A CONCISE AXIOMATIZATION OF $RM_{\to}$

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Let R be the system of relevant implication, and let $R_{\to}$ be its implicational fragment. $R_{\to}$ is given by the following independent axiom-schema, with the rules *modus ponens* and substitution [1, p. 88]:

(1) Cpp
(2) $CCpqCCqrCpr$
(3) $CpCCpqq$
(4) $CCpCpqCpq$

While Dunn's system RM may be generated by adding the simple formula $CpCpp$ to R [2], it was shown by Meyer and Parks [4] that its implicational fragment $RM_{\to}$ cannot be characterized by adding $CpCpp$ to $R_{\to}$. Rather, they show that an independent basis for $RM_{\to}$ consists of (2)–(4) above, plus the formula

(5) $CCCCCpqqprCCCCCqppqrr$

The system $RM_{\to}$ also coincides with the implicational fragment of the three-valued logic S of Sobociński [6]. This equivalence between S and $RM_{\to}$ was first shown by Parks [5], and the first independent axiomatization of S was given by Meyer and Parks [4]. The purpose of this note is to give a more concise independent basis for $RM_{\to}$ (and, hence, of the implicational fragment of S), consisting of (2) and (3), together with:

(6) $CCCpCCCqprqrr$

To prove this, we must show that (6) is a theorem of $RM_{\to}$ and that (2), (3) and (6) together entail both (4) and (5). The first claim is easy to prove, because $RM_{\to}$ has a simple three-element characteristic matrix, which may be found in [4]. So we may show that (6) is a theorem of $RM_{\to}$ by verifying that it takes only designated values on that matrix.

The second claim is established by the following condensed detachment proof, which was found by using William McCune's automated reasoning program, Otter [3]. In the proof, D·x·y means that the formula is the result of applying condensed detachment with x as major premise and y as minor.

1.	$CCpqCCqrCpr$	(2)
2.	$CpCCpqq$	(3)
3.	$CCCpCCCqprqrr$	(6)
4.	$CCCCpqCrquCCrpu$	D·1·1
5.	$CCpCqrCCuqCpCur$	D·4·4
6.	$CCpqCCCpruCCqru$	D·4·1
7.	$CCpCqrCqCpr$	D·5·2
8.	$CCCpqrCCCCpuuqr$	D·6·2
9.	$CCCCpCqrutCCCqCprut$	D·6·7
10.	$CCCpqrCCpuCCuqr$	D·7·6

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11.	$CCCCpqqCruCrCpu$	D·8·7
12.	$CCCCpqqrCCCputCCrut$	D·8·6
13.	$CCCCpqqrCCruCpu$	D·8·1
14.	$CCCCCpqrCqprr$	D·9·3
15.	$CCpqCCqrCCruCpu$	D·4·10
16.	$CCCCpqCCqrutCCCprut$	D·1·10
17.	$CCCpCCqCprruCCutCqt$	D·9·13
18.	$CCCCCpqCqpruCCCqpru$	D·12·14
19.	$CCCCCpqrCqpuCCurr$	D·10·14
20.	$CCCCpqCCqrCurtCCupt$	D·1·15
21.	$CCCCpqCCrputCCCrqut$	D·9·16
22.	$CCCCpqruCCqrCpu$	D·16·11
23.	$CCCCpqruCCtpCCCtqru$	D·16·4
24.	$CCCCpqruCCqtCCCptru$	D·22·21
25.	$CCCCCpqrprCqr$	D·22·3
26.	$CCCCCpqrpuCCurCqr$	D·10·25
27.	$CCCCCpqrCqprCCpqr$	D·26·18
28.	$CCpqCCqpCqp$	D·27·20
29.	$CCpCqrCCrqCpCqr$	D·5·28
30.	$CCCCpCqCrpCrCCqCrppuCqu$	D·17·29
31.	$CpCCpCpqq$	D·14·30
32.	$CCpCpqCpq\star$	D·7·31
33.	$CCCCpCpqruCCCpqru$	D·6·32
34.	$CCCCCpqrCCpqprr$	D·19·33
35.	$CCCCpqprCCCCCpquruu$	D·24·34
36.	$CCCCCpCCCqppqrprr$	D·35·3
37.	$CCCCCpCCCqppqrpuCCurr$	D·10·36
38.	$CCCCCCCpqqCqprqrr$	D·37·9
39.	$CCpCCCCqrrCrquCCCpruu$	D·23·38
40.	$CCCCCpqqruCCCCCqprquu\star$	D·21·39

The formula (4) is proven at line 32, and (5) is an instance of line 40. In addition to (6) above, the formula $CCCCCpqrCqprr$ also forms an independent three-basis for $RM_{\rightarrow}$ with (2) and (3).

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References

[1] Alan Ross Anderson and Nuel D. Belnap. *Entailment: The Logic of Relevance and Necessity*, volume 1. Princeton University Press, Princeton, New Jersey, 1975.

[2] Michael Dunn. Algebraic completeness results for R-Mingle and its extensions. *Journal of Symbolic Logic*, 35(1):1–13, 1970.

[3] William McCune. *OTTER: 3.0 Reference Manual and Guide*. Argonne National Laboratory, Argonne, Illinois, 1994.

[4] Robert K. Meyer and R. Zane Parks. Independent axioms for the implicational fragment of Sobociński's three-valued logic. *Zeitschrift für Mathematische Logik Grundlagen Mathematik*, 18:291–295, 1972.
[5] R. Zane Parks. A note on R-Mingle and Sobociński's three-valued logic. *Notre Dame Journal of Symbolic Logic*, XIII(2):227–228, 1972.
[6] B. Sobociński. Axiomatization of a partial system of three-valued calculus of propositions. *The Journal of Computing Systems*, 1:23–55, 1952.

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