ which would, in practice, be quite unrealistic. In reality the doctor could never hope to have more than partial, uncertain knowledge of his past patients.

A much more serious criticism however is that even in, for example, a very specialized medical field there would be so many signs, symptoms etc. that almost every patient would present a hitherto unseen combination. Hence if the doctor attempted to determine his belief that a patient had a particular disease by looking at the proportion of previous patients exhibiting the same combination of signs and symptoms who also had this disease then he would almost always land up with the indeterminate $0/0$.

A final shortcoming of this model is that it seems inapplicable in many situations in which we form beliefs, such as who will win the next Derby.

On the other hand, in situations in which it is appropriate, the urn model provides both a simple explanation of how the expert's beliefs could have arisen from his experience and a justification for the properties assumed of his belief function (i.e. being a probability function). Clearly for any assumptions about what properties belief should have the existence of such a model, explaining how such a belief function arises from the expert's experiences, is highly desirable – some might even say necessary. Unfortunately for many proposed notions of belief (such as those we shall encounter in Chapter 5) there is as yet no such satisfactory model. Instead the asserted properties of belief tend to be explained by appealing to our beliefs about beliefs.

The model we presented above could be viewed as a special case, albeit a rather natural one, of a 'possible worlds' interpretation of belief. That is we suppose that the 'real world' is one of the worlds in a non-empty set $\mathcal{W}$ of possible worlds w and that for $X \subseteq \mathcal{W}, \mu(X) \in [0,1]$ is the measure of our belief that the real world is in X. (By a *possible world* here we simply understand an object w for which some relation $w \models \theta$ is defined for $\theta \in SL$ so as to satisfy the usual inductive definition of truth. Note that we do not assume that if

$$w_1 \models \theta \Leftrightarrow w_2 \models \theta$$

for all $\theta \in SL$, then $w_1 = w_2$.) More precisely we assume of the function μ that it is defined on all subsets of $\mathcal{W}$, that $\mu(\mathcal{W}) = 1$ and $\mu(\emptyset) = 0$ and that for all $X, Y \subseteq \mathcal{W}$ with $X \cap Y = \emptyset$,

$$\mu(X \cup Y) = \mu(X) + \mu(Y),$$

i.e. μ is finitely additive.

In this context then we might imagine that an expression like $Bel(\phi) = c$ stands for

$$\mu\{w \in \mathcal{W} \mid w \models \phi\} = c.$$

For example in the above model $\mathcal{W} = M$, for $x \in M$,

$$x \models \phi \Leftrightarrow V_x(\phi) = 1$$

and for $X \subseteq M$,

$$\mu(X) = \frac{|X|}{|M|}.$$

With this interpretation it is easy to check that Bel is a probability function. Conversely if Bel is a probability function then it is generated by such a possible worlds approach, namely the possible worlds are the atoms α_i, $\alpha_i \models \phi$ has its usual meaning (i.e. equivalent to $\alpha_i \in S_\phi$) and $\mu(X) = Bel\ (\bigvee X)$, since in this case

$$Bel\ (\phi) = Bel\ (\bigvee S_\phi) = \mu(S_\phi) = \mu\{\alpha_i \mid \alpha_i \models \phi\}$$

as required.

Second Justification

A second justification for belief as probability emerges from some early work of Ramsey, de Finetti, Kemeny and Shimony (see [61], [19], [20], [37], [65]). The idea is to identify an expert's belief $Bel(\theta)$ in $\theta \in SL$ with his willingness to bet on θ being true in the real world, that is on $V(\theta) = 1$ where $V : L \to \{0, 1\}$ is the valuation which represents the true (but possibly unknown) state of the world.

To be precise suppose that $0 \le p \le 1$ and that the expert is required to make a choice, for stake $S > 0$, between

(i) gaining $S(1-p)$ if $V(\theta) = 1$ whilst losing Sp if $V(\theta) = 0$,

(ii) losing $S(1-p)$ if $V(\theta) = 1$ whilst gaining Sp if $V(\theta) = 0$.

Clearly if $p = 0$ he could not do better than to choose (i) whilst if $p = 1$ he could not do better than to choose (ii).

Furthermore if $0 \le p' < p \le 1$ then it would be irrational of the expert (i.e. against his best interests) to choose (i) for p and (ii) for p'. For suppose he was to make such choices. Then he stands to gain strictly more by choosing (i) at p' than he did by choosing it at p (since he would gain $S(1-p') > S(1-p)$ if successful and lose $Sp' < Sp$ if not). However he picked (ii) at p' so, rationally, he must believe that he stands to gain at least as much by picking (ii) at p' as (i) at p' and hence strictly more than (i) at p. But then, as above, since $p > p'$ he stands to gain more by picking (ii) at p than he did by picking (ii) at p' and hence by picking (i) at p. But this contradicts the rationality of his choice of (i) at p.

From this it follows that if β is the least upper bound of the set of $p \in [0, 1]$ for which the expert prefers (i) then for any $0 \le p < \beta$ he prefers (assuming as we do that he is rational) (i) at p and for any $\beta < p \le 1$ he prefers (ii) at p.

Thus β could be said to measure the expert's willingness to bet on θ, since if $0 \le p < \beta$ he prefers a bet which pays him if θ is true whilst if $\beta < p \le 1$ he prefers a bet which pays him if θ is false. Identifying belief with willingness to bet then leads to identifying $Bel(\theta)$ with β.

Now suppose that $Bel(\theta)$ is defined in this way for all $\theta \in SL$. Then it could be said that these values are rational or *fair* if it is not possible for an opponent to arrange a Dutch Book against him, that is arrange a finite set of bets (for various stakes) each of which the expert would agree to but whose combined effect would be to cause him certain loss no matter what V is.

To make this notion of fairness mathematically precise notice that if $p < Bel(\theta)$ then, as above, the expert would choose (i) and would stand to gain

$$S(1-p)V(\theta) - Sp(1 - V(\theta)) = S(V(\theta) - p),$$

where negative gain equals loss. Similarly if $Bel(\theta) < p$ then he would choose (ii) and would stand to gain

$$Sp(1 - V(\theta)) - S(1-p)V(\theta) = -S(V(\theta) - p).$$

With this observation we can define fairness (for $Bel : SL \longrightarrow [0,1]$ as usual) to mean that there do not exist $S_i, T_j > 0$, $\theta_i, \phi_j \in SL$, $p_i < Bel(\theta_i)$, $Bel(\phi_j) < q_j$ for $i = 1, ..., n$, $j = 1, ..., m$ such that for all valuations V

$$\sum_{i=1}^{n} S_i(V(\theta_i) - p_i) - \sum_{j=1}^{m} T_j(V(\phi_j) - q_j) < 0.$$

Theorem 3.1 *If the values $Bel(\theta)$ for $\theta \in SL$ are fair then Bel satisfies $(P1$–$2)$ and hence is a probability function.*

Proof To show $(P1)$ suppose that $\models \theta$ but that $Bel(\theta) < 1$, say $Bel(\theta) < q < 1$. Then since $V(\theta) = 1$ for all valuations V,

$$-1(V(\theta) - q) < 0$$

for all valuations V, contradicting fairness.

To show $(P2)$ suppose that $\models \neg(\phi \wedge \theta)$ but

$$Bel(\theta \vee \phi) \neq Bel(\theta) + Bel(\phi),$$

say $Bel(\theta \vee \phi) > Bel(\theta) + Bel(\phi)$. Pick $Bel(\theta \vee \phi) > p > q_1 + q_2$ where $q_1 > Bel(\theta), q_2 > Bel(\phi)$. Then since

$$V(\theta \vee \phi) = V(\theta) + V(\phi)$$

for all valuations V,

$$1(V(\theta \vee \phi) - p) - 1(V(\theta) - q_1) - 1(V(\phi) - q_2) = q_1 + q_2 - p < 0,$$

contradicting fairness. A similar argument shows that

$$Bel(\theta \vee \phi) \not< Bel(\theta) + Bel(\phi)$$

and hence $(P2)$ follows. □

In a similar fashion we could argue for identifying the expert's conditional belief, $Bel(\theta|\phi)$, with his willingness to bet on θ on condition that ϕ holds. (If ϕ does not hold then the bet is null and void.) Again, as above, if $p < Bel(\chi \mid \eta)$ then the expert would choose (i) and stand to gain

$$V(\eta)S(V(\chi) - p),$$

whilst if $p > Bel(\chi|\eta)$ he would choose (ii) and stand to gain

$$-V(\eta)S(V(\chi) - p).$$

Generalising the above definition then we define the values $Bel(\theta)$, $Bel(\theta|\phi)$, for $\theta, \phi \in SL$, to be fair if there do not exist $S_i, T_j, R_{i'}, U_{j'} > 0$, θ_i, ϕ_j, $\chi_{i'}$, $\eta_{i'}$, $\psi_{j'}$, $\lambda_{j'} \in SL$, $p_i < Bel(\theta_i)$, $q_j > Bel(\phi_j)$, $r_{i'} < Bel(\chi_{i'} \mid \eta_{i'})$, $u_{j'} > Bel(\psi_{j'} \mid \lambda_{j'})$ etc. such that for all valuations V,

$$\sum_{i=1}^{n} S_i(V(\theta_i) - p_i) - \sum_{j=1}^{m} T_j(V(\phi_j) - q_j) +$$

$$+ \sum_{i'=1}^{n'} R_{i'} V(\eta_{i'})(V(\chi_{i'}) - r_{i'}) - \sum_{j'=1}^{m'} U_{j'} V(\lambda_{j'})(V(\psi_{j'}) - u_{j'}) < 0.$$

Theorem 3.2 *If the values $Bel(\theta), Bel(\theta|\phi)$, $\theta, \phi \in SL$ are fair then Bel is a probability function and*

$$Bel(\theta \mid \phi) Bel(\phi) = Bel(\theta \wedge \phi).$$

Proof We already have ($P1$–2) from theorem 3.1. To complete the proof suppose, on the contrary, that

$$Bel(\theta \mid \phi) Bel(\phi) < Bel(\theta \wedge \phi).$$

If $Bel(\theta \mid \phi) < Bel(\theta \wedge \phi)$ then for $Bel(\theta \mid \phi) < p < r < Bel(\theta \wedge \phi)$,

$$(V(\theta \wedge \phi) - r) - V(\phi)(V(\theta) - p) = -r + V(\phi)p \leq p - r < 0$$

for any valuation V, since $V(\theta \wedge \phi) = V(\theta)V(\phi)$, contradicting fairness. Hence $Bel(\phi) \neq 1$. Also $Bel(\theta|\phi) < 1$ since by proposition 2.1, $Bel(\theta \wedge \phi) \leq Bel(\phi)$. Thus we can pick

$$p > Bel(\theta \mid \phi), \quad q > Bel(\phi), \quad r < Bel(\theta \wedge \phi)$$

such that $pq < r$. But then for any valuation V,

$$(V(\theta \wedge \phi) - r) - p(V(\phi) - q)) - V(\phi)(V(\theta) - p) = pq - r < 0$$

contradicting fairness.

A similar proof shows that the assumption

$$Bel(\theta \wedge \phi) < Bel(\theta \mid \phi) Bel(\phi)$$

also contradicts fairness and the required identity follows. □

To sum up then identifying belief with willingness to bet and imposing requirements of rationality forces Bel to be a probability function.

At this point it is natural to ask if the requirement of fairness does not put some additional conditions on Bel beyond simply being a probability function with the standard derived conditional probability. The following theorem shows that it does not.

Theorem 3.3 *Suppose that* $Bel : SL \longrightarrow [0,1]$ *satisfies (P1–2) and that for all* $\theta, \phi \in SL$ $Bel(\theta \mid \phi)$ *is defined and satisfies* $Bel(\theta \mid \phi)Bel(\phi) = Bel(\theta \wedge \phi)$. *Then the fairness condition is satisfied.*

Proof Suppose on the contrary that fairness fails. Then, referring back to the notation used in the definition of fairness, the inequality still holds if p_i is replaced by $Bel(\theta_i)$ etc. (Indeed this gives us an equivalent definition of fairness.) Combining the first and second and the third and fourth sums then we obtain an inequality

$$\sum_{\chi} G_\chi(V(\chi) - Bel(\chi)) + \sum_{\theta,\phi} H_{\theta,\phi} V(\phi)(V(\theta) - Bel(\theta \mid \phi)) < 0$$

which holds for all valuations V.

Now let α_i be an atom such that $Bel(\alpha_i) > 0$ and let V be the valuation such that $V(\alpha_i) = 1$, so $V(\alpha_j) = 0$ for $j \neq i$. Then for this valuation we obtain

$$\sum_{\alpha_i \in S_\chi} G_\chi - \sum_{\chi} G_\chi Bel(\chi) + \sum_{\substack{\alpha_i \in S_\phi \\ \alpha_i \in S_\theta}} H_{\theta,\phi} - \sum_{\substack{\theta \\ \alpha_i \in S_\phi}} H_{\theta,\phi} Bel(\theta \mid \phi) < 0.$$

Multiplying each such inequality by $Bel(\alpha_i)$ and summing over those α_i for which $Bel(\alpha_i) > 0$ gives

$$\sum_{Bel(\alpha_i)>0} \sum_{\alpha_i \in S_\chi} G_\chi Bel(\alpha_i) - \left(\sum_{\chi} G_\chi Bel(\chi)\right) \cdot \left(\sum_{Bel(\alpha_i)>0} Bel(\alpha_i)\right)$$
$$+ \sum_{Bel(\alpha_i)>0} \left(\sum_{\substack{\alpha_i \in S_\phi \\ \alpha_i \in S_\theta}} H_{\theta,\phi} Bel(\alpha_i)\right) - \sum_{\phi,\theta} H_{\theta,\phi} Bel(\theta \mid \phi) \left(\sum_{\substack{\alpha_i \in S_\phi \\ Bel(\alpha_i)>0}} Bel(\alpha_i)\right) < 0.$$

But the first two expressions are clearly both equal to

$$\sum_{\chi} G_\chi Bel(\chi),$$

whilst the last two are both equal to

$$\sum_{\theta,\phi} H_{\theta,\phi} Bel(\theta \wedge \phi),$$

hence giving the required contradiction. (To see this for the final expression notice that the sum

$$\sum_{\substack{\alpha_i \in S_\phi \\ Bel(\alpha_i)>0}} Bel(\alpha_i)$$

equals $Bel(\phi)$ whether or not it is empty, and $Bel(\theta \mid \phi) \cdot Bel(\phi) = Bel(\theta \wedge \phi)$.) $\square$

Third Justification

A third justification for belief as probability (or at least a scaled version of probability) appeared in a paper by R.T. Cox in the American Journal of Physics in 1946 [9]. Cox's proof is not, perhaps, as rigorous as some pedants might prefer and when an attempt is made to fill in all the details some of the attractiveness of the original is lost. Nevertheless his results certainly provide a valuable contribution to our understanding of the nature of belief.

We state here a rigorous version of Cox's main theorem which has aspects which are both stronger and weaker than the original. Slightly stronger versions still can be proved but the increased complications do not seem to justify doing so.

Just for the statement and proof of this theorem we shall assume that L is infinite (or alternatively that L, although finite, is variable).

Theorem 3.4 (Cox's theorem) *Suppose that whenever $\psi \in SL$ is consistent (i.e. non-contradictory) we can give a conditional belief $Bel(\theta \mid \psi) \in [0,1]$ to θ given ψ. Suppose further that for $\phi \wedge \psi$ consistent, $\theta, \phi, \psi \in SL$ etc.*

($Co1$) *If* $\models (\theta \leftrightarrow \theta')$, $\models (\psi \leftrightarrow \psi')$ *then* $Bel(\theta \mid \psi) = Bel(\theta' \mid \psi')$.

($Co2$) *If* $\models (\psi \rightarrow \theta)$ *then* $Bel(\theta \mid \psi) = 1$ *and* $Bel(\neg\theta \mid \psi) = 0$.

($Co3$) $Bel(\theta \wedge \phi \mid \psi) = F(Bel(\theta \mid \phi \wedge \psi), Bel(\phi \mid \psi))$ *for some continuous function* $F : [0,1]^2 \longrightarrow [0,1]$ *which is strictly increasing (in both coordinates) on* $(0,1]^2$.

($Co4$) $Bel(\neg\theta \mid \psi) = S(Bel(\theta \mid \psi))$ *for some decreasing function* $S : [0,1] \longrightarrow [0,1]$.

($Co5$) *For any* $0 \le \alpha, \beta, \gamma \le 1$ *and* $\epsilon > 0$ *there are* $\theta_1, \theta_2, \theta_3, \theta_4 \in SL$ *with* $\theta_1 \wedge \theta_2 \wedge \theta_3$ *consistent such that each of*

$$| Bel(\theta_4 \mid \theta_1 \wedge \theta_2 \wedge \theta_3) - \alpha |, \ | Bel(\theta_3 \mid \theta_1 \wedge \theta_2) - \beta |, \ | Bel(\theta_2 \mid \theta_1) - \gamma |$$

is less than ϵ.

Then there is a continuous, strictly increasing, surjective function $g : [0,1] \rightarrow [0,1]$ *such that* $gBel(\theta \mid T)$ *(where T is any tautology) satisfies* ($P1$–2) *and*

$$g\,Bel(\theta \mid \psi) \cdot g\,Bel(\psi \mid T) = g\,Bel(\theta \wedge \psi \mid T),$$

i.e. $gBel(\theta \mid \psi)$ *agrees with the conditional probability resulting from the probability function* $gBel(\ \mid T)$ *provided* $g\,Bel(\psi \mid T) \neq 0$.

Aside In his original, ($Co3$) and ($Co4$) were Cox's main assumptions. Cox justifies ($Co3$) by the example of a runner (of whom we assume ψ) and argues that our belief that he can run to a distant place (ϕ) and return (θ) should only be a function of our belief that he will get there, (Bel($\phi \mid \psi$)) and, having got there, that he will return ($Bel(\theta \mid \phi \wedge \psi)$). ($Co4$) is justified by the argument that as one's belief in θ given ψ increases from 0 to 1 so one's belief in $\neg\theta$ given ψ decreases from 1 to 0.

The assumption that $Bel(\theta \mid \phi)$ is defined whenever ϕ is consistent can be dropped in this theorem provided we strengthen ($Co1$), ($Co3$), ($Co4$) to assert that the existence of the right hand side of the equation implies the existence of the left hand side, and we strengthen ($Co2$), ($Co5$) to asserting that these conditional beliefs exist. The proof is essentially the same as the one we are about to give except that in our conclusion we need to assume that the relevant belief values are defined.

Cox appears to neglect what we perceive as a need for ($Co5$), as does J. Aczel in a rather similar result in [1]. The importance of ($Co5$) will become clear during the proof of the theorem.

Cox's theorem will be proved via a series of lemmas. Since only one of these, lemma 3.7, will be needed in later chapters the casual reader might be forgiven for skipping the rather involved proof and jumping straight to the easier material beyond.

In the lemmas which now follow we shall assume ($Co1$–5).

Lemma 3.5 *For $x, y, z \in [0, 1]$,*

$$F(F(x, y), z) = F(x, F(y, z)).$$

That is, as a binary operation, F is associative.

Proof For $\theta_1 \wedge \theta_2 \wedge \theta_3$ consistent,

$$\begin{aligned} Bel(\theta_4 \wedge \theta_3 \wedge \theta_2 \mid \theta_1) &= F(Bel(\theta_4 \wedge \theta_3 \mid \theta_1 \wedge \theta_2), Bel(\theta_2 \mid \theta_1)) \text{ by } (Co1), (Co3) \\ &= F(F(Bel(\theta_4 \mid \theta_1 \wedge \theta_2 \wedge \theta_3), Bel(\theta_3 \mid \theta_1 \wedge \theta_2)), Bel(\theta_2 \mid \theta_1)) \\ &\quad \text{by } (Co3). \end{aligned}$$

Also

$$\begin{aligned} Bel(\theta_4 \wedge \theta_3 \wedge \theta_2 \mid \theta_1) &= F(Bel(\theta_4 \mid \theta_1 \wedge \theta_2 \wedge \theta_3), Bel(\theta_3 \wedge \theta_2 \mid \theta_1)) \\ &= F(Bel(\theta_4 \mid \theta_1 \wedge \theta_2 \wedge \theta_3), F(Bel(\theta_3 \mid \theta_1 \wedge \theta_2), Bel(\theta_2 \mid \theta_1))). \end{aligned}$$

Putting $x = Bel(\theta_4 \mid \theta_1 \wedge \theta_2 \wedge \theta_3)$, $y = Bel(\theta_3 \mid \theta_1 \wedge \theta_2)$, $z = Bel(\theta_2 \mid \theta_1)$ gives $F(F(x, y), z) = F(z, F(y, z))$. By ($Co5$) the set of points $\langle x, y, z\rangle \in [0, 1]^3$ for which this holds is dense in $[0, 1]^3$ and, since a continuous function is determined by its value on any dense set of points, the continuity of F ensures that this identity holds everywhere on $[0, 1]^3$. □

Notice the critical use here of ($Co5$) and continuity to argue from a result about certain numbers of the form $Bel(\theta \mid \chi)$ to general, independent, variables in $[0, 1]$.

Lemma 3.6 *For* $x, y \in [0,1]$

$$F(x,1) = F(1,x) = x, \quad F(x,y) \leq x, y, \quad F(0,x) = F(x,0) = 0.$$

Proof For $\theta \wedge \psi$ consistent,

$$\begin{aligned} Bel(\theta \mid \psi) &= Bel(\theta \wedge \theta \mid \psi) \text{ by } (Co1) \\ &= F(Bel(\theta \mid \theta \wedge \psi), Bel(\theta \mid \psi)) = F(1, Bel(\theta \mid \psi)) \text{ by } (Co2). \end{aligned}$$

Hence by continuity of F and $(Co5)$, $x = F(1,x)$ holds for all $x \in [0,1]$.
Similarly $x = F(x,1)$ by using

$$\begin{aligned} Bel(\theta \mid \psi) &= Bel(\theta \wedge (\theta \vee \neg\theta) \mid \psi) \\ &= F(Bel(\theta \mid \psi \wedge (\theta \vee \neg\theta)), Bel(\theta \vee \neg\theta \mid \psi)) \\ &= F(Bel(\theta \mid \psi), 1) \text{ by } (Co1), (Co2). \end{aligned}$$

Hence by monotonicity of F, $F(x,y) \leq F(x,1) = x$, $F(x,y) \leq F(1,y) = y$ (using continuity for $x = 0$ or $y = 0$) and the last identities are now immediate. □

Lemma 3.7 *The structure* $([0,1], F, <)$ *is isomorphic to* $([0,1], \times, <)$*, where* $\times$ *denotes the usual multiplication, i.e. there is a 1–1 onto function* $g : [0,1] \longrightarrow [0,1]$ *such that for all* $x, y, \in [0,1]$,

$$\begin{aligned} x < y &\Leftrightarrow g(x) < g(y) \\ gF(x,y) &= g(x)g(y) \end{aligned}$$

Proof For notational simplicity we shall write $x \cdot y$ for $F(x,y)$ and $\dot{x}^n$ for $x \cdot x \cdot \ldots \cdot x$ n times. By associativity of F this is unambiguous. Fix $0 < \alpha < 1$.

Since $\dot{1}^m = 1 > \alpha > 0 = \dot{0}^m$ by continuity and (strict) monotonicity $(Co3)$ there is a unique β such that $\dot{\beta}^m = \alpha$ for $0 < m \in \mathbb{N}$ (= the set of natural numbers). Denote by $\dot{\alpha}^{\frac{n}{m}}$ the number $\dot{\beta}^n$. Then if $\dot{\delta}^{rm} = \alpha$ then $\dot{\delta}^{rm} = \dot{\beta}^m$ so by monotonicity $\beta = \dot{\delta}^r$.

Hence $\dot{\alpha}^{\frac{n}{m}} = \dot{\beta}^n = \dot{\delta}^{rn} = \dot{\alpha}^{\frac{rn}{rm}}$. Using this cancellation 'rule' we see that

$$\dot{\alpha}^{\frac{n}{m}} \cdot \dot{\alpha}^{\frac{r}{s}} = \dot{\alpha}^{\frac{ns}{ms}} \cdot \alpha^{\frac{mr}{ms}} = \dot{\alpha}^{(\frac{ns+mr}{ms})} = \dot{\alpha}^{(\frac{n}{m}+\frac{r}{s})}$$

and also if $\frac{n}{m} < \frac{r}{s}$ then $\dot{\alpha}^{\frac{r}{s}} = \dot{\alpha}^{\frac{n}{m}} \cdot \dot{\alpha}^{(\frac{r}{s}-\frac{n}{m})} < \dot{\alpha}^{\frac{n}{m}}$ by lemma 3.6 and monotonicity of F on $(0,1]^2$, since $0 < \dot{\alpha}^{\frac{r}{s}-\frac{n}{m}} < 1$.

Now notice that the sequence $\dot{\alpha}^n$ is decreasing so $\lim_{n\to\infty} \dot{\alpha}^n = \gamma$ for some $\gamma \geq 0$. If $0 < \gamma$ then, since $\gamma < \alpha < 1$, $\gamma < \dot{\gamma}^{\frac{1}{2}}$ so $\dot{\alpha}^n < \dot{\gamma}^{\frac{1}{2}}$ for some n. But then $\dot{\alpha}^{2n} < \gamma$, contradiction. Hence $\gamma = 0$. Similarly the sequence $\dot{\alpha}^{\frac{1}{n}}$ is increasing with limit 1.

Now given $0 < \beta < 1$ let

$$r = \sup\{\frac{p}{q} \mid \dot{\alpha}^{\frac{p}{q}} \geq \beta\} \in \mathbb{R},$$

where $p, q \in \mathbb{N},\ q \neq 0$. Notice that this set is non-empty since $\dot{\alpha}^{\frac{1}{2^n}} > \beta$ for some n. We claim that

$$\begin{aligned} \inf\{\dot{\alpha}^{\frac{p}{q}} \mid \frac{p}{q} < r\} &= \inf\{\dot{\alpha}^{\frac{p}{q}} \mid \frac{p}{q} \leq r\} = \beta \\ &= \sup\{\dot{\alpha}^{\frac{p}{q}} \mid \frac{p}{q} \geq r\} = \sup\{\dot{\alpha}^{\frac{p}{q}} \mid \frac{p}{q} > r\} \end{aligned} \tag{3.1}$$

For suppose not. Then there are γ_1, γ_2 such that

$$\inf\{\dot{\alpha}^{\frac{p}{q}} \mid \frac{p}{q} < r\} > \gamma_1 > \gamma_2 > \sup\{\dot{\alpha}^{\frac{p}{q}} \mid \frac{p}{q} > r\}.$$

Pick m such that $\gamma_2 < \dot{\alpha}^{\frac{1}{m}} \cdot \gamma_1$, which must be possible since $\lim_{n\to\infty} \dot{\alpha}^{\frac{1}{2^n}} = 1$ and $\gamma_2 < \gamma_1 = 1 \cdot \gamma_1$. Now pick $\frac{p}{q} > r > \frac{p}{q} - \frac{1}{m}$.

Then $\dot{\alpha}^{\frac{p}{q}} < \gamma_2 < \dot{\alpha}^{\frac{1}{m}} \cdot \gamma_1 < \dot{\alpha}^{\frac{1}{m}} \cdot \dot{\alpha}^{(\frac{p}{q} - \frac{1}{m})} = \dot{\alpha}^{\frac{p}{q}}$, contradiction.

Thus we have shown that for each $\beta \in (0,1)$ there is a unique $r \in (0,\infty)$ such that (3.1) holds, and conversely given $r \in (0,\infty)$ we can find $\beta \in (0,1)$ such that (3.1) holds, so we can unambiguously write $\dot{\alpha}^r$ for β. Furthermore by the result for rational r already proved, $\dot{\alpha}^{r_1} \cdot \dot{\alpha}^{r_2} = \dot{\alpha}^{(r_1+r_2)}$. It is now clear that if we define $g : [0,1] \longrightarrow [0,1]$ by $g(0) = 0,\ g(1) = 1,\ g(\dot{\alpha}^r) = (\frac{1}{2})^r$ for $r \in (0,\infty)$ then g is the required isomorphism. □

Remark In order to prove this lemma we have only used that F is associative and continuous on $[0,1]^2$, strictly increasing on $(0,1]^2$ and for all $x \in [0,1]$, $F(x,1) = F(1,x) = x$. We shall later have occasion to use this lemma in a different context in which these conditions hold.

Notice that in the proof of the lemma 3.7 we have a free choice of $\alpha \in (0,1)$. Notice also that since g is a strictly order-preserving function from $[0,1]$ onto $[0,1]$ it must be continuous since if $\gamma_n \in [0,1]$, $\lim_{n\to\infty} \gamma_n = \gamma$ and, say, $\beta = \liminf_{n\to\infty} g(\gamma_n) < g(\gamma)$ then $g(\gamma_n) < \beta + \frac{g(\gamma)-\beta}{2} < g(\gamma)$ for arbitrarily large n so $\gamma_n < g^{-1}(\beta + \frac{g(\gamma)-\beta}{2}) < \gamma$ for all such n, contradiction.

We now derive some properties of S. Towards this end let

$$B = \{Bel(\theta \mid \psi) \mid \theta, \psi \in SL \text{ and } \psi \text{ consistent}\}.$$

Then $0, 1 \in B$ and by $(Co5)$ B is dense in $[0,1]$.

Lemma 3.8 *$S(0) = 1$, S is onto $[0,1]$, strictly decreasing, continuous and S^2 is the identity.*

Proof By $(Co2)$ $S(0) = S(Bel(\theta \wedge \neg\theta \mid \theta \vee \neg\theta)) = Bel(\neg(\theta \wedge \neg\theta) \mid \theta \vee \neg\theta) = 1$.

To show the remaining parts we first show that S^2 is the identity on B. This follows since for $\theta, \psi \in SL$, ψ consistent,

$$S^2(Bel(\theta \mid \psi)) = S(Bel(\neg\theta \mid \psi)) = Bel(\neg\neg\theta \mid \psi) = Bel(\theta \mid \psi)$$

by $(Co4)$.

To show that S is onto $[0,1]$ suppose $\gamma \in [0,1]$ and let

$$\tau = \sup\{\beta \mid S(\beta) \geq \gamma\} = \inf\{\beta \mid S(\beta) < \gamma\}.$$

Then $S(\tau) = \gamma$. For if not, say $S(\tau) < \gamma$, then for some $\beta \in B$, $\gamma > \beta > S(\tau)$. Then since $\beta = S^2(\beta) > S(\tau)$, $S(\beta) < \tau$ so, by definition of τ, $\gamma \leq S^2(\beta) = \beta$, contradiction.

To show that S is strictly decreasing suppose on the contrary that $\gamma < \delta$ but $S(\gamma) = S(\delta)$. Pick $\beta, \tau \in B$ such that $\gamma < \beta < \tau < \delta$. Then $S(\gamma) = S(\beta) = S(\tau) = S(\delta)$ so $\beta = S^2(\beta) = S^2(\tau) = \tau$, contradiction. As in the above remark it now follows that S is continuous and hence S^2, being the identity on the dense set B, must be the identity on $[0,1]$. □

Remark If we assume that given any $0 < \beta, \gamma \leq 1$ we can find θ_3 and consistent $\theta_1 \wedge \theta_2$ such that $Bel(\theta_2 \mid \theta_1) = \beta$, $Bel(\theta_3 \mid \theta_1 \wedge \theta_2) = \gamma$ then it is no longer necessary to assume that S is decreasing, it can be derived. To see this suppose $\delta < \beta$. By continuity of F and lemma 3.6, $\delta = F(\gamma, \beta)$ for some $\gamma < 1$. Let $\theta_1, \theta_2, \theta_3$ be as above (clearly they also exist if $\gamma = 0$). Then

$$\delta = Bel(\theta_2 \wedge \theta_3 \mid \theta_1)$$

and

$$\begin{aligned} S(\beta) &= Bel(\neg\theta_2 \mid \theta_1) \\ &= Bel((\neg\theta_2 \vee \neg\theta_3) \wedge (\neg\theta_2 \vee \theta_3) \mid \theta_1) \\ &= F(Bel(\neg\theta_2 \vee \theta_3 \mid \theta_1 \wedge (\neg\theta_2 \vee \neg\theta_3)), Bel(\neg\theta_2 \vee \neg\theta_3 \mid \theta_1)), \end{aligned}$$

so, since F is increasing and $S(\delta) = Bel(\neg\theta_2 \vee \neg\theta_3 \mid \theta_1)$, by lemma 3.6, $S(\beta) \leq S(\delta)$.

Since S is continuous and $S(0) > 0$, $S(1) < 1$ we can pick $0 < \nu < 1$ such that $S(\nu) = \nu$. Let g be as in lemma 3.7 with $\alpha = \nu$. If we now define

$$\begin{aligned} Bel'(\theta \mid \psi) &= gBel(\theta \mid \psi), \\ F'(x,y) &= gF(g^{-1}(x), g^{-1}(y)) = xy \text{ by lemma 3.7}, \\ S'(x) &= gSg^{-1}(x), \end{aligned}$$

then $(Co1\text{–}5)$ hold with Bel', F', S' in place of Bel, F, S. We shall prove Cox's theorem for this g. Without loss of generality and to simplify the notation we may assume $Bel = Bel'$, $S = S'$, $F = F' =$ multiplication on $[0,1]$. Notice that now $S(\frac{1}{2}) = \frac{1}{2}$.

To simplify matters further we shall often henceforth write $1\dot{-}x$ for $S(x)$. Notice that $x = S^2(x) = 1\dot{-}(1\dot{-}x)$.

For $0 \leq u \leq v \leq 1$ set $v\dot{-}u = v(1\dot{-}\frac{u}{v})$ ($= 0$ if $v = 0$). Notice that $v\dot{-}u$ is increasing in v and decreasing in u. Our aim now is to show that $v\dot{-}u = v - u$. We shall first derive an important identity, involving S, which appears in Cox's original proof.

Lemma 3.9 *For* $0 < x \leq y < 1$,

$$yS\left(\frac{x}{y}\right) = S(x)S\left(\frac{S(y)}{S(x)}\right).$$

Proof First consider the case when $y = Bel(\theta \mid \psi)$, $\frac{x}{y} = Bel(\phi \mid \theta \wedge \psi)$ with $\theta \wedge \psi$ consistent. Then, since F is now multiplication, $x = Bel(\theta \wedge \phi \mid \psi)$ and $\psi \wedge \neg(\theta \wedge \phi)$ must be consistent since $S(x) > 0$. Hence

$$yS\left(\frac{x}{y}\right) = Bel(\neg\phi \mid \theta \wedge \psi)Bel(\theta \mid \psi) = Bel(\theta \wedge \neg\phi \mid \psi)$$

whilst

$$\begin{aligned} S(y) &= Bel(\neg\theta \mid \psi) = Bel((\neg\theta \vee \neg\phi) \wedge (\neg\theta \vee \phi) \mid \psi) \\ S(x) &= Bel(\neg\theta \vee \neg\phi \mid \psi) \end{aligned}$$

so, as above,

$$\begin{aligned} S(x)S\left(\frac{S(y)}{S(x)}\right) &= S(x) \cdot S(Bel(\neg\theta \vee \phi \mid \psi \wedge (\neg\theta \vee \neg\phi))) \\ &= S(x) \cdot Bel(\theta \wedge \neg\phi \mid \psi \wedge (\neg\theta \vee \neg\phi)) \\ &= Bel((\theta \wedge \neg\phi) \wedge (\neg\theta \vee \neg\phi) \mid \psi) \\ &= Bel(\theta \wedge \neg\phi \mid \psi). \end{aligned}$$

The result now follows by $(Co5)$ and continuity of S. □

Lemma 3.10 *For* $u, v, w \in [0,1]$ *if* $u \leq v$ *and* $w \leq v\dot{-}u$ *then* $w \leq v$ *and* $u \leq v\dot{-}w$ *and* $(v\dot{-}u)\dot{-}w = (v\dot{-}w)\dot{-}u$.

Proof If $w \leq v\dot{-}u$ then $w \leq v(1\dot{-}\frac{u}{v}) \leq v$ and $\frac{w}{v} \leq (1\dot{-}\frac{u}{v})$ for $v \neq 0$ so taking S of both sides gives $\frac{u}{v} \leq 1\dot{-}\frac{w}{v}$ as required. Finally

$$(v\dot{-}u)\dot{-}w = v(1\dot{-}\frac{u}{v})\left(1\dot{-}\frac{w}{v(1\dot{-}\frac{u}{v})}\right)$$

$$(v\dot{-}w)\dot{-}u = v(1\dot{-}\frac{w}{v})\left(1\dot{-}\frac{u}{v(1\dot{-}\frac{w}{v})}\right).$$

Putting $y = 1\dot{-}\frac{u}{v}, x = \frac{w}{v}$ these become $vyS(\frac{x}{y})$ and $vS(x)S\left(\frac{S(y)}{S(x)}\right)$ which are equal by lemma 3.9 for $u, w > 0$. The cases for $u = 0,\ w = 0$ follow by inspection. □

Lemma 3.11 *For* $0 \leq u \leq z \leq 1,\ z\dot{-}u \leq z$ *and* $z\dot{-}(z\dot{-}u) = u$.

Proof $z\dot{-}u = z(1\dot{-}\frac{u}{z}) \leq z$. Also

$$z\dot{-}(z\dot{-}u) = z\dot{-}z(1\dot{-}\frac{u}{z}) = z(1\dot{-}(1\dot{-}\frac{u}{z})) = z.\frac{u}{z} = u.$$

□

Now for $u, v \in [0,1]$ and $u \leq 1\dot{-}v$ define

$$u\dot{+}v = 1\dot{-}((1\dot{-}u)\dot{-}v).$$

We now derive a string of (expected) properties of $\dot{+}$.

Lemma 3.12 *Let* $u, v, w \in [0,1]$ *and* $u \leq 1\dot{-}v$. *Then*

(i) $v \leq 1\dot{-}u$.

(ii) $u\dot{+}v = v\dot{+}u$.

(iii) $(u\dot{+}v) \geq u, v$.

(iv) $(u\dot{+}v)\dot{-}u = v$.

(v) For $u \leq w$, $(w\dot{-}u)\dot{+}u = w$.

(vi) For $w \leq 1\dot{-}(u\dot{+}v)$, $\ w \leq 1\dot{-}v$ *and* $u \leq 1\dot{-}(v\dot{+}w)$ *and* $u\dot{+}(v\dot{+}w) = (u\dot{+}v)\dot{+}w$.

(vii) $wu \leq 1\dot{-}wv$ *and* $w(u\dot{+}v) = wu\dot{+}wv$.

Proof (i) is immediate since S is decreasing, and (ii) follows by (i) and lemma 3.10. For (iii) notice that $1\dot{-}u \geq (1\dot{-}u)\dot{-}v$ so that applying S to both sides gives the result. To show (iv) we have

$$\begin{aligned}(u\dot{+}v)\dot{-}u &= (1\dot{-}((1\dot{-}u)\dot{-}v))\dot{-}u \\ &= (1\dot{-}u)\dot{-}((1\dot{-}u)\dot{-}v) \quad \text{by lemma 3.10,} \\ &= v \quad \text{by lemma 3.11.}\end{aligned}$$

For (v),

$$\begin{aligned}(w\dot{-}u)\dot{+}u &= u\dot{+}(w\dot{-}u) = 1\dot{-}((1\dot{-}u)\dot{-}(w\dot{-}u)) \\ &= 1\dot{-}((1\dot{-}u)\dot{-}((1\dot{-}(1\dot{-}w))\dot{-}u)) \\ &= 1\dot{-}((1\dot{-}u)\dot{-}((1\dot{-}u)\dot{-}(1\dot{-}w))) \quad \text{by lemma 3.10,} \\ &= 1\dot{-}(1\dot{-}w) = w \quad \text{by lemma 3.11.}\end{aligned}$$

To show (vi) notice that if $w \leq 1\dot{-}(u\dot{+}v)$ then

$$1\dot{-}w \geq u\dot{+}v \geq v.$$

Hence $w \leq 1\dot{-}v$. Also $(1\dot{-}w)\dot{-}v \geq (u\dot{+}v)\dot{-}v = u$ which gives $u \leq 1\dot{-}(w\dot{+}v)$. Finally then

$$\begin{aligned}u\dot{+}(v\dot{+}w) &= 1\dot{-}((1\dot{-}u)\dot{-}(1\dot{-}((1\dot{-}v)\dot{-}w))) \\ &= 1\dot{-}((1\dot{-}(1\dot{-}((1\dot{-}v)\dot{-}w)))\dot{-}u) \\ &= 1\dot{-}(((1\dot{-}v)\dot{-}w)\dot{-}u)\end{aligned}$$

and by lemma 3.10 the u, v, w can be permuted to give the answer. To show (vii), $wu \leq u \leq 1\dot{-}v \leq 1\dot{-}wv$ since $wv \leq v$. Also

$$\begin{aligned}wu &= w((u\dot{+}v)\dot{-}v) = w(u\dot{+}v)\left(1\dot{-}\tfrac{v}{u\dot{+}v}\right) \\ &= w(u\dot{+}v)\left(1\dot{-}\tfrac{wv}{w(u\dot{+}v)}\right) \quad (= 0 \text{ if } w = 0) \\ &= w(u\dot{+}v)\dot{-}wv,\end{aligned}$$

and the result follows by adding wv to both sides. □

We shall now show that $\dot{+}$ and $+$ are the same thing, first in some special cases and then in general.

Lemma 3.13 *For* $n > 0$, $\frac{1}{2^n}\dot{+}\frac{1}{2^n} = \frac{1}{2^{n-1}}$.

Proof $\frac{1}{2}\dot{+}\frac{1}{2} = 1\dot{-}((1\dot{-}\frac{1}{2})\dot{-}\frac{1}{2}) = 1\dot{-}(\frac{1}{2}\dot{-}\frac{1}{2}) = 1$. For $n > 1$ we can straightforwardly use induction and lemma 3.12(vii). □

For $m \leq 2^n$ let $\dot{m}\left(\frac{1}{2^n}\right)$ stand for $\frac{1}{2^n}\dot{+}\frac{1}{2^n}\dot{+}...\dot{+}\frac{1}{2^n}$ m times. Since $\dot{2^n}\left(\frac{1}{2^n}\right) = 1$, by lemma 3.12 the additions here are easily seen to be well defined.

Lemma 3.14 *For* $n, m > 0$ *and* $m < 2^n$ *we have* $\dot{m}\left(\frac{1}{2^n}\right) = \frac{m}{2^n}$.

Proof Suppose $\dot{m}\left(\frac{1}{2^n}\right) < \frac{m}{2^n}$ (the other case is similar), say,

$$\dot{m}\left(\frac{1}{2^n}\right) < \frac{1}{2^{\frac{p}{q}}} < \frac{m}{2^n} \text{ with } p, q \in \mathbb{N},\ q > 0.$$

Then by lemma 3.12 (vii),

$$\left(\dot{m}\left(\frac{1}{2^n}\right)\right)^q = \dot{m}^q\left(\frac{1}{2^{nq}}\right) < \frac{1}{2^p} < \frac{m^q}{2^{nq}}.$$

But since $m^q > 2^{nq-p}$,

$$\dot{m}^q\left(\frac{1}{2^{nq}}\right) > \dot{2}^{nq-p}\left(\frac{1}{2^{nq}}\right) = \frac{2^{nq-p}}{2^{nq}} = \frac{1}{2^p}$$

by lemma 3.13, giving the required contradiction. □

Lemma 3.15 *For* $0 \leq x \leq y \leq 1,\ \ y\dot{-}x = y - x.$

Proof By lemma 3.12 (iv) and lemma 3.14 this is true for x, y of the form $\frac{m}{2^n}$ $(n, m > 0,\ m < 2^n)$ and hence by continuity for all x, y. □

Corollary 3.16 *For* $x, y \in [0,1]$ *with* $x \leq 1\dot{-}y,\ x\dot{+}y = x + y.$

Proof Immediate from the definition of $\dot{+}$ and lemma 3.15. □

We are ready to complete the proof of Cox's theorem.

Proof of theorem 3.4

That $Bel(\theta \mid T) = 1$ for $\models \theta$ is clear from $(Co2)$. If $\models \neg(\theta \wedge \phi)$ then either $\theta \vee \phi$ is contradictory, in which case each of $Bel(\theta \vee \phi \mid T)$, $Bel(\theta \mid T)$, $Bel(\phi \mid T)$ is zero, or else

$$Bel(\theta \vee \phi \mid T) - Bel(\theta \mid T) = Bel(\theta \vee \phi \mid T) - Bel((\theta \vee \phi) \wedge \neg\phi \mid T)$$

by $(Co1)$ since $\models \theta \leftrightarrow ((\theta \vee \phi) \wedge \neg\phi)$,

$$\begin{aligned} &= Bel(\theta \vee \phi \mid T) - Bel(\neg\phi \mid \theta \vee \phi) Bel(\theta \vee \phi \mid T) \\ &\quad \text{since } F \text{ is multiplication,} \\ &= Bel(\theta \vee \phi \mid T) Bel(\phi \mid \theta \vee \phi) \text{ by lemma 3.15} \\ &= Bel(\phi \wedge (\theta \vee \phi) \mid T) = Bel(\phi \mid T) \text{ by } (Co1). \end{aligned}$$

Either way we obtain $(P2)$ for Bel.

Finally since F is multiplication, if $Bel(\psi \mid T) \neq 0$ then ψ is consistent and by $(Co3)$, $Bel(\theta \wedge \psi \mid T) = Bel(\theta \mid \psi) Bel(\psi \mid T)$ as required. □

Criticism

Despite these arguments for belief as probability, or scaled probability, the unpleasant fact is that if, as in example $\mathbb{E}$, one does elicit such knowledge and belief values from the expert then the set K very often turns out to be (seriously) inconsistent with belief as (scaled) probability and indeed with the other interpretations given in the next chapters. (On this point see [68] and also, for contrary views, [28], [51].)

We shall later see some further criticisms of belief as probability in Chapter 10. For the present however we can at least say that the results of this chapter have provided strong arguments in favour of belief as probability being the ideal.