SOLUTIONS TO SOME OPEN PROBLEMS FROM SLANEY

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ABSTRACT. In response to a paper by Harris & Fitelson [1], Slaney [6] states several open questions concerning possible strategies for proving distributivity in a wide class of positive sentential logics. In this note, I provide answers to all of Slaney's open questions. The result is a better understanding of the class of positive logics in which distributivity holds.

1. INTRODUCTION

Harris & Fitelson [1] used Otter to prove distributivity in $Ł_{\aleph_0}$ and other non-classical sentential logics. Their proofs involved axiomatizations in terms of implication ($\to$) and negation ($\neg$). Slaney [6] showed how to prove these results in the positive fragments of these logics, which involve only implication ($\to$), conjunction ($\wedge$), and disjunction ($\vee$). Slaney also provided a much more general framework for thinking about distributivity in a wide class of positive logics. This led him to state several open questions regarding strategies for establishing distributivity in this broad class of non-classical (positive) logics. In this note, I will provide answers to all of Slaney's open questions. All of these results were obtained using (various) automated reasoning tools.[1]

2. SLANEY'S THREE (BACKGROUND) POSITIVE LOGICS

Slaney [6] presents a large class of (positive) logics, which involve various combinations of the following axioms and rules (*i.e.*, axiom and rule *schemata*).[2]

(AxK)	$\vdash A \to (B \to A)$
(AxB)	$\vdash (B \to C) \to ((A \to B) \to (A \to C))$
(AxŁ)	$\vdash (A \to (B \to B)) \to (B \to (A \to A))$
(AxTO)	$\vdash ((A \to B) \to (B \to A)) \to (B \to A)$
(AxC)	$\vdash (A \to (B \to C)) \to (B \to (A \to C))$
(AxI)	$\vdash A \to A$
(AxB′)	$\vdash (A \to B) \to ((B \to C) \to (A \to C))$

[1] I used a combination of `prover9` [3], `Otter` [4], `E` [5], and `Vampire` [2] to solve Slaney's open problems. All proofs are presented in the APPENDIX, in `Otter` format.

[2] Here, I follow Slaney's [6] notation and nomenclature, which differs slightly from that of Harris & Fitelson [1].

(Det)	From $\vdash A \to B$ and $\vdash A$, infer $\vdash B$
(Ax∧E1)	$\vdash (A \wedge B) \to A$
(Ax∧E2)	$\vdash (A \wedge B) \to B$
(Ax∧I)	$\vdash ((A \to B) \wedge (A \to C)) \to (A \to (B \wedge C))$
(Ax∨I1)	$\vdash A \to (A \vee B)$
(Ax∨I2)	$\vdash B \to (A \vee B)$
(Ax∨E)	$\vdash ((A \to C) \wedge (B \to C)) \to ((A \vee B) \to C)$
(Adj)	From $\vdash A$ and $\vdash B$, infer $\vdash A \wedge B$

Specifically, Slaney's open questions involve the following three (background) positive logics.

(1) TW^+[AxŁ, AxTO], the pure implicational fragment of which ($TW^\to$) is given by the axioms AxB, AxI, AxB′, AxŁ, and AxTO, and the rule Det. The full logic TW^+[AxŁ, AxTO] is then obtained by adding all of the axioms and rules for conjunction and disjunction to this implicational base. In other words, TW^+[AxŁ, AxTO] is given by: AxB, AxI, AxB′, AxŁ, AxTO, Det, Ax∧E1, Ax∧E2, Ax∧I, Ax∨I1, Ax∨I2, Ax∨E, and Adj.

(2) $BCK^\to$[AxŁ], which consists of the axioms AxK, AxB, AxC, and AxŁ, and the rule Det.

(3) $TW^\to$[AxŁ], which consists of the axioms AxB, AxI, AxB′, and AxŁ, and the rule Det.

3. FOUR OTHER PRINCIPLES IMPLICATED IN SLANEY'S OPEN QUESTIONS

In addition to these three background positive logics, Slaney's open questions also involve the following four additional axioms/theorems and rules:

(Dist)	$\vdash (A \wedge (B \vee C)) \to ((A \wedge B) \vee (A \wedge C))$
(RTO)	$\vdash (A \to B) \vee (B \to A)$
(IO)	$\vdash ((A \to B) \to B)) \to (A \vee B)$
(Resid)	$\vdash (A \circ B) \to C \iff \vdash A \to (B \to C)$[3]

4. SLANEY'S (SIX) OPEN QUESTIONS AND THEIR SOLUTIONS

Slaney's first four open questions involve the background positive logic TW^+[AxŁ, AxTO]. The first two of these open questions are as follows.

(1) Is (Dist) provable in TW^+[AxŁ, AxTO]?
(2) Is (RTO) provable in TW^+[AxŁ, AxTO]?

[3] The meaning of "$p \iff q$" is "From p, infer q; and, from q, infer p." Thus, (Resid) adds a new "fusion" connective '$\circ$,' which obeys the two-way rule of inference in question.

Slaney [6, *p*. 65] notes that affirmative answers to both questions (1) and (2) are forthcoming, *if* it is possible to prove (IO) in TW⁺[AxŁ, AxTO]. Our first theorem therefore implies affirmative answers to both (1) and (2).[4]

Theorem 1. *(IO) is provable in TW⁺[AxŁ, AxTO].*

Slaney's next two open questions regarding TW⁺[AxŁ, AxTO] involve the addition of a fusion operator '$\circ$' to TW⁺[AxŁ, AxTO], *via* the (Resid) rule.

3. Is the addition of fusion a conservative extension of the positive logic TW⁺[AxŁ, AxTO]? That is, does the addition of (Resid) to TW⁺[AxŁ, AxTO] imply *no new theorems* involving *only* $\langle \to, \wedge, \vee \rangle$?

4. If the answer to (3) is *negative* (*i.e.*, if new $\langle \to, \wedge, \vee \rangle$-theorems *are* derivable upon adding (Resid) to TW⁺[AxŁ, AxTO]), then does the addition of (Resid) to TW⁺[AxŁ, AxTO] allow us to prove *both* (AxK) *and* (AxC)?

Our second theorem implies both a negative answer to (3) and a positive answer to (4).

Theorem 2. *(AxK) and (AxC) are provable in TW⁺[AxŁ, AxTO] + (Resid).*

Slaney's fifth open question involves the background positive logic BCK⁻[AxŁ].

5. Is the addition of fusion, with its two-way rule (Resid), enough to generate a(nother) negation-free proof of (AxTO) from BCK⁻[AxŁ]? In other words, is (AxTO) provable in BCK⁻[AxŁ] + (Resid)?

Our third theorem implies an affirmative answer to (5).

Theorem 3. *(AxTO) is provable in BCK⁻[AxŁ] + (Resid).*

That brings us to Slaney's sixth (and final) open question (implicitly asked on page 66), which involves his third background positive logic TW⁻[AxŁ].

6. Is (AxK) provable in TW⁻[AxŁ] + (Resid)?

Our fourth (and final) theorem implies an affirmative answer to (6).

Theorem 4. *(AxK) is provable in TW⁻[AxŁ] + (Resid).*

Appendix: Proofs of Theorems

In this Appendix, I provide Otter proofs of our four theorems. Instead of using infix notation involving $\langle \to, \wedge, \vee, \circ \rangle$, I will use prefix notation involving ⟨i, and, or, f⟩. That is to say, we will adopt the following Otter notation:

$\vdash A \to B \mapsto$ p(i(A,B))

$A \wedge B \mapsto$ and(A,B)

[4]See the Appendix for Otter proofs of all theorems reported in this paper.

$A \vee B \mapsto$ or(A,B)

$A \circ B \mapsto$ f(A,B)

See Harris & Fitelson [1] for further explanation of how our Otter proof objects are to be interpreted (and related to more traditional presentations of proofs in sentential logics). The proofs presented here are the shortest/simplest proofs I was able to find using Otter.

Otter Proof of Theorem 1.[5]

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Length of proof is 36.  Level of proof is 14.

---------------- PROOF ----------------

38 [] -p(i(A,B))| -p(A)|p(B)   # label(Det).
40 [] p(i(i(A,B),i(i(B,C),i(A,C))))   # label(AxBp).
41 [] p(i(i(i(X,Y),Y),i(i(Y,X),X)))   # label(AxL).
42 [] p(i(i(i(X,Y),i(Y,X)),i(Y,X)))   # label(AxTO).
43 [] p(i(X,or(X,Y)))   # label(AxorI1).
44 [] p(i(Y,or(X,Y)))   # label(AxorI2).
51 [hyper,38,40,40] p(i(i(i(i(A,B),i(C,B)),D),i(i(C,A),D))).
52 [hyper,38,40,41] p(i(i(i(i(A,B),B),C),i(i(i(B,A),A),C))).
53 [hyper,38,40,42] p(i(i(i(A,B),C),i(i(i(B,A),i(A,B)),C))).
54 [hyper,38,40,43] p(i(i(or(A,B),C),i(A,C))).
55 [hyper,38,40,44] p(i(i(or(A,B),C),i(B,C))).
56 [hyper,38,51,51] p(i(i(A,i(B,C)),i(i(D,B),i(A,i(D,C))))).
57 [hyper,38,51,42] p(i(i(A,A),i(A,A))).
58 [hyper,38,51,52] p(i(i(A,i(B,C)),i(i(i(C,B),B),i(A,C)))).
59 [hyper,38,52,53] p(i(i(i(A,B),B),i(i(i(A,B),i(B,A)),A))).
60 [hyper,38,40,54] p(i(i(i(A,B),C),i(i(or(A,D),B),C))).
61 [hyper,38,51,56] p(i(i(A,B),i(i(C,A),i(i(B,D),i(C,D))))).
62 [hyper,38,56,52] p(i(i(A,i(i(B,C),C)),i(i(i(i(C,B),B),D),i(A,D)))).
63 [hyper,38,42,57] p(i(A,A)).
64 [hyper,38,51,58] p(i(i(A,B),i(i(i(C,A),A),i(i(B,C),C)))).
65 [hyper,38,58,56] p(i(i(i(i(A,i(B,C)),i(B,D)),i(B,D)),i(i(A,i(D,C)),i(A,i(B,C))))).
66 [hyper,38,56,59] p(i(i(A,i(i(B,C),i(C,B))),i(i(i(B,C),C),i(A,B)))).
67 [hyper,38,62,60] p(i(i(i(i(A,or(B,C)),or(B,C)),D),i(i(i(B,A),A),D))).
68 [hyper,38,62,56] p(i(i(i(i(i(A,i(B,C)),B),B),D),i(i(A,i(i(A,i(B,C)),C)),D))).
69 [hyper,38,56,64] p(i(i(A,i(i(B,C),C)),i(i(C,D),i(A,i(i(D,B),B))))).
70 [hyper,38,64,55] p(i(i(i(A,i(or(B,C),D)),i(or(B,C),D)),i(i(i(C,D),A),A))).
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[5]In fact, this Otter proof establishes something *stronger* than Theorem 1. It shows that (IO) is derivable from {Det, AxB′, AXŁ, AxTO, Ax∨I1, Ax∨I2}. An Otter input file which verifies this proof is available from http://fitelson.org/slaney_theorem_1.in.

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71 [hyper,38,56,66] p(i(i(A,i(i(B,C),C)),i(i(D,i(i(B,C),i(C,B))),i(A,i(D,B))))).
72 [hyper,38,68,63] p(i(i(A,i(i(A,i(B,C)),C)),i(i(i(A,i(B,C)),B),B))).
73 [hyper,38,51,71] p(i(i(i(A,B),C),i(i(D,i(i(A,B),i(B,A))),i(i(C,B),i(D,A))))).
74 [hyper,38,42,72] p(i(i(i(A,i(A,A)),A),A)).
75 [hyper,38,69,73] p(i(i(i(i(A,B),i(B,A)),C),i(i(i(A,B),A),i(i(C,B),B)))).
76 [hyper,38,61,74] p(i(i(A,i(i(B,i(B,B)),B)),i(i(B,C),i(A,C)))).
77 [hyper,38,75,42] p(i(i(i(A,B),A),i(i(i(B,A),B),B))).
78 [hyper,38,40,76] p(i(i(i(i(A,B),i(C,B)),D),i(i(C,i(i(A,i(A,A)),A)),D))).
79 [hyper,38,51,77] p(i(i(A,i(A,B)),i(i(i(B,i(A,B)),B),B))).
80 [hyper,38,78,42] p(i(i(A,i(i(A,i(A,A)),A)),i(A,A))).
81 [hyper,38,58,79] p(i(i(i(A,i(i(A,i(B,A)),A)),i(i(A,i(B,A)),A)),i(i(B,i(B,A)),A))).
82 [hyper,38,65,81] p(i(i(A,i(A,A)),i(A,i(i(A,i(A,A)),A)))).
83 [hyper,38,40,82] p(i(i(i(A,i(i(A,i(A,A)),A)),B),i(i(A,i(A,A)),B))).
84 [hyper,38,83,80] p(i(i(A,i(A,A)),i(A,A))).
85 [hyper,38,70,84] p(i(i(i(A,or(B,A)),or(B,A)),or(B,A))).
86 [hyper,38,67,85] p(i(i(i(A,B),B),or(A,B))).

------------ end of proof -------------
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Otter Proof of Theorem 2.[6]

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Length of proof is 74.  Level of proof is 24.

---------------- PROOF ----------------

75 [] -p(i(A,B))| -p(A)|p(B)   # label(Det).
76 [] -p(i(f(A,B),C))|p(i(A,i(B,C)))   # label(Resid1).
77 [] p(i(f(A,B),C))| -p(i(A,i(B,C)))   # label(Resid2).
79 [] p(i(i(A,B),i(i(B,C),i(A,C))))   # label(AxBp).
80 [] p(i(i(i(X,Y),Y),i(i(Y,X),X)))   # label(AxL).
81 [] p(i(i(i(X,Y),i(Y,X)),i(Y,X)))   # label(AxTO).
88 [hyper,77,79] p(i(f(i(A,B),i(B,C)),i(A,C))).
89 [hyper,75,79,79] p(i(i(i(i(A,B),i(C,B)),D),i(i(C,A),D))).
90 [hyper,75,79,80] p(i(i(i(i(A,B),B),C),i(i(i(B,A),A),C))).
91 [hyper,75,79,81] p(i(i(i(A,B),C),i(i(i(B,A),i(A,B)),C))).
92 [hyper,75,79,88] p(i(i(i(A,B),C),i(f(i(A,D),i(D,B)),C))).
93 [hyper,75,89,89] p(i(i(A,i(B,C)),i(i(D,B),i(A,i(D,C))))).
94 [hyper,75,89,81] p(i(i(A,A),i(A,A))).
95 [hyper,75,89,90] p(i(i(A,i(B,C)),i(i(i(C,B),B),i(A,C)))).
96 [hyper,75,79,90] p(i(i(i(i(i(A,B),B),C),D),i(i(i(i(B,A),A),C),D))).
97 [hyper,75,90,89] p(i(i(i(i(A,B),i(C,B)),i(C,B)),i(i(A,C),i(A,B)))).
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[6]In fact, this Otter proof establishes something *stronger* than Theorem 2. It shows that (AxC) and (AxK) are both derivable from {Det, Resid, AxB′, AxŁ, AxTO}. An Otter input file which verifies this proof is available from http://fitelson.org/slaney_theorem_2.in.


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98 [hyper,75,89,91] p(i(i(A,B),i(i(i(C,B),i(B,C)),i(A,C)))).
99 [hyper,75,92,81] p(i(f(i(i(A,B),C),i(C,i(B,A))),i(B,A))).
100 [hyper,75,92,80] p(i(f(i(i(A,B),C),i(C,B)),i(i(B,A),A))).
101 [hyper,75,93,91] p(i(i(A,i(i(B,C),i(C,B))),i(i(i(C,B),D),i(A,D)))).
102 [hyper,75,93,90] p(i(i(A,i(i(B,C),C)),i(i(i(i(C,B),B),D),i(A,D)))).
103 [hyper,75,81,94] p(i(A,A)).
104 [hyper,75,93,95] p(i(i(A,i(i(B,C),C)),i(i(D,i(C,B)),i(A,i(D,B))))).
105 [hyper,75,79,95] p(i(i(i(i(i(A,B),B),i(C,A)),D),i(i(C,i(B,A)),D))).
106 [hyper,75,95,93] p(i(i(i(i(A,i(B,C)),i(B,D)),i(B,D)),i(i(A,i(D,C)),i(A,i(B,C))))).
107 [hyper,75,93,98] p(i(i(A,i(i(B,C),i(C,B))),i(i(D,C),i(A,i(D,B))))).
108 [hyper,75,79,98] p(i(i(i(i(i(A,B),i(B,A)),i(C,A)),D),i(i(C,B),D))).
109 [hyper,76,99] p(i(i(i(A,B),C),i(i(C,i(B,A)),i(B,A)))).
110 [hyper,76,100] p(i(i(i(A,B),C),i(i(C,B),i(i(B,A),A)))).
111 [hyper,76,103] p(i(A,i(B,f(A,B)))).
112 [hyper,75,91,104] p(i(i(i(i(i(A,B),B),C),i(C,i(i(A,B),B))),i(i(D,i(B,A)),i(C,i(D,A))))).
113 [hyper,75,105,106] p(i(i(A,i(i(A,B),B)),i(i(B,i(B,B)),i(B,i(A,B))))).
114 [hyper,75,102,109] p(i(i(i(i(i(A,B),C),C),D),i(i(i(B,A),C),D))).
115 [hyper,75,90,109] p(i(i(i(A,B),B),i(i(A,i(A,B)),i(A,B)))).
116 [hyper,75,89,109] p(i(i(A,B),i(i(i(A,C),i(C,B)),i(C,B)))).
117 [hyper,75,110,79] p(i(i(i(i(A,B),i(C,B)),A),i(i(A,C),C))).
118 [hyper,75,79,111] p(i(i(i(A,f(B,A)),C),i(B,C))).
119 [hyper,75,114,108] p(i(i(i(i(A,B),i(B,A)),i(C,B)),i(i(C,A),i(C,B)))).
120 [hyper,75,114,97] p(i(i(i(A,B),i(C,A)),i(i(B,C),i(B,A)))).
121 [hyper,75,102,115] p(i(i(i(i(i(A,B),A),A),C),i(i(i(A,B),B),C))).
122 [hyper,75,102,116] p(i(i(i(i(i(A,B),i(C,A)),i(C,A)),D),i(i(C,B),D))).
123 [hyper,75,118,79] p(i(A,i(i(f(A,B),C),i(B,C)))).
124 [hyper,75,108,119] p(i(i(A,B),i(i(A,B),i(A,B)))).
125 [hyper,75,89,120] p(i(i(A,B),i(i(B,A),i(B,B)))).
126 [hyper,75,120,109] p(i(i(A,i(A,i(B,B))),i(A,i(B,B)))).
127 [hyper,75,121,117] p(i(i(i(i(A,B),B),B),i(i(i(A,B),A),A))).
128 [hyper,75,93,124] p(i(i(A,i(B,C)),i(i(B,C),i(A,i(B,C))))).
129 [hyper,75,90,125] p(i(i(i(A,B),B),i(i(A,i(B,A)),i(A,A)))).
130 [hyper,75,122,127] p(i(i(A,i(A,B)),i(i(i(B,i(A,B)),B),B))).
131 [hyper,75,105,127] p(i(i(A,i(i(A,B),B)),i(i(i(B,i(A,B)),B),B))).
132 [hyper,75,96,127] p(i(i(i(i(A,B),B),A),i(i(i(B,A),B),B))).
133 [hyper,75,89,128] p(i(i(A,B),i(i(A,C),i(i(B,C),i(A,C))))).
134 [hyper,75,129,126] p(i(i(A,i(i(A,i(B,B)),A)),i(A,A))).
135 [hyper,75,131,130] p(i(i(i(A,i(i(A,i(A,A)),A)),A),A)).
136 [hyper,75,93,132] p(i(i(A,i(i(B,C),B)),i(i(i(i(C,B),B),C),i(A,B)))).
137 [hyper,75,118,133] p(i(A,i(i(B,C),i(i(f(A,B),C),i(B,C))))).
138 [hyper,75,123,134] p(i(i(f(i(i(A,i(i(A,i(B,B)),A)),i(A,A)),C),D),i(C,D))).
139 [hyper,75,92,135] p(i(f(i(i(A,i(i(A,i(A,A)),A)),B),i(B,A)),A)).
140 [hyper,75,136,88] p(i(i(i(i(A,B),B),A),i(f(i(i(B,A),C),i(C,B)),B))).
141 [hyper,75,79,137] p(i(i(i(i(A,B),i(i(f(C,A),B),i(A,B))),D),i(C,D))).
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142 [hyper,75,138,139] p(i(i(i(A,A),A),A)).
143 [hyper,75,140,142] p(i(f(i(i(A,A),B),i(B,A)),A)).
144 [hyper,76,143] p(i(i(i(A,A),B),i(i(B,A),A))).
145 [hyper,75,93,144] p(i(i(A,i(B,C)),i(i(i(C,C),B),i(A,C)))).
146 [hyper,75,89,144] p(i(i(A,B),i(i(i(A,B),B),B))).
147 [hyper,75,93,145] p(i(i(A,i(i(B,B),C)),i(i(D,i(C,B)),i(A,i(D,B))))).
148 [hyper,75,145,146] p(i(i(i(A,A),i(i(B,A),A)),i(i(B,A),A))).
149 [hyper,75,97,148] p(i(i(A,i(B,A)),i(A,A))).
150 [hyper,75,141,149] p(i(A,i(i(B,C),i(B,C)))).
152 [hyper,75,107,150] p(i(i(A,B),i(C,i(A,B)))).
153 [hyper,75,101,150] p(i(i(i(A,A),B),i(C,B))).
155 [hyper,75,147,152] p(i(i(A,i(i(B,C),D)),i(i(B,C),i(A,D)))).
157 [hyper,75,79,152] p(i(i(i(A,i(B,C)),D),i(i(B,C),D))).
160 [hyper,75,153,153] p(i(A,i(B,i(C,C)))).
163 [hyper,75,155,95] p(i(i(i(A,B),B),i(i(C,i(B,A)),i(C,A)))).
170 [hyper,75,163,160] p(i(i(A,i(i(B,i(C,C)),D)),i(A,D))).
173 [hyper,75,170,113] p(i(i(A,i(i(A,B),B)),i(B,i(A,B)))).
174 [hyper,75,157,173] p(i(i(i(A,B),B),i(B,i(A,B)))).
176 [hyper,75,112,174] p(i(i(A,i(B,C)),i(B,i(A,C)))).
177 [hyper,75,81,174] p(i(A,i(B,A))).

------------ end of proof -------------
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`Otter` Proof of Theorem 3.[7]

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Length of proof is 28.  Level of proof is 13.

---------------- PROOF ----------------

29 [] -p(i(A,B))| -p(A)|p(B)   # label(Det).
30 [] -p(i(f(A,B),C))|p(i(A,i(B,C)))   # label(Resid1).
31 [] p(i(f(A,B),C))| -p(i(A,i(B,C)))   # label(Resid2).
33 [] p(i(i(A,B),i(i(B,C),i(A,C))))   # label(AxBp).
34 [] p(i(i(i(A,B),B),i(i(B,A),A)))   # label(AxL).
35 [] p(i(i(A,i(B,C)),i(B,i(A,C))))   # label(AxC).
36 [hyper,29,33,33] p(i(i(i(i(A,B),i(C,B)),D),i(i(C,A),D))).
37 [hyper,29,35,35] p(i(A,i(i(B,i(A,C)),i(B,C)))).
38 [hyper,29,33,35] p(i(i(i(A,i(B,C)),D),i(i(B,i(A,C)),D))).
39 [hyper,29,35,34] p(i(i(A,B),i(i(i(B,A),A),B))).
40 [hyper,29,35,33] p(i(i(A,B),i(i(C,A),i(C,B)))).
41 [hyper,29,36,36] p(i(i(A,i(B,C)),i(i(D,B),i(A,i(D,C))))).
42 [hyper,29,33,37] p(i(i(i(i(A,i(B,C)),i(A,C)),D),i(B,D))).
```

[7]In fact, this `Otter` proof establishes something *stronger* than Theorem 3. It shows that (AxTO) is derivable from {Det, Resid, AxB′, AxC, AxŁ}. An `Otter` input file which verifies this proof is available from `http://fitelson.org/slaney_theorem_3.in`.


```
43 [hyper,29,38,35] p(i(i(A,i(B,C)),i(A,i(B,C)))).
44 [hyper,29,38,33] p(i(i(A,i(B,C)),i(i(i(A,C),D),i(B,D)))).
45 [hyper,29,41,39] p(i(i(A,i(i(B,C),C)),i(i(C,B),i(A,B)))).
47 [hyper,29,42,35] p(i(A,i(B,i(i(B,i(A,C)),C)))).
48 [hyper,29,35,43] p(i(A,i(i(A,i(B,C)),i(B,C)))).
49 [hyper,29,44,43] p(i(i(i(i(A,i(B,C)),i(B,C)),D),i(A,D))).
50 [hyper,31,47] p(i(f(A,B),i(i(B,i(A,C)),C))).
51 [hyper,29,45,48] p(i(i(i(A,B),C),i(C,C))).
52 [hyper,29,44,50] p(i(i(i(f(A,B),C),D),i(i(B,i(A,C)),D))).
53 [hyper,29,51,49] p(i(i(A,B),i(A,B))).
54 [hyper,29,44,53] p(i(i(i(i(A,B),B),C),i(A,C))).
56 [hyper,29,54,51] p(i(A,i(B,B))).
57 [hyper,29,56,56] p(i(A,A)).
58 [hyper,29,35,56] p(i(A,i(B,A))).
59 [hyper,30,57] p(i(A,i(B,f(A,B)))).
60 [hyper,31,58] p(i(f(A,B),A)).
61 [hyper,29,41,59] p(i(i(A,B),i(C,i(A,f(C,B))))).
62 [hyper,29,40,60] p(i(i(A,f(B,C)),i(A,B))).
63 [hyper,29,61,62] p(i(A,i(i(B,f(C,D)),f(A,i(B,C))))).
64 [hyper,29,45,63] p(i(i(f(A,i(B,A)),B),i(A,B))).
65 [hyper,29,52,64] p(i(i(i(A,B),i(B,A)),i(B,A))).

------------ end of proof -------------
```

`Otter` Proof of Theorem 4.[8]

```
Length of proof is 52.  Level of proof is 26.

---------------- PROOF ----------------

53 [] -p(i(A,B))| -p(A)|p(B)   # label(Det).
54 [] -p(i(f(A,B),C))|p(i(A,i(B,C)))   # label(Resid1).
55 [] p(i(f(A,B),C))| -p(i(A,i(B,C)))   # label(Resid2).
57 [] p(i(A,A))   # label(AxI).
58 [] p(i(i(A,B),i(i(B,C),i(A,C))))   # label(AxBp).
59 [] p(i(i(i(A,B),B),i(i(B,A),A)))   # label(AxL).
66 [hyper,55,57] p(i(f(i(A,B),A),B)).
67 [hyper,54,57] p(i(A,i(B,f(A,B)))).
68 [hyper,55,58] p(i(f(i(A,B),i(B,C)),i(A,C))).
69 [hyper,53,58,58] p(i(i(i(i(A,B),i(C,B)),D),i(i(C,A),D))).
```

[8]In fact, this `Otter` proof establishes something *stronger* than Theorem 4. It shows that (AxK) is derivable from {Det, Resid, AxI, AxB′, AxŁ}. An `Otter` input file which verifies this proof is available from `http://fitelson.org/slaney_theorem_4.in`.

```
70 [hyper,53,58,66] p(i(i(A,B),i(f(i(C,A),C),B))).
71 [hyper,53,58,67] p(i(i(i(A,f(B,A)),C),i(B,C))).
72 [hyper,53,58,68] p(i(i(i(A,B),C),i(f(i(A,D),i(D,B)),C))).
73 [hyper,53,69,69] p(i(i(A,i(B,C)),i(i(D,B),i(A,i(D,C))))).
74 [hyper,53,70,66] p(i(f(i(A,f(i(B,C),B)),A),C)).
75 [hyper,53,70,59] p(i(f(i(A,i(i(B,C),C)),A),i(i(C,B),B))).
76 [hyper,53,69,71] p(i(i(A,B),i(C,i(A,f(C,B))))).
77 [hyper,53,69,73] p(i(i(A,B),i(i(C,A),i(i(B,D),i(C,D))))).
78 [hyper,53,73,59] p(i(i(A,i(B,C)),i(i(i(C,B),B),i(A,C)))).
79 [hyper,54,74] p(i(i(A,f(i(B,C),B)),i(A,C))).
80 [hyper,54,75] p(i(i(A,i(i(B,C),C)),i(A,i(i(C,B),B)))).
81 [hyper,53,73,78] p(i(i(A,i(i(B,C),C)),i(i(D,i(C,B)),i(A,i(D,B))))).
82 [hyper,53,69,78] p(i(i(A,B),i(i(i(C,A),A),i(i(B,C),C)))).
83 [hyper,53,58,78] p(i(i(i(i(i(A,B),B),i(C,A)),D),i(i(C,i(B,A)),D))).
84 [hyper,53,78,77] p(i(i(i(i(i(A,B),i(C,B)),i(C,D)),i(C,D)),i(i(D,A),i(i(A,B),i(C,B))))).
85 [hyper,53,78,73] p(i(i(i(i(A,i(B,C)),i(B,D)),i(B,D)),i(i(A,i(D,C)),i(A,i(B,C))))).
86 [hyper,53,76,79] p(i(A,i(i(B,f(i(C,D),C)),f(A,i(B,D))))).
87 [hyper,53,58,80] p(i(i(i(A,i(i(B,C),C)),D),i(i(A,i(i(C,B),B)),D))).
88 [hyper,53,71,82] p(i(A,i(i(i(B,C),C),i(i(f(A,C),B),B)))).
89 [hyper,53,83,85] p(i(i(A,i(i(A,B),B)),i(i(B,i(B,B)),i(B,i(A,B))))).
90 [hyper,53,81,86] p(i(i(A,i(f(i(i(B,C),C),i(B,C)),B)),i(i(i(B,C),C),i(A,B)))).
91 [hyper,53,73,88] p(i(i(A,i(i(B,C),C)),i(D,i(A,i(i(f(D,C),B),B))))).
92 [hyper,53,87,89] p(i(i(A,i(i(B,A),A)),i(i(B,i(B,B)),i(B,i(A,B))))).
93 [hyper,53,90,72] p(i(i(i(A,A),A),i(i(i(i(A,A),A),A),A))).
94 [hyper,53,87,91] p(i(i(A,i(i(B,C),C)),i(D,i(A,i(i(f(D,B),C),C))))).
95 [hyper,53,89,93] p(i(i(A,i(A,A)),i(A,i(i(i(A,A),A),A)))).
96 [hyper,53,94,93] p(i(A,i(i(i(B,B),B),i(i(f(A,i(i(B,B),B)),B),B)))).
97 [hyper,53,95,58] p(i(i(A,A),i(i(i(i(A,A),i(A,A)),i(A,A)),i(A,A)))).
98 [hyper,55,96] p(i(f(A,i(i(B,B),B)),i(i(f(A,i(i(B,B),B)),B),B))).
99 [hyper,53,97,57] p(i(i(i(i(A,A),i(A,A)),i(A,A)),i(A,A))).
100 [hyper,53,89,98] p(i(i(A,i(A,A)),i(A,i(f(B,i(i(A,A),A)),A)))).
102 [hyper,53,59,99] p(i(i(i(A,A),i(i(A,A),i(A,A))),i(i(A,A),i(A,A)))).
103 [hyper,53,100,58] p(i(i(A,A),i(f(B,i(i(i(A,A),i(A,A)),i(A,A))),i(A,A)))).
104 [hyper,53,103,57] p(i(f(A,i(i(i(B,B),i(B,B)),i(B,B))),i(B,B))).
105 [hyper,54,104] p(i(A,i(i(i(i(B,B),i(B,B)),i(B,B)),i(B,B)))).
107 [hyper,53,58,105] p(i(i(i(i(i(i(A,A),i(A,A)),i(A,A)),i(A,A)),B),i(C,B))).
108 [hyper,53,107,84] p(i(A,i(i(B,B),i(i(B,B),i(B,B))))).
109 [hyper,53,58,108] p(i(i(i(i(A,A),i(i(A,A),i(A,A))),B),i(C,B))).
110 [hyper,53,109,102] p(i(A,i(i(B,B),i(B,B)))).
112 [hyper,53,73,110] p(i(i(A,i(B,B)),i(C,i(A,i(B,B))))).
116 [hyper,53,92,112] p(i(i(A,i(A,A)),i(A,i(i(B,i(C,C)),A)))).
```



```
122 [hyper,53,116,110] p(i(i(A,A),i(i(B,i(C,C)),i(A,A)))).
125 [hyper,53,73,122] p(i(i(A,i(B,i(C,C))),i(i(D,D),i(A,i(D,D))))).
131 [hyper,53,125,110] p(i(i(A,A),i(B,i(A,A)))).
137 [hyper,53,131,57] p(i(A,i(B,B))).
142 [hyper,53,116,137] p(i(A,i(i(B,i(C,C)),A))).
155 [hyper,53,73,142] p(i(i(A,i(B,i(C,C))),i(D,i(A,D)))).
173 [hyper,53,155,137] p(i(A,i(B,A))).

------------ end of proof -------------
```

References

[1] Harris, K. and Fitelson, B.: Distributivity in $Ł_{\aleph_0}$ and Other Sentential Logics, *Journal of Automated Reasoning*, **27**: 141–156 (2001).

[2] Kovacs, L. and Voronkov, A.: First order theorem proving and Vampire, in Sharygina, N., Veith, H. (*eds.*), *Proceedings of the 25th International Conference on Computer-Aided Verification* (CAV), volume 8044 of Lecture Notes in Computer Science, pp. 1âĂŞ35. Springer (2013).

[3] McCune, W.: Prover9 and Mace4, available from `http://www.cs.unm.edu/~mccune/prover9`.

[4] McCune, W.: `Otter 3.3` Reference Manual, Technical Report `ANL/MSC-TM-263`, Argonne National Laboratory, Argonne, IL, USA, 2003.

[5] Schulz, S.: System Description: `E 1.8`, in McMillan, K., Middeldorp, A. and Voronkov, A. (*eds.*), *Logic for Programming, Artificial Intelligence, and Reasoning: 19th International Conference, LPAR-19*, Stellenbosch, South Africa, December 14-19, 2013, Proceedings. Vol. 8312. Springer, 2013.

[6] Slaney, J.: More Proofs of an Axiom of Łukasiewicz, *Journal of Automated Reasoning*, **29**: 59–66 (2002).

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