Harmony and Separability in Classical Logic

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Plan of the talk

(1) In the first part of the talk, I'll introduce three possible (indeed, actual) interpretations of the inferentialist thesis that rules determine meanings.

- I'll argue that two key proof-theoretic requirements – harmony and separabilty – directly stem from at least two of these interpretations.

(2) In the second part, I'll rehearse the well-known complaint that classical logic, unlike intuitionistic logic, fails to live up to the inferentialist's standards.

  - Boricic (1985) and Read (2000) provide harmonious and separable formalizations of classical logic in a SET-SET, multiple-conclusion, framework.
  - Smiley (1996) and Rumfitt (2000) show that, even within a SET-FMLA framework, classical logic can be made harmonious and separable provided we add to our formalization of logic rules for denying complex statements.
  - However, it is controversial whether *bilateralist* and *multiple-conclusion* formalizations of logic represent our actual logical practice, which is standardly taken to be single-conclusion and assertion-based.

(3) In the third part of the talk, I'll introduce a harmonious and separable formalization of classical logic – one that is both *assertion-based* and *single-conclusion*.

Logical inferentialism

(a) The meaning of a logical constant is fully determined by the rules for its correct use (a semantic thesis).

> *The meaning of [the logical constants] can be exhaustively determined by the rules of inference in which these signs occur. (Popper, 1947, p. 220)*
>
> *Formal (or logical) signs are those whose full sense can be given by laying down rules of development for the propositions expressed by their help. (Kneale, 1956, pp. 254-5)*
>
> *The meaning of [a] logical constant can be completely determined by laying down the fundamental laws governing it. (Dummett, 1991, p. 247)*

(b) To understand a logical operator $ is to be willing to infer according to/grasp/implicitly know the $ meaning-constitutive rules (an epistemological thesis).

Two aspects of use

- Dummett individuates two semantically relevant aspects of the correct use of a sentence:

> *crudely expressed, there are always two aspects of the use of a given form of sentence: the conditions under which an utterance of that sentence is appropriate ... and the consequences of an utterance of it ... (Dummett, 1973, p. 396)*

- In the case of complex sentences, these two aspects are respectively represented by I- and E-rules.
- Hence, thesis (a) becomes the claim that the meanings of the logical constants is fully determined by their I- and E-rules.
- Two comments:
  - (i) I take it that what logical inferentialists are after is a story about the 'logical meanings' of the English words 'and', 'if', 'or', 'not', 'some' etc. – they're not trying to give a *complete* account of what, say, 'if' and 'and' mean in English.
  - (ii) In keeping with Gentzen and Prawitz, I will take I- and E-rules to be, in Dummett's terminology, pure.

> *to understand* $\ulcorner A \vee B \urcorner$*, one need not understand* $\ulcorner A \wedge B \urcorner$ *or* $\ulcorner A \to B \urcorner$*. (Dummett, 1991, p. 223)*

Rules determine truth-conditions

(1) I- and E-rules determine meanings in the sense of determining the truth-conditions of the logical operators (*rules first!*).

- For instance, $\wedge$-I and $\wedge$-E force $\wedge$ to denote the truth-function it denotes, on the assumption that they are truth-preserving.

> *The semantic explanatory route does not go from [...] "objects" or meanings to the laws concerning them [...], but the other way around, from [laws] to meanings. (Coffa, 1991, p. 267)*

- Two (more) comments:

  (i) Insofar as the logics we're interested in are complete, there's no *extensional* difference between (1) and the more traditional view that *truth-functions come first*, and validate rules.

  (ii) (b) and (1) fit nicely with the Fregean theses that to know the meaning of a linguistic expression is to grasp its sense and that sense determines reference (see MacFarlane, 2005/2009).

I-rules are in some sense complete

(2) I- and E-rules determine meanings in the sense that I-rules *exhaust in principle the grounds for asserting complex statements*, they are "collectively in a certain sense *complete*" (Dummett, 1991, p. 252).

- I-rules specify in principle *necessary and sufficient* conditions for assertion: "I-rules are (jointly) necessary as well as severally sufficient" (Read, 2010, p. 562).
- Dummett calls this the Fundamental Assumption (1991, pp. 252-254):

  (FA) If a complex statement can be introduced, then it can *in principle* be introduced by an argument ending with an application of one of the *I-rules* for its main logical operator (a canonical argument).

- Given the Fundamental Assumption, A's canonical grounds and A itself must have the same set of consequences, i.e. B follows from A's canonical grounds iff B follows from A itself.

> *Proof: Let $\mathcal{CG}[A]$ be A's canonical grounds, and suppose that B follows from A. Since A follows from $\mathcal{CG}[A]$, B itself follows from $\mathcal{CG}[A]$. Now suppose that B follows from $\mathcal{CG}[A]$, and assume A. By the Fundamental Assumption, $\mathcal{CG}[A]$ itself follows. Hence, we may conclude B, as required.* ■

Separability

(3) I- and E-rules can also be thought to determine meanings in the sense that the totality of \$'s correct uses is derivable from \$'s meaning-constitutive rules (alone).

> *[every correct use of a logical constant is], in some sense to be specified, derivable and/or justified on the basis of the putatively meaning-conferring rule or rules. (Milne, 1991, pp. 49-50)*

- Suppose logic is analytic, in the sense that *logically valid inferences are valid in virtue of the meaning of the logical vocabulary figuring in them*.
- Suppose that the *meaning* of a logical operator is given by (a subset of) its I- and E-rules.
- Then, Milne (2002, p. 521) argues that acceptable formalizations of logic ought to be separable.

Definition

(Separability) Whenever $\Gamma \vdash A$, there is a derivation of A from Γ which only involves operational rules for the logical vocabulary figuring in either Γ or A (plus possibly *structural* rules)

Conservativeness

Separability is essentially equivalent to the requirement that the rules for a logical operator should yield a conservative extension of the systems to which they are added – Dummett (1991, p. 252) calls this total harmony.

Definition

(Conservativeness) Let S and S' be two systems with language L and L' respectively, where $S \subseteq S'$ and $L \subseteq L'$. Then, S' is *conservative* over S if, for all $A \in L$ and for all sets Γ of L-sentences, $\Gamma \vdash_{S'} A$ only if $\Gamma \vdash_S A$.

Fact

A system S composed by a set of structural rules S_s and a set of operational rules S_o is separable if and only if, for each logical operator \$, the \$ rules yield a conservative extension of $S_s \cup S_o - \{\$\text{-}I, \$\text{-}E\}$.

- Notice that (3) is stronger than (2): separability requires that \$-sentences that are provable at all must be provable by means of a proof each of whose steps is taken into accordance with some \$-rule – *not just the last step*, as the Fundamental Assumption demands.

E-rules: neither too strong nor too weak

- We find several accounts of harmony in the literature:
  - Intrinsic harmony (defined, but never really endorsed, by Prawitz and Dummett);
  - Strong intrinsic harmony (well-known in Computer Science departments);
  - GE harmony (Lorenzen, Negri and von Plato, Read, Schroeder-Heister);
  - Harmony as reflective equilibrium (Tennant, but see Steinberger 2010, 2011 for a compelling criticism).
- I briefly compare strong intrinsic harmony and GE harmony in an Appendix.
- For the sake of simplicity, I'll identify harmony with strong intrinsic harmony, to which we now turn.

Maximum formulae

Definition

(Maximum formula) A formula occurrence occurring in a derivation Π that is both the consequence of an application of a $\$$ I-rule and the major premise of an application of a $\$$ E-rule is a maximum formula in Π.

Example

(Maximum formula)

$$\dfrac{\dfrac{\begin{matrix}\Gamma_0\\ \Pi_0\\ A\end{matrix}}{A \vee B}\vee\text{-I} \qquad \begin{matrix}\Gamma_1,[A]^i\\ \Pi_1\\ C\end{matrix} \qquad \begin{matrix}\Gamma_2,[B]^i\\ \Pi_2\\ C\end{matrix}}{C}\vee\text{-E}, i$$

Local soundness (or intrinsic harmony)

Definition

(Local soundness) The $\$$ E-rules are locally sound iff derivations containing a maximum formula $\$(A_1, \ldots, A_n)$ can always be transformed in derivations of the same conclusions from the same undischarged assumptions which avoid the detour through $\$(A_1, \ldots, A_n)$.

Example

($\to$-reduction)

$$\dfrac{\dfrac{\begin{matrix}\Gamma_0,[A]^i\\ \Pi_0\\ B\end{matrix}}{A\to B}\to\text{-I}, i \qquad \begin{matrix}\Gamma_1\\ \Pi_1\\ A\end{matrix}}{B}\to\text{-E} \qquad \rightsquigarrow_r \qquad \begin{matrix}\Gamma_1\\ \Pi_1\\ \underbrace{\Gamma_0, A}\\ \Pi_0\\ B\end{matrix}$$

- Locally sound rules are guaranteed to be 'not too strong':

  by a E-rule one essentially only restores what had already been established by the major premiss of the application of a I-rule. (Prawitz, 1965, p. 33)

Local completeness

Definition

(Local completeness) The E-rules for $\$$ are locally complete if and only if "every derivation of a formula A with principal operator $\$$ can be expanded to one containing an application of an E-rule of $\$$, and applications of all I-rules of $\$$ each with conclusion A" (Francez and Dyckhoff, 2009, p. 9).

Example

($\to$-expansion)

$$\begin{matrix}\Pi\\ A\to B\end{matrix} \qquad \rightsquigarrow_e \qquad \dfrac{\dfrac{\begin{matrix}\Pi\\ A\to B\end{matrix} \qquad [A]^i}{B}\to\text{-E}}{A\to B}\to\text{-I}, i$$

- Locally complete E-rules are guaranteed to be 'not too weak':

  we can apply the elimination rules to a judgment to recover enough knowledge to permit reconstruction of the original judgment. (Pfenning and Davies, 2001, p. 3)

Strong intrinsic harmony

Definition

(Strong Intrinsic harmony) A pair of $ I- and E-rules satisfies strong intrinsic harmony iff the $ E-rules are both locally sound and locally complete.

Example

(Intuitionistic negation)

$$\dfrac{\dfrac{\begin{matrix}\Gamma_0,[A]^i\\ \Pi_0\\ \bot\end{matrix}}{\neg A}\,\neg\text{-I}, i \qquad \begin{matrix}\Gamma_1\\ \Pi_1\\ A\end{matrix}}{\bot}\,\neg\text{-E} \quad \rightsquigarrow_r \quad \begin{matrix}\Gamma_1\\ \Pi_1\\ \underbrace{\Gamma_0, A}\\ \Pi_0\\ \bot\end{matrix} \qquad\qquad \begin{matrix}\Pi\\ \neg A\end{matrix} \quad \rightsquigarrow_e \quad \dfrac{\dfrac{\begin{matrix}\Pi\\ \neg A\end{matrix} \qquad [A]^i}{\bot}\,\neg\text{-E}}{\neg A}\,\neg\text{-I}, i$$

Fact

Minimal logic is harmonious.

Some well-known facts

Fact (Prawitz 1965)

Standard (single-conclusion and assertion-based) formalizations of IPL are separable.

Fact (Prawitz 1965, 1977)

Standard formalizations of IPL are arguably harmonious.

Fact

Standard formalizations of CPL violate the Fundamental Assumption.

Fact

The classical rules for negation are not harmonious: the harmonious rules are given by the intuitionistic pair $\{\neg\text{-}I, \neg\text{-}E\}$. The classical extra rule we need to get classical logic sits out there in the cold!

Theorem (Leblanc 1966)

If $\bot$ is a propositional constant and either double negation elimination or classical reductio (or some equivalent rule) are taken to partly determine the meaning of classical negation, then no semantically complete single-conclusion formalization of CPL is separable.

A proof-theoretic mess!

Plainly, the classical rule [of double negation elimination] is not in harmony with the introduction rule. (Dummett, 1991, p. 291)

Clearly [. . .] we know procedures that make all intuitionistic inference rules acceptable [. . .] but not any procedure that makes the rule of double negation elimination acceptable. (Prawitz, 1977, p. 39)

The meaning of the intuitionistic logical constants can be explained in a very direct way, without any apparatus of semantic theory, in terms of the use made of them in [our deductive] practice. (Dummett, 1991, p. 299)

Three assumptions

- Can classical logic be made harmonious and separable without giving up the assumption that a formalization of logic (suitable for an inferentialist justification) ought to be assertion-based and keep conclusions single?
- I think we can respond in the affirmative, although some objectionable assumptions must be made:

  (1) $\bot$ is not a propositional constant, but rather a logical punctuation sign (Tennant, 1999; Rumfitt, 2000; Steinberger2008).

  (2) we can help ourselves to higher-order rules (rules in which *derivations* are discharged) in our formalization of logic (Schroeder-Heister, 1984; Read, 2010).

  (3) In order to fully respect the Fundamental Assumption, we must substitute the standard rules for disjunction with new classical rules.

Prawitz's and Dummett's accounts of $\bot$

- Prawitz and others take $\bot$ to be a propositional constant whose meaning is determined by the *empty set of canonical grounds* and EFQ.

  > *the introduction rule for $\bot$ is empty, i.e. it is the rule that says that there is no introduction whose conclusion is $\bot$. (Prawitz, 2005, p. 685)*

  > *$\bot$ has no introduction rule. The immediate grounds for deriving $\bot$ are empty. (Negri and von Plato, 2001, p. 8)*

- Dummett agrees that $\bot$ has content, but thinks that it could be asserted if we could assert every atom of the language:

$$(\bot I_D)\ \frac{P_1 \qquad P_2 \qquad P_3 \quad \ldots}{\bot},$$

  where the P_n are all the atoms of the language.

- (If there is a denumerable infinity of atoms, Dummett's rules isn't recursive.)

$\bot$ as a logical punctuation sign

- Absurdity is a logical punctuation sign – it does not mean anything, it marks the end of a deduction in which some contradiction has been reached.

  > *an occurrence of '$\bot$' is appropriate only within a proof ... as a kind of structural punctuation mark. It tells us where a story being spun out gets tied up in a particular kind of knot—the knot of a patent absurdity, or self contradiction. (Tennant, 1999, p. 204)*

- Rumfitt (2000) and Steinberger (2008) defend and elaborate Tennant's account.

- Some problems with the standard view that $\bot$ has content:
  - Dummett's rule isn't very popular, and isn't really viable (it isn't recursive, we may conceive of languages all of whose atoms are true etc.);
  - Inferential justifications of $\bot$ all make use of EFQ in the metalanguage (see Tennant, 1999).
  - See Steinberger (2008, MS) for a sustained defense of the logical punctuation sign approach.

Classical disjunction

- The standard I-rules for $\vee$ hardly represent canonical ways to introduce disjunctions:

  > *nearly always when we assert the disjunction of A and B in ordinary language, we do not so because we already know that A is true, or because we already know that B is true. Rather, we assert the disjunction because we have some reason for thinking that it is highly unlikely, perhaps even impossible, that both A and B will fail to be true. (Soames, 2003, p. 207)*

- Inferentialists might want to consider the following impure but harmonious rules for disjunction:

$$\vee\text{-I}^*,\, n\ \frac{\begin{matrix}[\neg A, \neg B]^n\\ \vdots\\ \bot\end{matrix}}{A \vee B} \qquad\qquad \vee\text{-E}^*\ \frac{A \vee B \qquad \neg A \qquad \neg B}{\bot}.$$

- The standard rules are easily derivable given classical reductio.
- LEM can be now proved by means of an argument ending with an application of $\vee$-I, as the Fundamental Assumption requires (assume $\neg A$ and $\neg\neg A$ and apply $\vee$-I*).

Schroeder-Heister on assumptions of axioms and rules

- Peter Schroeder-Heister once suggested that assumptions be accounted for as *ad hoc* or *temporary axioms*:

  > *Assumptions in sentential calculi technically work like additional axioms. ... But whereas "genuine" axioms ... are usually assumed to be valid ... assumptions bear an ad hoc character: they are considered only within the context of certain derivations. (Schroeder-Heister, 1984, p. 1284)*

- If assumptions are temporary axioms, we may as well add temporary rules to our system:

  > *Instead of considering only ad hoc axioms (i.e. assumption formulas) we can also regard ad hoc inference rules, that is, inference rules ... used as assumptions. Assumption rules technically work like additional basic rules ... (Schroeder-Heister, 1984, p. 1285)*

Two new rules

- Given (i) the logical punctuation sign account of $\bot$ and (ii) the availability of higher-order rules, classical reductio can be rewritten as a structural rule:

$$\dfrac{\begin{array}{c}[A \Rightarrow \bot]^n \\ \vdots \\ \bot\end{array}}{A}\ \mathrm{CR}^{ho}, n\ .$$

- Our proposed impure classical rules for disjunction can be turned into the following admittedly awkward, but pure, and still harmonious, rules:

$$\dfrac{\begin{array}{c}[A \Rightarrow \bot, B \Rightarrow \bot]^n \\ \vdots \\ \bot\end{array}}{A \vee B}\ \vee\text{-}\mathrm{I}^*_p, n \qquad \dfrac{A \vee B \quad \begin{array}{c}[A]^n \\ \vdots \\ \bot\end{array} \quad \begin{array}{c}[B]^n \\ \vdots \\ \bot\end{array}}{\bot}\ \vee\text{-}\mathrm{E}^*_p, n\ .$$

A new normalization theorem

Definition

Formulae of **CPL*** are built up from atoms and from the standard binary connectives $\wedge$, $\vee$, $\rightarrow$, and the unary connective $\neg$. The rules for $\wedge$, $\rightarrow$, and $\neg$ are the standard ones: $\wedge$-I, $\wedge$-E, $\rightarrow$-I, $\rightarrow$-E, $\neg$-I, $\neg$-E. The rules for $\vee$ are $\vee$-I^*_p and $\vee$-E^*_p. There is a structural rule: CR^{ho}.

Fact

CPL* *satisfies strong intrinsic harmony.*

Definition

(Normal derivation) A derivation in **CPL*** is normal iff it contains no maximum formulae.

Theorem

(Normalization) If $\Gamma \vdash_{\mathbf{CPL}^*} A$, then there is a normal deduction in **CPL*** *of A from Γ.*

Two key corollaries

Corollary

(Subformula property) Each formula occurring in a normal deduction Π of A from Γ is a subformula of A or of one of the formulae in Γ.

Corollary

(Separation property) Any normal deduction only consists of applications of the rules for the connectives occurring in the undischarged assumptions, if any, or in the conclusion, plus possibly CR^{ho}.

- This result improves on Prawitz's original 1965 normalization result for **CPL**, for at least two reasons:
  - Normalization is proved for a full formalization of **CPL**, i.e. there are no defined connectives.
  - Normalization entails the subformula property, and hence separability.
- It improves on Stalmarck 1991 normalization result for **CPL**, too: Stalmarck proved normalization for a full formalization of **CPL**, but his result does not yield subformula property and separability.

Objections and some sketchy replies

- You have surreptitiously switched from a SET-FMLA framework to a SET-SET framework (David Matkinson, p.c.).
  - This is true, but conclusions are kept single.
  - Pending further argument that a SET-SET framework with single conclusions is not faithful to our actual logical practice, this is not really an objection.
- Classicality has been reintroduced by brute force, just by adding an extra *structural rule* – the meaning of classical negation still isn't fully determined by your I- and E-rules.
  - That classicality is ultimately a structural feature of a logical system is something that multiple-conclusion logics, bilateral formalizations of **CPL** and **CPL*** all have in common.
  - The objection at best points to a more general problem, viz. that part of the inferential role of the logical constants is determined by the structural rules of the system.
- We never assume rules in ordinary practice (Florian Steinberger).
  - Fair enough. But notice that CR^{ho} has the following (perhaps not o innatural) form: we add R $(A \Rightarrow \bot)$ to our system, get stuck, abandon R, and get sentence (A) out of the whose process.

Three ways of being a classical inferentialist

- Go multiple-conclusions:
  - SET-SET framework.
  - Multiple-conclusions are not really found in our inferential practice.
- Go bilateralist:
  - SET-SET or SET-FMLA framework.
  - Conclusions are kept single.
  - We're given rules for denying complex statements.
  - The relata of the consequence relations are pairs of the form ⟨speech-act, propositional item⟩ (e.g. $\langle +, A\rangle$ or $\langle -, A\rangle$).
  - But are they? Can we assume $-A$?
- Go higher-order:
  - SET-SET framework.
  - Conclusions are kept single.
  - The relata of the consequence relations can be just sentences, or propositions.

Concluding remarks

- Classicality is a structural feature of logical systems (e.g. multiple-conclusions, classical structural rules for denial, CR^{ho}).
- It is some kind of structural-metaphysical assumption – intuitionists are right in saying that it is not given by the (harmonious) negation rules alone.
  - But then, could we not have added LEM or Classical Dilemma as metaphysical assumptions to intuitionistic logic?
  - Could not 'classical logicians' factorize their logical commitments into purely logical ones (intuitionistic logic) and metaphysical ones (say LEM or Classical Dilemma)?
  - They might, but **CPL*** offers a cleaner way of doing this – classicality is likewise added on the top, so to speak, but this does not perturb the proof-theoretic virtues of the resulting logical system.
- Debates about logical revision should focus on our justification, if there can be any, for the structural rules of the logic: if we are prepared to accept higher-order rules, there are no purely proof-theoretic grounds for dismissing classical logic.

The Generalized Inversion Principle

- Recall our argument that, if the Fundamental Assumption holds, A and its canonical grounds must have the same set of consequences.
- We may take this argument seriously, and define a notion of harmony out of it – one that, unlike strong intrinsic harmony, allows us to produce harmonious E-rules given arbitrary I-rules, in keeping with Gentzen's well-known remark that

  > *it should be possible to display the E-inferences as unique functions of their corresponding I-inferences, on the basis of certain requirements. (Gentzen, 1934, p. 80)*

- Following Lorenzen (1955), Sara Negri and Jan von Plato (2001) suggest the following recipe for deriving harmonious- E-rules from arbitrary I-rules:

Generalized Inversion Principle

Whatever follows from the canonical grounds for asserting A must also follow from A.

GE harmony

- Let $\$$ be A's main logical operator, and let $\pi_1, \dots \pi_m$ be the severally sufficient and jointly necessary grounds for asserting A, where each π_i schematically represents either a sentence or a derivation.
- Then, $\$$'s I-rules are as follows:

$$\dfrac{\pi_1 \quad \dots \quad \pi_j}{A}\ \$\text{-I}_1 \quad \dots \quad \dfrac{\pi_k \quad \dots \quad \pi_m}{A}\ \$\text{-I}_o$$

- And here is a rough approximation of what we may call the GE schema:

$$\dfrac{A \qquad \begin{matrix}[\pi_1]^k \\ \vdots \\ C\end{matrix} \qquad \dots \qquad \begin{matrix}[\pi_m]^k \\ \vdots \\ C\end{matrix}}{C}\ \$\text{-E}, k$$

Definition

(GE harmony) A pair of I- and E-rules is GE harmonious if and only if the E-rule has been induced from the I-rule by means of (a suitable representation of) the GE schema.

Examples

Example

(Conjunction)

$$\dfrac{A \wedge B \qquad \begin{array}{c}[A, B]^k \\ \vdots \\ C\end{array}}{C} \wedge\text{-E}_{GE},\ k$$

The standard rule of $\wedge$-E can be derived as a special case, setting C equal to A, B. (Notice that one could equally have *two* generalized E-rules.)

Example

(Implication: higher-order)

$$\dfrac{A \to B \qquad \begin{array}{c}[A \Rightarrow B]^k \\ \vdots \\ C\end{array}}{C} \to\text{-E}_{GE},\ k$$

Tonk and Bullet

Example

(Prior's `tonk`)

$$\texttt{tonk}\text{-I}\ \dfrac{A}{A\ \texttt{tonk}\ B} \qquad\qquad \texttt{tonk}\text{-E}\ \dfrac{A\ \texttt{tonk}\ B}{B}\ .$$

Example

(Bullet)

$$\bullet\text{-I},\ i\ \dfrac{\begin{array}{c}[\bullet]^i \\ \vdots \\ \bot\end{array}}{\bullet} \qquad\qquad \bullet\text{-E}\ \dfrac{\bullet \qquad \bullet}{\bot}\ .$$

Some well-known facts

Fact

`tonk` and $\bullet$ yield non-separable systems.

Fact

`tonk`'s rules are not harmonious.

Fact

$\bullet$ is GE harmonious: its E-rule can be rewritten in a GE format.

Example

(GE Bullet)

$$\bullet\text{-I},\ i\ \dfrac{\begin{array}{c}[\bullet]^i \\ \vdots \\ \bot\end{array}}{\bullet} \qquad\qquad \bullet\text{-E}_{GE},\ i\ \dfrac{\bullet \qquad \begin{array}{c}[\bullet \Rightarrow \bot]^i \\ \vdots \\ C\end{array}}{C}\ .$$

Fact

$\bullet$-E is locally complete, but it is not locally sound, if reductions must not only remove maximum formulae, but also reduce the degree of complexity of the original derivations.

Example

(Bullet: reduction)

$$\bullet\text{-E}\ \dfrac{\bullet\text{-I},\ n\ \dfrac{\begin{array}{c}[\bullet]^n \\ \Pi_1 \\ \bot\end{array}}{\bullet} \qquad \begin{array}{c}\Pi_2 \\ \bullet\end{array}}{\bot} \quad \rightsquigarrow_r \quad \begin{array}{c}\Pi_2 \\ \bullet \\ \Pi_1 \\ \bot\end{array} \qquad\qquad \begin{array}{c}\Pi \\ \bullet\end{array} \quad \rightsquigarrow_e \quad \bullet\text{-I},\ n\ \dfrac{\bullet\text{-E}\ \dfrac{\begin{array}{c}\Pi \\ \bullet\end{array} \qquad [\bullet]^n}{\bot}}{\bullet}\ .$$

- Read (2010) takes this to be an argument for GE harmony and against intrinsic harmony, on the assumption that connectives like $\bullet$ are meaningful and coherent:

> *GE-harmony does not prevent one introducing inconsistent connectives with harmonious rules. Dummett and others claim that harmony should ensure consistency. Not so: the I-rule can itself be inconsistent (but not necessarily incoherent). ... Coherence and consistency are different. (Read, 2010, pp. 570-1)*

- But we could turn tables: perhaps $\bullet$ is meaningless (its I-rule is viciously circular), and it is a bad thing to validate the $\bullet$ rules, so harmony can't be GE harmony.