

Two Approaches to Doxastic Representation

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- Bayesian Theory provides a unified and elegant framework for representing the doxastic states of rational agents.
- Agents may have various types of epistemic attitudes — among these are the following three types: (1) *numerical credences*, (2) confidence *orderings*, and (3) (full) *beliefs*.
- I prefer to start with confidence orderings, and think of the other states in terms of their relation(s) with them.
- I will focus on (logically omniscient) agents S who form attitudes over some finite Boolean algebra \mathcal{B} of (classical, possible worlds) propositions. Let's start with orderings.
- Confidence orderings are induced by binary relations ' $p \succeq q$ ', which may be interpreted as ' S is at least as confident in p as she is in q .' I will assume that \succeq induces a total preorder (i.e., a complete, transitive order) over \mathcal{B} .
- Various *rational requirements* have been proposed for \succeq . See Halpern's *Reasoning About Uncertainty* [10] for a survey.

- The most fundamental requirement on \succeq is (R_0) .
The Fundamental Requirement (R_0) . The relation \succeq (a total preorder on \mathcal{B}) should satisfy the following two constraints:
(A₁) $\top \succ \perp$.
(A₂) For all $p, q \in \mathcal{B}$, if p entails q then $q \succeq p$.
- (R_0) is widely accepted as a rational requirement for \succeq .
- Once we move to stronger requirements on \succeq , things get more controversial. Here are some additional requirements that Bayesians accept — over and above T.F.R. (R_0) .
(A₃) If p entails q and $\langle q, r \rangle$ are mutually exclusive, then:
$$q \succ p \implies q \vee r \succ p \vee r$$
- ☞ A direct, *accuracy-dominance* argument for $(R_0) + (A_3)$ can be given (Joyce-style) [8]. Here's a stronger req., due to de Finetti:
(A₄) If $\langle p, q \rangle$ and $\langle p, r \rangle$ are mutually exclusive, then:
$$q \succeq r \iff p \vee q \succeq p \vee r$$

- (R_0) holds iff \succeq is representable by some *capacity*, i.e., by some function $f : \mathcal{B} \mapsto \mathbb{R}^+$ s.t. (a) $f(\top) > f(\perp)$, and (b) $f(p)$ is *monotonically decreasing wrt logical strength of p*.
- $(R_0) + (A_3)$ iff \succeq is representable by a *belief function* [1].
- $(R_0) + (A_4)$ iff \succeq is a *qualitative probability* relation [10].
- $(R_0) + (A_4)$ does *not* imply that \succeq is representable by a probability function [14]. But, Bayesians typically assume this stronger condition [16] is a rational requirement on \succeq .
- Bayesians offer *arguments* for the Pr-representability of \succeq . There are *direct* arguments [12, 6], and *indirect* arguments, *via* arguments for (numerical) probabilism for *credences* [13].
- When it comes to (full) *belief*, Bayesians typically adhere to some version of the *Lockean thesis*. Lockean theses can be stated either in terms of an agent's confidence ordering (\succeq), or in terms of their degrees of belief/credences (b).

- $\mathbf{B} \stackrel{\text{def}}{=} \text{the set of propositions (in } \mathcal{B} \text{) that the agent believes.}$
- Hawthorne [11] endorses this \succeq -based Lockean thesis:

$$p \in \mathbf{B} \iff p \succeq \top$$

- If we assume that \succeq is Pr-representable (as Hawthorne does), then this implies that \mathbf{B} is *deductively cogent*. But, it also requires that agents be *certain* of things they believe.
- Most Bayesians opt for a weaker Lockean thesis [9], which is expressed in terms of the agent's credence function.

$$p \in \mathbf{B} \iff b(p) \geq t$$

- Typically, Bayesians assume a (non-extremal) Lockean threshold $t \in (1/2, 1)$, which may depend on context, *etc.*
- A Bayesian can use *epistemic utility theory* to *argue for* the Lockean thesis on the grounds that Lockean agents have \mathbf{B} 's which *maximize (by the lights of b) expected accuracy* [3, 4].

- Let $\mathbb{R}_\infty^+ \stackrel{\text{def}}{=} \mathbb{R}^+ \cup \{+\infty\}$ be the set containing the non-negative reals, plus $+\infty$ (which will denote “having infinite rank”).
- A *ranking function* (on a finite Boolean algebra \mathbf{B}) is a function $\kappa : \mathbf{B} \mapsto \mathbb{R}_\infty^+$ which satisfies these 3 axioms [17].
 - (1) $\kappa(\top) = 0.$
 - (2) $\kappa(\perp) = +\infty.$
 - (3) $\kappa(p \vee q) = \min(\kappa(p), \kappa(q)).$
- Intuitively, κ can be thought of as a measure of “degree of disbelief,” where tautologies have the lowest (zero) degree, and contradictions have the highest (infinite) degree.
- For each ranking function κ , we can define a corresponding *positive ranking function*: $\beta(p) \stackrel{\text{def}}{=} \kappa(\neg p)$, which can then be interpreted as “degree of belief.” The axioms for β are:
 - (1) $\beta(\top) = +\infty.$
 - (2) $\beta(\perp) = 0.$
 - (3) $\beta(p \wedge q) = \min(\beta(p), \beta(q)).$

- Here are a few easy consequences of Spohn's definition of β .
 - (I) $\beta(\top) > \beta(\perp).$
 - (II) If p entails q , then $\beta(q) \geq \beta(p).$
 - (III) For all p , *either* $\beta(p) = 0$ *or* $\beta(\neg p) = 0$ *or both*.
- (I) and (II) imply that β is a *capacity*.
- Thus, *if* one's confidence ranking \succeq is representable by some β , *then* \succeq satisfies our Fundamental Requirement (R_0).
- However, (R_0) is *not sufficient* to ensure β -representability. Indeed, Pr-representability isn't even sufficient for this!
- We can exploit (III) to show this. Consider the following \succeq -relation on $\mathcal{B} = \{P, \neg P, \top, \perp\}$, where P is contingent.

$$\top \succ P \sim \neg P \succ \perp$$

- \succeq is representable by a *unique* Pr-function (think: *fair coin*). If \succeq were representable by some β , then — by (III) — we would have to have *either* $P \sim \perp$ *or* $\neg P \sim \perp$ *or both*.

- Because the “fair coin ordering” seems permissible (as an ordering of comparative confidence), ranking functions (β 's) seem ill-suited to furnishing rational requirements for \succeq 's.
- Moreover, some β 's induce \succeq orderings that *violate* (A_4). [Although, β -representability of \succeq *does* entail (A_3).] See Extras.
- It would be nice to have a set of qualitative \succeq -constraints which *characterizes representability by a β -function* (akin to what Dana Scott [16] did for Pr-representability of \succeq).
- If β -functions are not suitable for representing/rationally determining confidence rankings \succeq , then what are they useful for (*qua* epistemic representational tools)?
- It is not for nothing that Spohn's book is called *The Laws of Belief*. Epistemologically, the main application of β -functions is to the representation of (full) *belief*.
- Like a Lockean, a β -theorist may (and Spohn, in fact, *does*) represent belief states *via a β -threshold constraint*.

- The natural way to use a β to represent an agent's *belief set* \mathbf{B} (i.e., the set of propositions in \mathcal{B} that the agent *believes*) is:

$$p \in \mathbf{B} \iff \beta(p) > 0$$

- Here, we adopt the interpretation that $\beta(p) = 0$ corresponds to *non-belief* in p . On this reading, (III) makes good sense, since then (III) is a plausible rational constraint on belief. [Even *Lockeans* will accept *this* reading of (III)!]
- Indeed, (I)–(III) will *all* be plausible requirements, when taken as constraints on *belief* — understood in this way.
- However, this approach has *other* consequences for rational requirements on belief that many will find *too demanding*.
- Most notably, this approach will imply that *all belief sets* \mathbf{B} *should be deductively cogent* (i.e., \mathbf{B} should be *consistent and closed under conjunction*). Closure under conjunction follows from (3). Then, consistency follows from (II) and (2).

- It's clear that ranking function theory is a very fruitful and elegant theory, with lots of interesting applications.
- But, why not just use Bayesian theory instead? After all,
 - Bayesian theory gives us a unified account of the representation of *all types* of doxastic states.
 - Bayesian theory furnishes *arguments* in favor of the rational requirements it endorses — for *all types* of states [7].
 - Even if one wants *deductive cogency* as a requirement for belief sets, one can have this as well [11, 15]. Although, Bayesian theory also gives *reasons to doubt* that cogency is a general rational requirement for belief [9, 2, 5].
 - Some \geq 's that *are* Pr-representable *cannot be represented by any* β . Moreover, some β 's induce orderings \geq that *violate* (A_4) — see Extras. So, it seems that β 's are not well suited for representing/constraining rational confidence orderings.
 - But, then, it seems β 's are (from this *epistemic* perspective, at least) *merely* numerical tools for representing *cogent* \mathbf{B} 's.

- This table depicts a Boolean algebra (\mathcal{B}) generated by a language with two atoms (X, Y). The second column depicts a ranking function β over \mathcal{B} . The ordering \geq induced by β *violates* (A_4), where $p \stackrel{\text{def}}{=} X \wedge Y$, $q \stackrel{\text{def}}{=} \neg X$, and $r \stackrel{\text{def}}{=} \neg Y$.

Basic Disjunction (d)	$\beta(d)$
$X \vee Y$	1
$X \vee \neg Y$	1
$\neg X \vee Y$	0
$\neg X \vee \neg Y$	0

- Every contingent $p \in \mathcal{B}$ can be expressed in CNF, as a conjunction of some of the d 's. Thus all values of β are determined by this table, together with the min rule for $\&$.²
 - $\beta(q) = \beta(\neg X) = \min(0, 0) = 0 \geq \beta(r) = \beta(\neg Y) = \min(1, 0)$.
 - $\beta(p \vee q) = \beta(\neg X \vee Y) = 0 < \beta(p \vee r) = \beta(X \vee \neg Y) = 1$.

²Stipulating $\beta(\top) \stackrel{\text{def}}{=} +\infty$ and $\beta(\perp) \stackrel{\text{def}}{=} 0$ completes the definition of β .

- **Theorem.** β -representability of \geq entails (A_3).
- Suppose, for *reductio*, that we have a counterexample to (A_3) which is representable by some β function.
- This counterexample can be represented using three “atoms” (X, Y, Z), which induce the following set of 8 basic disjunctions (d), that are assigned *ranking masses* m_i .

Basic Disjunction (d)	$\beta(d)$
$X \vee Y \vee Z$	m_1
$X \vee Y \vee \neg Z$	m_2
$X \vee \neg Y \vee Z$	m_3
$X \vee \neg Y \vee \neg Z$	m_4
$\neg X \vee Y \vee Z$	m_5
$\neg X \vee Y \vee \neg Z$	m_6
$\neg X \vee \neg Y \vee Z$	m_7
$\neg X \vee \neg Y \vee \neg Z$	m_8

- A counterexample to (A_3) must satisfy the following:

(1) $X \models Y$. In ranking theory, this imposes two conditions.

(1.1) $\beta(x) = 0$ for all $x \in \mathcal{B}$ such that $x \models X \& \neg Y$.

(1.2) $\beta(x) = +\infty$ for all $x \in \mathcal{B}$ such that $\neg X \vee Y \models x$.

Algebraically, (1.1) and (1.2) reduce to the following.

(1.1) $\min(m_1, m_2, m_3, m_4, m_7, m_8) = 0$.

(1.2) $m_5 = +\infty$, and $m_6 = +\infty$.

(2) $Y \models \neg Z$. In ranking theory, this imposes two conditions.

(2.1) $\beta(x) = 0$ for all $x \in \mathcal{B}$ such that $x \models Y \& Z$.

(2.2) $\beta(x) = +\infty$ for all $x \in \mathcal{B}$ such that $\neg Y \vee \neg Z \models x$.

Algebraically, (2.1) and (2.2) reduce to the following.

(2.1) $\min(m_1, m_2, m_3, m_5, m_6, m_7) = 0$.

(2.2) $m_4 = +\infty$, and $m_8 = +\infty$.

- A counterexample to (A_3) must also satisfy:

(3) $\beta(Y) > \beta(X)$. Algebraically, (3) reduces to the following.

(3) $\min(m_1, m_2, m_5, m_6) > \min(m_1, m_2, m_3, m_4)$.

(4) $\beta(Y \vee Z) \leq \beta(X \vee Z)$. Algebraically, (4) reduces to:


(4) $\min(m_1, m_5) \leq \min(m_1, m_3)$.

- (1.2) and (2.2) allow us to reduce (3) and (4) to:

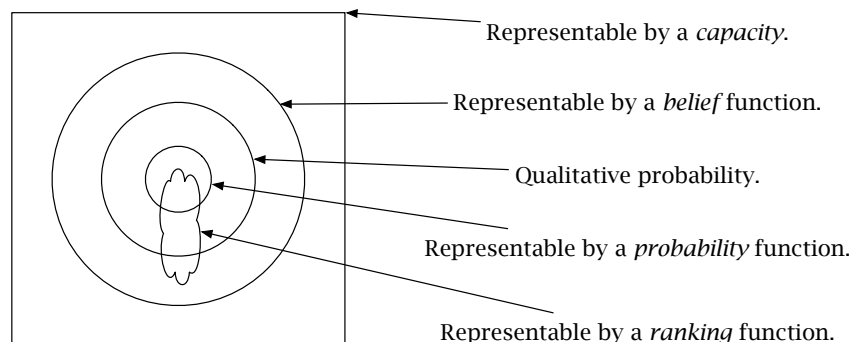
(3) $\min(m_1, m_2, +\infty, +\infty) > \min(m_1, m_2, m_3, +\infty)$.

(4) $\min(m_1, +\infty) \leq \min(m_1, m_3)$.

- Now, (4) implies that $m_1 \leq m_3$. Therefore, (3) implies $\min(m_1, m_2) > \min(m_1, m_2)$. *Contradiction.* \square

 **Theorem.** All β -representable \geq 's satisfy (A_3) . Therefore, β -representability \Rightarrow D-S/belief function-representability.³

³I have implemented a decision procedure for β -theory in *Mathematica*.



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