- In the first edition of LFP, Carnap [3] undertakes a precise probabilistic explication of the concept of confirmation. This is where modern confirmation theory was born (in sin).
- Carnap was interested mainly in *quantitative* confirmation (which he took to be fundamental). But, he also gave (derivative) qualitative and comparative explications:
  - Qualitative. E inductively supports H.
  - Comparative. E supports H *more strongly than* E' supports H'.
  - Quantitative. E inductively supports H *to degree* r.
- Carnap begins by clarifying the *explicandum* (the informal "inductive support" concept) in various ways, including:
  - Qualitative. ($\star$) E gives some (positive) evidence for H.
- Note two things. First, ($\star$) sounds *epistemic* (not *logical*). Second, ($\star$) sounds like it involves (positive) *relevance*.
- Strangely, Carnap proceeds (in LFP_1) to offer a *logical* account of confirmation that does *not* involve relevance.
- These were the two original sins of Bayesian confirmation...

- In the 1st ed. of LFP, Carnap characterizes "the degree to which E confirms H" as $\mathfrak{c}(H, E) = \Pr(H \mid E)$, which leads to:

  Quantitative. $\Pr(H \mid E) = r$.

  Comparative. $\Pr(H \mid E) > \Pr(H' \mid E')$.

  Qualitative. $\Pr(H \mid E) > t$ (typically, with "threshold" $t > \frac{1}{2}$).
  - Doesn't sound like ($\star$). More on this dissonance below.
- Like Hempel [14], Carnap wanted a *logical* explication of confirmation (as a relation between sentences in FO$\mathcal{L}$s).
- For Carnap, this meant that the probability functions used in confirmation theory must *themselves* be "logical".
- This leads naturally to the Carnapian project of providing a "logical explication" of conditional probability $\Pr(\cdot \mid \cdot)$ *itself*.
- Here, Carnap was strongly influenced by Keynes [16], who believed there were (probabilistic) "partial entailments". I'm skeptical [11] (as are most, but not all, modern Bayesians).
- Hempel's theory of confirmation [14] satisfies the following:

(SCC) If E confirms H_1 and $H_1 \vDash H_2$, then E confirms H_2.

- In LFP_1, Carnap describes a counterexample to Hempel's (SCC), which presupposes a more ($\star$)-like qualitative conception of confirmation. There, he presupposes:

  Qualitative. E confirms H iff $\Pr(H \mid E) > \Pr(H)$.
- This *probabilistic relevance* conception *violates* (SCC), whereas the previous Pr-threshold conception *implies* (SCC).
- Popper [22] notes this tension in LFP. Largely in response to Popper, Carnap published a second edition of LFP [2], which includes a preface acknowledging an "*ambiguity*" in LFP_1:
  - **Firmness**. The degree to which E confirms$_f$ H:
  $$\mathfrak{c}_f(H, E) = \Pr(H \mid E).$$
  - **Increase in Firmness**. The degree to which E confirms$_i$ H:
  $$\mathfrak{c}_i(H, E) = \mathfrak{f}[\Pr(H \mid E), \Pr(H)]$$

  $\mathfrak{f}$ measures "the degree to which E *increases* the Pr of H."
- The 1st ed. of LFP was mainly about firmness, and the 2nd edition only adds the preface, which says very little about $\mathfrak{c}_i$. Specifically, no function $\mathfrak{f}$ is rigorously defended there.

- $\mathfrak{c}_i$ is more similar to ($\ast$) than $\mathfrak{c}_f$ is. To see this, note that we can have $\Pr(H \mid E) > t$ *even if* E ***lowers*** *the probability of* H.
- Example: Let H be the hypothesis that John does *not* have HIV, and let E be a *positive* test result for HIV from a highly reliable test. Plausibly, in such cases, we could have both:
  - $\Pr(H \mid E) > t$, for just about any threshold value t, but
  - $\Pr(H \mid E) < \Pr(H)$, since E *lowers* the probability of H.
- So, if we adopt Carnap's $\mathfrak{c}_f$-explication, then we must say that E confirms H in such cases. But, in ($\ast$)-terms, this implies E provides some *positive evidential support for* H!
- I take it we don't want to say *that*. Intuitively, what we want to say here is that, while H is (still) *highly probable given* E, (nonetheless) E provides (strong!) evidence ***against*** H.
- Carnap [2] seems to appreciate this dissonance, when he concedes $\mathfrak{c}_i$ is (in some settings) "more interesting" than $\mathfrak{c}_f$.
- Contemporary Bayesians would agree with this. They've since embraced a probabilistic relevance conception [23].

- Salmon's discussion in C&R revolves mainly around some *prima facie* puzzling features of probabilistic relevance.
- To wit, consider the following three principles.

  Conjunction Introduction (**And**). If E confirms H_1 and E confirms H_2, then E confirms $H_1 \,\&\, H_2$.

  Disjunction Introduction (**Or**). If E confirms H_1 and E confirms H_2, then E confirms $H_1 \vee H_2$.

  And/Or. If E confirms H_1 and E confirms H_2, then *either* E confirms $H_1 \,\&\, H_2$ *or* E confirms $H_1 \vee H_2$.
- *Neither* confirms$_f$ *nor* confirms$_i$ satisfies **And**.
- It is easy to see that confirms$_f$ satisfies **Or** (hence **And/Or**). Indeed, confirms$_f$ satisfies the stronger principle: If *either* E confirms$_f$ H_1 *or* E confirms$_f$ H_2, then E confirms$_f$ $H_1 \vee H_2$.
- Interestingly, confirms$_i$ *also* satisfies **And/Or** [3, T71–2a(1)].
- Confirms$_f$ and confirms$_i$ diverge drastically wrt **Or**. While high probability *obviously satisfies* **Or**, probabilistic *relevance violates* **Or**! This is the main focus of C&R.

- In order to understand Salmon's approach to the anomalies, we'll need to introduce his measure of "degree of relevance." Salmon [23] adopts the following measure.

  Carnap's Difference Measure [2, preface] of the degree to which E is relevant to H, given background knowledge K.

  $$d(H, E \mid K) \overset{\text{def}}{=} \Pr(H \mid E \,\&\, K) - \Pr(H \mid K)$$

☞ Salmon's basic insight is that many of the anomalous cases occur because *the degree to which* E *is relevant to* H_1 (H_2) *depends on whether* H_2 (H_1) *is already known.*

- To wit, Salmon proposes the following single, elegant sufficient condition for *both* **And** *and* **Or**.

  Salmon's Sufficient Condition for *both* And *and* Or:

  $$d(H_1, E \mid H_2) = d(H_1, E \mid \top)$$
- This condition can be weakened, disjunctively, to: *either* $d(H_1, E \mid H_2) = d(H_1, E \mid \top)$ *or* $d(H_2, E \mid H_1) = d(H_2, E \mid \top)$.
- Moreover, this result doesn't depend on *which relevance measure one uses*. [More on this robustness property, below.]

- Interestingly, Salmon doesn't discuss **And/Or** in C&R.[1] But, **And/Or** gives us another (odd) sufficient condition for **Or**.

  Another Sufficient Condition for Or:

  E ***dis****confirms* $H_1 \,\&\, H_2$, *i.e.*, $d(H_1 \,\&\, H_2, E \mid \top) < 0$.
- For some reason, **And** has received more attention in the literature since C&R than **Or** has. Other sufficient conditions for **And** have been presented in the literature.
- Recently, Koscholke has pointed out that **And/Or** yields "E *dis*confirms $H_1 \vee H_2$" as a (odd) sufficient for **And** [17].
- Cohen [6] gives the following sufficient condition for **And**.

  Cohen's Sufficient Condition for And:

  $$d(H_2, H_1 \mid E) \geq 0 \text{ and } d(H_2, H_1 \mid {\sim}E) \leq 0$$
- The weakest sufficient condition for **And** I have seen is the following condition (inspired by Salmon's analysis).

[1] This was surprising to me, since §4 of C&R is about Duhem's problem, which involves cases in which E *disconfirms a conjunction* ($H \,\&\, A$).

Weakest known Sufficient Condition for And:

Either $d(H_2, E \mid H_1) \geq 0$ *or* $d(H_1, E \mid H_2) \geq 0$

☞ So, 50 years on, Salmon's analysis of the anomalies (still) provides the ingredients for the weakest known sufficient (but still unnecessary) conditions for *each* of **And** *and* **Or**.

- I mentioned that Salmon's analyses in C&R do not depend on one's choice of relevance measure. That is not entirely true. It is true that his weakest sufficient conditions for of **And** and **Or** remain sufficient for other choices of measure.
- But, there is one claim he makes in his analysis which *does* depend on choice of relevance measure. To wit:

  $$d(H, E \mid \top) - d(H, E \mid K) = d(H, K \mid \top) - d(H, K \mid E)$$
- This *fails* for other (difference) measures [4, 20, 21].

  $$s(H, E \mid K) \overset{\text{def}}{=} \Pr(H \mid E \,\&\, K) - \Pr(H \mid {\sim}E \,\&\, K)$$

  $$n(H, E \mid K) \overset{\text{def}}{=} \Pr(E \mid H \,\&\, K) - \Pr(E \mid {\sim}H \,\&\, K)$$

  $$m(H, E \mid K) \overset{\text{def}}{=} \Pr(E \mid H \,\&\, K) - \Pr(E \mid K)$$

- Finally, we can estimate the *prevalence* of each of Salmon's two anomalies *via* Monte Carlo simulation [31].
- First, we randomly generate a set $\mathcal{D}$ of 10^7 probability distributions over $\{E, H_1, H_2\}$. Then, we select from $\mathcal{D}$ those on which (C) E confirms$_i$ each of H_1 and H_2 (individually).
- Finally, we calculate the proportion of distributions in C which violate **And** and **Or**, respectively.
- ☞ This verifies that violations of **And** and **Or** are *anomalous*. The proportion of C-distributions violating **Or** (or **And**) is only ≈10%. [For confirms$_f$, the **And** failure rate is ≈50%!]
- Koscholke [18] has done simulations for **And**, and other patterns of inference involving confirms$_i$ (*e.g.*, transitivity fails ≈35% of the time). His **And** results confirm ours.
- Vincenzo Crupi and I [7] have performed a similar analysis regarding the prevalence of Simpson's Paradox [13]. Only ≈3% of cases in which both $d(H_1, E \mid H_2) > 0$ and $d(H_1, E \mid {\sim}H_2) > 0$ are cases in which $d(H_1, E \mid \top) \leq 0$.

- When it comes to *quantitative* judgments, Bayesians use various *relevance measures* $\mathfrak{c}$ of degree of confirmation.
- These are much like the candidate functions $\mathfrak{f}$ we saw in connection with Carnapian $\mathfrak{c}_i$, but defined relative to subjective probabilities rather than "logical" probabilities.
- There are *many comparatively distinct* measures. See [9, 10] and [28] for philosophical and psychological discussion.
- Once we choose a measure $\mathfrak{c}(H, E)$ of the degree to which E confirms H, we can explicate comparative confirmation relations. *E.g.*, E favors H_1 over H_2 iff $\mathfrak{c}(H_1, E) > \mathfrak{c}(H_2, E)$.
- Note: $\Pr(H \mid E)$ is a *bad* candidate for $\mathfrak{c}(H, E)$ in this context. It implies "E favors H_1 over H_2," in some cases where E is negatively relevant to H_1 but positively relevant to H_2 [22]!
- In the context of comparative confirmation, there is ongoing philosophical/theoretical debate about the appropriate choice of $\mathfrak{c}$ (*e.g.*, the Likelihoodism debate [12]).
- An account is *robust* if it does not depend on choice of $\mathfrak{c}$.

- Tversky and Kahneman [30] discuss the following example, which was the first example of the "conjunction fallacy":
  - (E) Linda is 31, single, outspoken and very bright. She majored in philosophy. As a student, she was deeply concerned with issues of discrimination and social justice and she also participated in antinuclear demonstrations.
- Is it more probable, given E, that Linda is (H_1) a bank teller, or (H_1 and H_2) a bank teller *and* an active feminist?
- Most say "H_1 and H_2" is more probable (given E) than H_1. On its face, this violates comparative probability theory, since $X \vDash Y$ implies $\Pr(X \mid E) \leq \Pr(Y \mid E)$, and $H_1 \,\&\, H_2 \vDash H_1$.
- Experiments have been done to ensure subjects understand "H_1 and H$_2$" in the experiment as a *conjunction* $H_1 \,\&\, H_2$, and H_1 as a *conjunct* thereof (*not* as $H_1 \,\&\, {\sim}H_2$) [25, 27].
- At the same time, the "fallacy" persists when people are queried about *betting odds* rather than *probabilities* [25, 1].
- Comparative Bayesian confirmation can be helpful [19]. Detailed accounts have recently been developed [8, 29].

- It *is* possible to have $\mathfrak{c}(H_1 \,\&\, H_2, E) > \mathfrak{c}(H_1, E)$ *even though* $H_1 \,\&\, H_2 \vDash H_1$. And, intuitively, this is true in the Linda case.
- As Tversky & Kahneman *themselves* [30] say: "feminist bank teller is a better hypothesis about Linda than bank teller".
- ☞ Bayesian confirmation theory can explain why [8].

  Theorem*. For all Bayesian relevance measures* $\mathfrak{c}$*, if*
  (i) $\mathfrak{c}(H_2, E \mid H_1) > 0$ *and*
  (ii) $\mathfrak{c}(H_1, E) \leq 0$,
  then $\mathfrak{c}(H_1 \,\&\, H_2, E) > \mathfrak{c}(H_1, E)$.

- The first inequality (i) has been empirically well established in several traditional (Linda-like) CF cases [24].
- More recently, inequality (ii) has also been empirically confirmed in traditional CF cases, and the account has been extended to explain other kinds of CF cases as well [29].
- Even more recent work on CF has suggested that other notions of relevance may also be explanatorily useful [5].

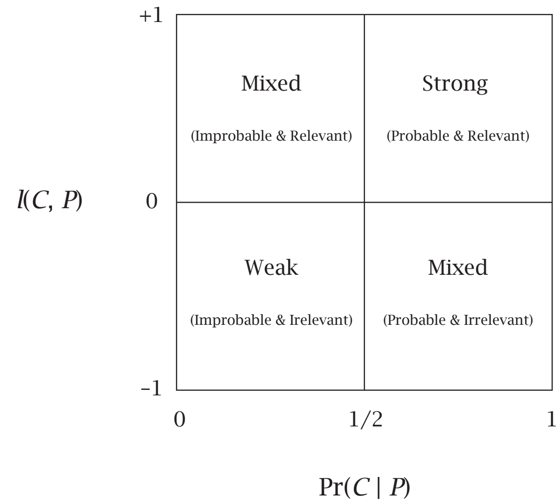
- The literature on inductive argument strength tends to focus only on conditional probability. That is, the strength of $P \therefore C$ is usually assumed to be given by $\Pr(C \mid P)$ [26].
- I have proposed, instead, that we should think of argument strength as having two (not generally commensurable) dimensions: Probability and Relevance [11].
- I have proposed the following desideratum for a relevance measure $\mathfrak{c}(C, P)$ — in the context of *inductive logic*.

$$\mathfrak{c}(C,P) \text{ should be } \begin{cases} +1 & \Leftarrow P \text{ entails } C \text{ (non-trivially).} \\ > 0 \text{ (confirmation)} & \Rightarrow \Pr(C \mid P) > \Pr(C). \\ = 0 \text{ (irrelevance)} & \Rightarrow \Pr(C \mid P) = \Pr(C). \\ < 0 \text{ (disconfirmation)} & \Rightarrow \Pr(C \mid P) < \Pr(C). \\ -1 & \Leftarrow P \text{ entails } {\sim}C \text{ (non-trivially).} \end{cases}$$

- This leads, nearly uniquely [15], to the following *inductive-logical relevance measure* (suppressing K).

$$l(C,P) \stackrel{\text{def}}{=} \frac{\Pr(P \mid C) - \Pr(P \mid {\sim}C)}{\Pr(P \mid C) + \Pr(P \mid {\sim}C)}$$

☞ We can then represent the strength of any argument $P \therefore C$ (relative to any Pr-function) as a point in the following 4-quadrant, 2-D map. I'll close with some examples.



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