Two Approaches to Doxastic Representation

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- Bayesian Theory provides a unified and elegant framework for representing the doxastic states of rational agents.
- Agents may have various types of epistemic attitudes — among these are the following three types: (1) *numerical* credences, (2) confidence *orderings*, and (3) (full) *beliefs*.
- I prefer to start with confidence orderings, and think of the other states in terms of their relation(s) with them.
- I will focus on (logically omniscient) agents S who form attitudes over some finite Boolean algebra $\mathcal{B}$ of (classical, possible worlds) propositions. Let's start with orderings.
- Confidence orderings are induced by binary relations $\ulcorner p \succeq q \urcorner$, which may be interpreted as $\ulcorner S$ is at least as confident in p as she is in $q.\urcorner$ I will assume that $\succeq$ induces a total preorder (*i.e.*, a complete, transitive order) over $\mathcal{B}$.
- Various *rational requirements* have been proposed for $\succeq$. See Halpern's *Reasoning About Uncertainty* [10] for a survey.

- The most fundamental requirement on $\succeq$ is (R_0).

  The Fundamental Requirement (R_0). The relation $\succeq$ (a total preorder on $\mathcal{B}$) should satisfy the following two contraints:

  (A_1) $\top \succ \bot$.

  (A_2) For all $p, q \in \mathcal{B}$, if p entails q then $q \succeq p$.
- (R_0) is widely accepted as a rational requirement for $\succeq$.
- Once we move to stronger requirements on $\succeq$, things get more controversial. Here are some additional requirements that Bayesians accept — over and above T.F.R. (R_0).

  (A_3) If p entails q and $\langle q, r\rangle$ are mutually exclusive, then:

$$q \succ p \Longrightarrow q \vee r \succ p \vee r.$$

☞ A direct, *accuracy-dominance* argument for $(R_0) + (A_3)$ can be given (Joyce-style) [8]. Here's a stronger req., due to de Finetti:

(A_4) If $\langle p, q\rangle$ and $\langle p, r\rangle$ are mutually exclusive, then:

$$q \succeq r \Longleftrightarrow p \vee q \succeq p \vee r$$

- (R_0) holds iff $\succeq$ is representable by some *capacity*, *i.e.*, by some function $f : \mathcal{B} \mapsto \mathbb{R}^+$ s.t. (a) $f(\top) > f(\bot)$, and (b) $f(p)$ is *monotonically decreasing wrt logical strength of* p.
- $(R_0) + (A_3)$ iff $\succeq$ is representable by a *belief function* [1].
- $(R_0) + (A_4)$ iff $\succeq$ is a *qualitative probability* relation [10].
- $(R_0) + (A_4)$ does *not* imply that $\succeq$ is representable by a probability function [14]. But, Bayesians typically assume this stronger condition [16] *is* a rational requirement on $\succeq$.
- Bayesians offer *arguments* for the Pr-representability of $\succeq$. There are *direct* arguments [12, 6], and *indirect* arguments, *via* arguments for (numerical) probabilism for *credences* [13].
- When it comes to (full) *belief*, Bayesians typically adhere to some version of the *Lockean thesis*. Lockean theses can be stated either in terms of an agent's confidence ordering ($\succeq$), or in terms of their degrees of belief/credences (b).

- $\mathbf{B} \stackrel{\text{def}}{=}$ the set of propositions (in $\mathcal{B}$) that the agent *believes*.
- Hawthorne [11] endorses this $\succeq$-based Lockean thesis:

$$p \in \mathbf{B} \iff p \succeq \top$$

- If we assume that $\succeq$ is Pr-representable (as Hawthorne does), then this implies that $\mathbf{B}$ is *deductively cogent*. But, it also requires that agents be *certain* of things they believe.
- Most Bayesians opt for a weaker Lockean thesis [9], which is expressed in terms of the agent's credence function.

$$p \in \mathbf{B} \iff b(p) \geq t$$

- Typically, Bayesians assume a (non-extremal) Lockean threshold $t \in (1/2, 1)$, which may depend on context, *etc.*
- A Bayesian can use *epistemic utility theory* to *argue for* the Lockean thesis on the grounds that Lockean agents have $\mathbf{B}$'s which *maximize (by the lights of b) expected accuracy* [3, 4].

- Let $\mathbb{R}^+_\infty \stackrel{\text{def}}{=} \mathbb{R}^+ \cup \{+\infty\}$ be the set containing the non-negative reals, plus $+\infty$ (which will denote "having infinite rank").
- A *ranking function* (on a finite Boolean algebra $\mathbf{B}$) is a function $\kappa : \mathbf{B} \mapsto \mathbb{R}^+_\infty$ which satisfies these 3 axioms [17].
  (1) $\kappa(\top) = 0$.
  (2) $\kappa(\bot) = +\infty$.
  (3) $\kappa(p \vee q) = \min(\kappa(p), \kappa(q))$.
- Intuitively, κ can be thought of as a measure of "degree of *dis*belief," where tautologies have the lowest (zero) degree, and contradictions have the highest (infinite) degree.
- For each ranking funtion κ, we can define a corresponding *positive* ranking function: $\beta(p) \stackrel{\text{def}}{=} \kappa(\neg p)$, which can then be interpreted as "degree of belief." The axioms for β are:
  (1) $\beta(\top) = +\infty$.
  (2) $\beta(\bot) = 0$.
  (3) $\beta(p \wedge q) = \min(\beta(p), \beta(q))$.

- Here are a few easy consequences of Spohn's definition of β.
  (I) $\beta(\top) > \beta(\bot)$.
  (II) If p entails q, then $\beta(q) \geq \beta(p)$.
  (III) For all p, *either* $\beta(p) = 0$ *or* $\beta(\neg p) = 0$ *or both.*
- (I) and (II) imply that β is a *capacity*.
- Thus, *if* one's confidence ranking $\succeq$ is representable by some β, *then* $\succeq$ satisfies our Fundamental Requirement (R_0).
- However, (R_0) is *not sufficient* to ensure β-representability. Indeed, Pr-representability isn't even sufficient for this!
- We can exploit (III) to show this. Consider the following $\succeq$-relation on $\mathcal{B} = \{P, \neg P, \top, \bot\}$, where P is contingent.

$$\top \succ P \sim \neg P \succ \bot$$

- $\succeq$ is representable by a *unique* Pr-function (think: *fair coin*). If $\succeq$ *were* representable by some β, then — by (III) — we would have to have *either* $P \sim \bot$ *or* $\neg P \sim \bot$ *or both*.

- Because the "fair coin ordering" seems permissible (as an ordering of comparative confidence), ranking functions (β's) seem ill-suited to furnishing rational requirements for $\succeq$'s.
- ☞ Moreover, some β's induce $\succeq$ orderings that *violate* (A_4). [Although, β-representability of $\succeq$ *does* entail (A_3).] See Extras.
- It would be nice to have a set of qualitative $\succeq$-constraints which *characterizes representability by a β-function* (akin to what Dana Scott [16] did for Pr-representability of $\succeq$).
- If β-functions are not suitable for representing/rationally determining confidence rankings $\succeq$, then what are they useful for (*qua* epistemic representational tools)?
- It is not for nothing that Spohn's book is called *The Laws of **Belief***. Epistemologically, the main application of β-functions is to the representation of (full) *belief*.
- Like a Lockean, a β-theorist may (and Spohn, in fact, *does*) represent belief states *via a β-threshold constraint*.

- The natural way to use a β to represent an agent's *belief set* **B** (*i.e.*, the set of propositions in $\mathcal{B}$ that the agent *believes*) is:

$$p \in \mathbf{B} \iff \beta(p) > 0$$

- Here, we adopt the interpretation that $\beta(p) = 0$ corresponds to *non*-belief in p. On this reading, (III) makes good sense, since then (III) *is* a plausible rational constraint on belief. [Even *Lockeans* will accept *this* reading of (III)!]
- Indeed, (I)–(III) will *all* be plausible requirements, when taken as constraints on *belief* — understood in this way.
- However, this approach has *other* consequences for rational requirements on belief that many will find *too demanding*.
- Most notably, this approach will imply that *all belief sets* **B** *should be deductively cogent* (*i.e.*, **B** should be *consistent and closed under conjunction*). Closure under conjunction follows from (3). Then, consistency follows from (II) and (2).

- It's clear that ranking function theory is a very fruitful and elegant theory, with lots of interesting applications.
- But, why not just use Bayesian theory instead? After all,
  - Bayesian theory gives us a unified account of the representation of *all types* of doxastic states.
  - Bayesian theory furnishes *arguments* in favor of the rational requirements it endorses — for *all* types of states [7].
  - Even if one wants *deductive cogency* as a requirement for belief sets, one can have this as well [11, 15]. Although, Bayesian theory also gives *reasons to doubt* that cogency *is* a general rational requirement for belief [9, 2, 5].
  - Some $\succeq$'s that *are* Pr-representable *cannot be represented by any* β. Moreover, some β's induce orderings $\succeq$ that *violate* (A_4) — see Extras. So, it seems that β's are not well suited for representing/constraining rational confidence orderings.
  - But, then, it seems β's are (from this *epistemic* perspective, at least) *merely* numerical tools for representing *cogent* **B**'s.

- This table depicts a Boolean algebra ($\mathcal{B}$) generated by a language with two atoms (X, Y). The second column depicts a ranking function β over $\mathcal{B}$. The ordering $\succeq$ induced by β *violates* (A_4), where $p \stackrel{\text{def}}{=} X \wedge Y$, $q \stackrel{\text{def}}{=} \neg X$, and $r \stackrel{\text{def}}{=} \neg Y$.

Basic Disjunction (d)	$\beta(d)$
$X \vee Y$	1
$X \vee \neg Y$	1
$\neg X \vee Y$	0
$\neg X \vee \neg Y$	0

- Every contingent $p \in \mathcal{B}$ can be expressed in CNF, as a conjunction of some of the d's. Thus all values of β are determined by this table, together with the min rule for &.[2]
  - $\beta(q) = \beta(\neg X) = \min(0, 0) = 0 \geq \beta(r) = \beta(\neg Y) = \min(1, 0)$.
  - $\beta(p \vee q) = \beta(\neg X \vee Y) = 0 < \beta(p \vee r) = \beta(X \vee \neg Y) = 1$.

[2]Stipulating $\beta(\top) \stackrel{\text{def}}{=} +\infty$ and $\beta(\bot) \stackrel{\text{def}}{=} 0$ completes the definition of β.

- **Theorem.** β-representability of $\succeq$ entails (A_3).
- Suppose, for *reductio*, that we have a counterexample to (A_3) which *is* representable by some β function.
- This counterexample can be represented using three "atoms" (X, Y, Z), which induce the following set of 8 basic disjunctions (d), that are assigned *ranking masses* $\mathfrak{m}_i$.

Basic Disjunction (d)	$\beta(d)$
$X \vee Y \vee Z$	$\mathfrak{m}_1$
$X \vee Y \vee \neg Z$	$\mathfrak{m}_2$
$X \vee \neg Y \vee Z$	$\mathfrak{m}_3$
$X \vee \neg Y \vee \neg Z$	$\mathfrak{m}_4$
$\neg X \vee Y \vee Z$	$\mathfrak{m}_5$
$\neg X \vee Y \vee \neg Z$	$\mathfrak{m}_6$
$\neg X \vee \neg Y \vee Z$	$\mathfrak{m}_7$
$\neg X \vee \neg Y \vee \neg Z$	$\mathfrak{m}_8$

- A counterexample to (A_3) must satisfy the following:
  - (1) $X \vDash Y$. In ranking theory, this imposes two conditions.
    - (1.1) $\beta(x) = 0$ for all $x \in \mathcal{B}$ such that $x \vDash X \,\&\, \neg Y$.
    - (1.2) $\beta(x) = +\infty$ for all $x \in \mathcal{B}$ such that $\neg X \vee Y \vDash x$.

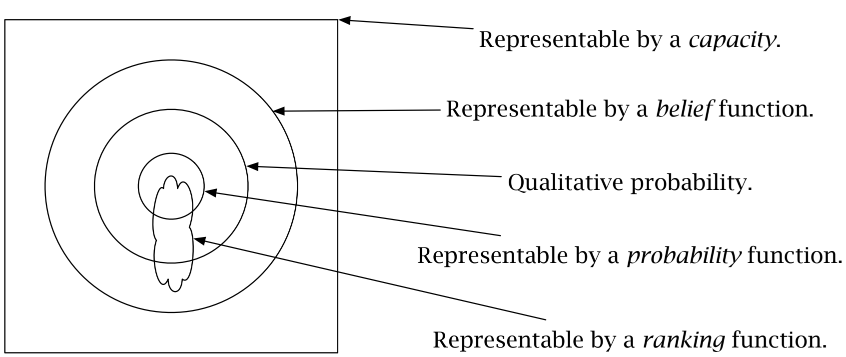
    Algebraically, (1.1) and (1.2) reduce to the following.
    - (1.1) $\min(\mathfrak{m}_1, \mathfrak{m}_2, \mathfrak{m}_3, \mathfrak{m}_4, \mathfrak{m}_7, \mathfrak{m}_8) = 0$.
    - (1.2) $\mathfrak{m}_5 = +\infty$, and $\mathfrak{m}_6 = +\infty$.
  - (2) $Y \vDash \neg Z$. In ranking theory, this imposes two conditions.
    - (2.1) $\beta(x) = 0$ for all $x \in \mathcal{B}$ such that $x \vDash Y \,\&\, Z$.
    - (2.2) $\beta(x) = +\infty$ for all $x \in \mathcal{B}$ such that $\neg Y \vee \neg Z \vDash x$.

    Algebraically, (2.1) and (2.2) reduce to the following.
    - (2.1) $\min(\mathfrak{m}_1, \mathfrak{m}_2, \mathfrak{m}_3, \mathfrak{m}_5, \mathfrak{m}_6, \mathfrak{m}_7) = 0$.
    - (2.2) $\mathfrak{m}_4 = +\infty$, and $\mathfrak{m}_8 = +\infty$.

- A counterexample to (A_3) must also satisfy:
  - (3) $\beta(Y) > \beta(X)$. Algebraically, (3) reduces to the following.
    - (3) $\min(\mathfrak{m}_1, \mathfrak{m}_2, \mathfrak{m}_5, \mathfrak{m}_6) > \min(\mathfrak{m}_1, \mathfrak{m}_2, \mathfrak{m}_3, \mathfrak{m}_4)$.
  - (4) $\beta(Y \vee Z) \leq \beta(X \vee Z)$. Algebraically, (4) reduces to:
    - (4) $\min(\mathfrak{m}_1, \mathfrak{m}_5) \leq \min(\mathfrak{m}_1, \mathfrak{m}_3)$.
- (1.2) and (2.2) allow us to reduce (3) and (4) to:
  - (3) $\min(\mathfrak{m}_1, \mathfrak{m}_2, +\infty, +\infty) > \min(\mathfrak{m}_1, \mathfrak{m}_2, \mathfrak{m}_3, +\infty)$.
  - (4) $\min(\mathfrak{m}_1, +\infty) \leq \min(\mathfrak{m}_1, \mathfrak{m}_3)$.
- Now, (4) implies that $\mathfrak{m}_1 \leq \mathfrak{m}_3$. Therefore, (3) implies $\min(\mathfrak{m}_1, \mathfrak{m}_2) > \min(\mathfrak{m}_1, \mathfrak{m}_2)$. *Contradiction.* □

☞ **Theorem.** *All β-representable $\succeq$'s satisfy (A_3). Therefore, β-representability $\Longrightarrow$ D–S/belief function-representability.*[3]

[3] I have implemented a decision procedure for β-theory in *Mathematica*.



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