Quantitative Aspects of Simpson's Paradox

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First, we load in the **PrSAT** *Mathematica* package, which can be downloaded for free from the following webpage:

http://fitelson.org/PrSAT/

This will provide all the functions we need to verify/perform all the theoretical results/simulations discussed in the paper.

In[ ]:=
```
<< PrSAT`
```

1 Simpson's Paradox & Bayesian Confirmation (Qualitative)

Toy Example (Graduate Admissions, Simplified)

	E	$\sim E$	Overall
K	$40/50 = \Pr(H \mid E \,\&\, K)$	$60/80 = \Pr(H \mid \sim E \,\&\, K)$	$100/130 = \Pr(H \mid K)$
$\sim K$	$20/80 = \Pr(H \mid E \,\&\, \sim K)$	$10/50 = \Pr(H \mid \sim E \,\&\, \sim K)$	$30/130 = \Pr(H \mid \sim K)$
Overall	$60/130 = \Pr(H \mid E)$	$70/130 = \Pr(H \mid \sim E)$	$130/260 = \Pr(H)$

In[ ]:=
```
TOY = PrSAT[{
    Pr[𝕀 | 𝔼 ∧ 𝕂] == 40/50,
    Pr[𝕀 | ¬ 𝔼 ∧ 𝕂] == 60/80,
    Pr[𝕀 | 𝔼 ∧ ¬ 𝕂] == 20/80,
    Pr[𝕀 | ¬ 𝔼 ∧ ¬ 𝕂] == 10/50,
    Pr[𝕀 | 𝔼] == 60/130,
    Pr[𝕀 | ¬ 𝔼] == 70/130,
    Pr[𝕀 | 𝕂] == 100/130,
    Pr[𝕀 | ¬ 𝕂] == 30/130,
    Pr[𝕀] == 130/260
  }]
```

Out[ ]=
$$\left\{\{\mathbb{E} \to \{a_2, a_5, a_6, a_8\}, \mathbb{I} \to \{a_3, a_5, a_7, a_8\}, \mathbb{K} \to \{a_4, a_6, a_7, a_8\}, \Omega \to \{a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8\}\}, \left\{a_1 \to \frac{2}{13}, a_2 \to \frac{3}{13}, a_3 \to \frac{1}{26}, a_4 \to \frac{1}{13}, a_5 \to \frac{1}{13}, a_6 \to \frac{1}{26}, a_7 \to \frac{3}{13}, a_8 \to \frac{2}{13}\right\}\right\}$$

In[ ]:=
```
STT[TOY]
```

Out[ ]=

𝔼	𝕀	𝕂	var	Pr
T	T	T	a_8	$\frac{2}{13}$
T	T	F	a_5	$\frac{1}{13}$
T	F	T	a_6	$\frac{1}{26}$
T	F	F	a_2	$\frac{3}{13}$
F	T	T	a_7	$\frac{3}{13}$
F	T	F	a_3	$\frac{1}{26}$
F	F	T	a_4	$\frac{1}{13}$
F	F	F	a_1	$\frac{2}{13}$

𝔼 is being female, 𝕀 is being admitted into grad school, and 𝕂 is applying to the “easy” department (i.e., the one that’s easier to get into, overall).

Verification that this is indeed a Simpson’s Paradox case, using the `EvaluateProbability` function.

In[∘]:=
```
EvaluateProbability[
  {Pr[ℍ | 𝔼 ∧ 𝕂] > Pr[ℍ | 𝕂], Pr[ℍ | 𝔼 ∧ ¬ 𝕂] > Pr[ℍ | ¬ 𝕂], Pr[ℍ | 𝔼] < Pr[ℍ]}, TOY] // N
```

Out[∘]= {True, True, True}

Because Pr[ℍ | 𝔼 & 𝕂] > Pr[ℍ | 𝕂], Pr[ℍ | 𝔼 & ¬𝕂] > Pr[ℍ | ¬𝕂], and Pr[ℍ | 𝔼] < Pr[ℍ], this is indeed a Simpson Paradox.

Real-World Case (Kidney Treatments)

	E	$\sim E$	Overall
K	$81/87 = \Pr(H \mid E \,\&\, K)$	$234/270 = \Pr(H \mid \sim E \,\&\, K)$	$315/357 = \Pr(H \mid K)$
$\sim K$	$192/263 = \Pr(H \mid E \,\&\, \sim K)$	$55/80 = \Pr(H \mid \sim E \,\&\, \sim K)$	$247/343 = \Pr(H \mid \sim K)$
Overall	$273/350 = \Pr(H \mid E)$	$289/350 = \Pr(H \mid \sim E)$	$562/700 = \Pr(H)$

In[∘]:=
```
RWC = PrSAT[{
    Pr[ℍ | 𝔼 ∧ 𝕂] == 81/87,
    Pr[ℍ | ¬ 𝔼 ∧ 𝕂] == 234/270,
    Pr[ℍ | 𝔼 ∧ ¬ 𝕂] == 192/263,
    Pr[ℍ | ¬ 𝔼 ∧ ¬ 𝕂] == 55/80,
    Pr[ℍ | 𝔼] == 273/350,
    Pr[ℍ | ¬ 𝔼] == 289/350,
    Pr[ℍ | 𝕂] == 315/357,
    Pr[ℍ | ¬ 𝕂] == 247/343,
    Pr[ℍ] == 562/700
  }]
```

Out[∘]= $\left\{\{\mathbb{E} \to \{a_2, a_5, a_6, a_8\}, \mathbb{H} \to \{a_3, a_5, a_7, a_8\}, \mathbb{K} \to \{a_4, a_6, a_7, a_8\}, \Omega \to \{a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8\}\}, \left\{a_1 \to \frac{1}{28}, a_2 \to \frac{71}{700}, a_3 \to \frac{11}{140}, a_4 \to \frac{9}{175}, a_5 \to \frac{48}{175}, a_6 \to \frac{3}{350}, a_7 \to \frac{117}{350}, a_8 \to \frac{81}{700}\right\}\right\}$

In[∘]:=
```
STT[RWC]
```

Out[∘]=

$\mathbb{E}$	$\mathbb{H}$	$\mathbb{K}$	var	Pr
T	T	T	a_8	$\frac{81}{700}$
T	T	F	a_5	$\frac{48}{175}$
T	F	T	a_6	$\frac{3}{350}$
T	F	F	a_2	$\frac{71}{700}$
F	T	T	a_7	$\frac{117}{350}$
F	T	F	a_3	$\frac{11}{140}$
F	F	T	a_4	$\frac{9}{175}$
F	F	F	a_1	$\frac{1}{28}$

$\mathbb{E}$ is receiving an open surgical procedure, $\mathbb{H}$ is having a successful procedure, and $\mathbb{K}$ is having small kidney stones.

Verification that this is indeed a Simpson's Paradox case, using **EvaluateProbability**.

In[∘]:=
```
EvaluateProbability[
  {Pr[𝔼 | 𝔼 ∧ 𝕂] > Pr[𝔼 | 𝕂], Pr[𝔼 | 𝔼 ∧ ¬ 𝕂] > Pr[𝔼 | ¬ 𝕂], Pr[𝔼 | 𝔼] < Pr[𝔼]}, RWC] // N
```

Out[∘]=
```
{True, True, True}
```

Because Pr[$\mathbb{H}$ | $\mathbb{E}$ & $\mathbb{K}$] > Pr[$\mathbb{H}$ | $\mathbb{K}$], Pr[$\mathbb{H}$ | $\mathbb{E}$ & ¬$\mathbb{K}$] > Pr[$\mathbb{H}$ | ¬$\mathbb{K}$], and Pr[$\mathbb{H}$ | $\mathbb{E}$] < Pr[$\mathbb{H}$], this is also a Simpson Paradox.

2 Simpson's Paradox & Bayesian Confirmation (Quantitative)

Verifying the Quantitative Features of our Toy Example and our Real-World Example

First, we define our five confirmation measures (d, r, l, s, and z), as follows:

In[∘]:=
```
d[ℍ_, 𝔼_, 𝕂_] := Pr[ℍ | 𝔼 ∧ 𝕂] - Pr[ℍ | 𝕂];
r[ℍ_, 𝔼_, 𝕂_] := (Pr[ℍ | 𝔼 ∧ 𝕂] - Pr[ℍ | 𝕂])/(Pr[ℍ | 𝔼 ∧ 𝕂] + Pr[ℍ | 𝕂]);
ℓ[ℍ_, 𝔼_, 𝕂_] := (Pr[𝔼 | ℍ ∧ 𝕂] - Pr[𝔼 | ¬ ℍ ∧ 𝕂])/(Pr[𝔼 | ℍ ∧ 𝕂] + Pr[𝔼 | ¬ ℍ ∧ 𝕂]);
s[ℍ_, 𝔼_, 𝕂_] := Pr[ℍ | 𝔼 ∧ 𝕂] - Pr[ℍ | ¬ 𝔼 ∧ 𝕂];
zc[ℍ_, 𝔼_, 𝕂_] := (Pr[ℍ | 𝔼 ∧ 𝕂] - Pr[ℍ | 𝕂])/Pr[¬ ℍ | 𝕂];
zdc[ℍ_, 𝔼_, 𝕂_] := (Pr[ℍ | 𝔼 ∧ 𝕂] - Pr[ℍ | 𝕂])/Pr[ℍ | 𝕂];
T = ℍ ∨ ¬ ℍ;
```

Then, we can define a quantitative Simpson Paradox, of strength *t*, according to each of the five confirmation measures *d*, *r*, *l*, *s*, and *z*, ($\mathbf{QSP}_c^\tau$) respectively, as follows.

In[∘]:=
```
QSPd[τ_] := {d[ℍ, 𝔼, 𝕂] ≥ τ,
   d[ℍ, 𝔼, ¬ 𝕂] ≥ τ,
   d[ℍ, 𝔼, T] ≤ -τ};
QSPr[τ_] := {r[ℍ, 𝔼, 𝕂] ≥ τ,
   r[ℍ, 𝔼, ¬ 𝕂] ≥ τ,
   r[ℍ, 𝔼, T] ≤ -τ};
QSPl[τ_] := {ℓ[ℍ, 𝔼, 𝕂] ≥ τ,
   ℓ[ℍ, 𝔼, ¬ 𝕂] ≥ τ,
   ℓ[ℍ, 𝔼, T] ≤ -τ};
QSPs[τ_] := {s[ℍ, 𝔼, 𝕂] ≥ τ,
   s[ℍ, 𝔼, ¬ 𝕂] ≥ τ,
   s[ℍ, 𝔼, T] ≤ -τ};
QSPz[τ_] := {zc[ℍ, 𝔼, 𝕂] ≥ τ,
   zc[ℍ, 𝔼, ¬ 𝕂] ≥ τ,
   zdc[ℍ, 𝔼, T] ≤ -τ};
```

Now, we can verify that our toy example is an instance of ($\mathbf{QSP}_d^{0.01}$) using the `EvaluateProbability` function.

In[∘]:=
```
EvaluateProbability[QSPd[0.01], TOY]
```

Out[∘]=
```
{True, True, True}
```

Similarly, we can verify that our to example is also an instance of ($\mathbf{QSP}_r^{0.019}$), ($\mathbf{QSP}_l^{0.053}$), ($\mathbf{QSP}_s^{0.05}$), and ($\mathbf{QSP}_z^{0.025}$), respectively, as follows:

In[∘]:=
```
EvaluateProbability[QSPr[0.019], TOY]
```

Out[∘]=
```
{True, True, True}
```

In[∘]:=
```
EvaluateProbability[QSPl[0.052], TOY]
```

Out[∘]=
```
{True, True, True}
```

In[∘]:=
```
EvaluateProbability[QSPs[0.05], TOY]
```
Out[∘]=
```
{True, True, True}
```

In[∘]:=
```
EvaluateProbability[QSPz[0.025], TOY]
```
Out[∘]=
```
{True, True, True}
```

Next, we can do the same analysis to verify that our real-world example is an instance of $(\mathbf{QSP}_d^{0.009})$, $(\mathbf{QSP}_r^{0.0068})$, $(\mathbf{QSP}_l^{0.024})$, $(\mathbf{QSP}_s^{0.042})$, and $(\mathbf{QSP}_z^{0.028})$, respectively.

In[∘]:=
```
EvaluateProbability[QSPd[0.009], RWC]
```
Out[∘]=
```
{True, True, True}
```

In[∘]:=
```
EvaluateProbability[QSPr[0.0068], RWC]
```
Out[∘]=
```
{True, True, True}
```

In[∘]:=
```
EvaluateProbability[QSPl[0.024], RWC]
```
Out[∘]=
```
{True, True, True}
```

In[∘]:=
```
EvaluateProbability[QSPs[0.042], RWC]
```
Out[∘]=
```
{True, True, True}
```

In[∘]:=
```
EvaluateProbability[QSPz[0.028], RWC]
```
Out[∘]=
```
{True, True, True}
```

Verifying the Elements of our Main Result (Theorem)

The following function can be used to generate (regular) probability distributions such that $(\mathbf{QSP}_l^{1-\epsilon})$, for arbitrarily small ϵ.

In[∘]:=
```
StrongQSPl[ϵ_] :=
 PrSAT[{ℓ[𝕀, 𝔼, 𝕂] == 1 - ϵ, ℓ[𝕀, 𝔼, ¬ 𝕂] == 1 - ϵ, ℓ[𝕀, 𝔼, T] == -(1 - ϵ), Pr[𝕂] == 1/2},
  Probabilities → Regular, BypassSearch → True]
```

In[∘]:=
```
SL = StrongQSPl[1/1000000];
```

In[∘]:=
```
STT[N[SL, 8]]
```

Out[∘]=

𝔼	ℍ	𝕂	var	Pr
T	T	T	a_8	$7.4676183 \times 10^{-60}$
T	T	F	a_5	$6.5685866 \times 10^{-47}$
T	F	T	a_6	$1.9259299 \times 10^{-33}$
T	F	F	a_2	$2.4074147 \times 10^{-46}$
F	T	T	a_7	$9.6935277 \times 10^{-34}$
F	T	F	a_3	6.8212064×10^{-8}
F	F	T	a_4	0.50000000
F	F	F	a_1	0.49999993

In[∘]:=
```
N[EvaluateProbability[{ℓ[ℍ, 𝔼, 𝕂], ℓ[ℍ, 𝔼, ¬ 𝕂], ℓ[ℍ, 𝔼, T]}, SL], 8]
```

Out[∘]= {0.99999900, 0.99999900, -0.99999900}

The following function can be used to generate (regular) probability distributions such that ($\mathbf{QSP}_r^{1-\epsilon}$), for arbitrarily small ϵ.

In[∘]:=
```
StrongQSPr[ϵ_] :=
 PrSAT[{r[ℍ, 𝔼, 𝕂] == 1 - ϵ, r[ℍ, 𝔼, ¬ 𝕂] == 1 - ϵ, r[ℍ, 𝔼, T] == -(1 - ϵ), Pr[𝕂] == 1/2},
  Probabilities → Regular, BypassSearch → True]
```

In[∘]:=
```
SR = StrongQSPr[1/1000000];
```

In[∘]:=
```
STT[N[SR, 8]]
```

Out[∘]=

𝔼	ℍ	𝕂	var	Pr
T	T	T	a_8	$7.8303654 \times 10^{-54}$
T	T	F	a_5	$3.2842933 \times 10^{-47}$
T	F	T	a_6	$4.0389678 \times 10^{-27}$
T	F	F	a_2	1.3552517×10^{-7}
F	T	T	a_7	$4.8467638 \times 10^{-34}$
F	T	F	a_3	$2.7741661 \times 10^{-47}$
F	F	T	a_4	0.50000000
F	F	F	a_1	0.49999986

In[∘]:=
```
N[EvaluateProbability[{r[ℍ, 𝔼, 𝕂], r[ℍ, 𝔼, ¬ 𝕂], r[ℍ, 𝔼, T]}, SR], 8]
```

Out[∘]= {0.99999900, 0.99999900, -0.99999900}

The following function can be used to generate (regular) probability distributions such that ($\mathbf{QSP}_z^{1/2-\epsilon}$), for arbitrarily small ϵ.

In[∘]:=
```
StrongQSPz[ϵ_] :=
 PrSAT[{zc[ℍ, 𝔼, 𝕂] == 1/2 - ϵ, zc[ℍ, 𝔼, ¬ 𝕂] == 1/2 - ϵ, zdc[ℍ, 𝔼, T] == -(1/2 - ϵ),
   Pr[ℍ && 𝔼 && 𝕂] == Pr[ℍ && ¬ 𝔼 && 𝕂], Pr[𝕂] == ϵ}, BypassSearch → True]
```

In[∘]:=
```
SZ = StrongQSPz[1/1000000];
```

In[∘]:=
```
STT[N[SZ, 8]]
```

Out[∘]=

𝔼	ℍ	𝕂	var	Pr
T	T	T	a_8	$1.1368684 \times 10^{-13}$
T	T	F	a_5	$3.2526065 \times 10^{-19}$
T	F	T	a_6	$1.1368724 \times 10^{-13}$
T	F	F	a_2	$2.1826907 \times 10^{-25}$
F	T	T	a_7	$1.1368684 \times 10^{-13}$
F	T	F	a_3	0.99999766
F	F	T	a_4	9.9999966×10^{-7}
F	F	F	a_1	1.3421129×10^{-6}

In[∘]:=
```
N[EvaluateProbability[{zc[ℍ, 𝔼, 𝕂], zc[ℍ, 𝔼, ¬ 𝕂], zdc[ℍ, 𝔼, T]}, SZ], 8]
```

Out[∘]= {0.49999900, 0.49999900, -0.49999900}

The following function can be used to generate (regular) probability distributions such that ($\mathbf{QSP}_d^{1/3-\epsilon}$), for arbitrarily small ϵ.

In[∘]:=
```
StrongQSPd[ϵ_] := PrSAT[{d[ℍ, 𝔼, 𝕂] == 1/3 - ϵ, d[ℍ, 𝔼, ¬ 𝕂] == 1/3 - ϵ,
   d[ℍ, 𝔼, T] == -(1/3 - ϵ), Pr[ℍ && 𝔼 && 𝕂] == Pr[ℍ && ¬ 𝔼 && 𝕂], Pr[𝕂] == ϵ},
  Probabilities → Regular, BypassSearch → True]
```

In[∘]:=
```
SD = StrongQSPd[1/1000000];
```

In[∘]:=
```
STT[N[SD, 8]]
```

Out[∘]=

𝔼	ℍ	𝕂	var	Pr
T	T	T	a_8	$5.6843419 \times 10^{-13}$
T	T	F	a_5	$1.5178830 \times 10^{-18}$
T	F	T	a_6	$1.1368677 \times 10^{-12}$
T	F	F	a_2	$9.1538865 \times 10^{-25}$
F	T	T	a_7	$5.6843419 \times 10^{-13}$
F	T	F	a_3	0.66666640
F	F	T	a_4	9.9999773×10^{-7}
F	F	F	a_1	0.33333260

In[•]:=
```
N[EvaluateProbability[{d[ℍ, 𝔼, 𝕂], d[ℍ, 𝔼, ¬ 𝕂], d[ℍ, 𝔼, T]}, SD], 8]
```
Out[•]= {0.33333233, 0.33333233, -0.33333233}

The following function can be used to generate probability distributions such that ($\mathbf{QSP}_s^{1/3-\epsilon}$), for arbitrarily small ϵ.

In[•]:=
```
StrongQSPs[ϵ_] := PrSAT[
  {s[ℍ, 𝔼, 𝕂] == 1/3 - ϵ, s[ℍ, 𝔼, ¬ 𝕂] == 1/3 - ϵ, s[ℍ, 𝔼, T] == -(1/3 - ϵ)}, BypassSearch → True]
```

In[•]:=
```
SS = StrongQSPs[1/1000000];
```

In[•]:=
```
N[STT[SS], 8]
```

Out[•]=

𝔼	ℍ	𝕂	var	Pr
T	T	T	a_8	0.15625000
T	T	F	a_5	5.9604645×10^{-7}
T	F	T	a_6	0.31249952
T	F	F	a_2	0
F	T	T	a_7	$8.6775811 \times 10^{-13}$
F	T	F	a_3	0.35416669
F	F	T	a_4	6.4802253×10^{-7}
F	F	F	a_1	0.17708255

In[•]:=
```
N[EvaluateProbability[{s[ℍ, 𝔼, 𝕂], s[ℍ, 𝔼, ¬ 𝕂], s[ℍ, 𝔼, T]}, SS], 8]
```
Out[•]= {0.33333233, 0.33333233, -0.33333233}

The remaining parts of our **Theorem** can be verified by showing that no cases of ($\mathbf{QSP}_l^1$), ($\mathbf{QSP}_r^1$), ($\mathbf{QSP}_z^{1/2}$), ($\mathbf{QSP}_d^{1/3}$), or ($\mathbf{QSP}_s^{1/3}$) exist.

In[•]:=
```
PrSAT[{l[ℍ, 𝔼, 𝕂] == 1, l[ℍ, 𝔼, ¬ 𝕂] == 1, l[ℍ, 𝔼, T] == -1}, BypassSearch → True]
```
Out[•]= { }

In[•]:=
```
PrSAT[{r[ℍ, 𝔼, 𝕂] == 1, r[ℍ, 𝔼, ¬ 𝕂] == 1, r[ℍ, 𝔼, T] == -1}, BypassSearch → True]
```
Out[•]= { }

In[•]:=
```
PrSAT[{zc[ℍ, 𝔼, 𝕂] == 1/2, zc[ℍ, 𝔼, ¬ 𝕂] == 1/2, zdc[ℍ, 𝔼, T] == -1/2}, BypassSearch → True]
```
Out[•]= { }

In[•]:=
```
PrSAT[{d[ℍ, 𝔼, 𝕂] == 1/3, d[ℍ, 𝔼, ¬ 𝕂] == 1/3, d[ℍ, 𝔼, T] == -1/3}, BypassSearch → True]
```
Out[•]= { }

In[•]:=
```
PrSAT[{s[ℍ, 𝔼, 𝕂] == 1/3, s[ℍ, 𝔼, ¬ 𝕂] == 1/3, s[ℍ, 𝔼, T] == -1/3}, BypassSearch → True]
```

Out[•]= { }

3 Estimating the Prevalence and Strength of Simpson's Paradoxes

The following function samples *n* probability distributions at random (assuming a uniform distribution over all possible distributions).

In[•]:=
```
Distributions[n_] := Module[{M},
   M := {{𝔼 → {a2, a5, a6, a8}, ℍ → {a3, a5, a7, a8},
      𝕂 → {a4, a6, a7, a8}, Ω → {a1, a2, a3, a4, a5, a6, a7, a8}},
     MapThread[Rule, {{a1, a2, a3, a4, a5, a6, a7, a8}, RandomModel[8]}]};
   ParallelTable[M, {n}]
  ];
```

This will provide our raw simulation data (which is a random sample of 10 million distributions).

In[•]:=
```
DISTs = Distributions[10 000 000];
```

The raw data file (containing all 10 million distributions in our random sample) can be downloaded (in compressed form) from http://fitelson.org/DISTs.nb.zip.
Note: even this compressed version of the file is rather large (1.01 GB). Uncompressed, the file is 12.89 GB.

Next, we estimate the probability of obtaining a qualitative Simpson's Paradox, using our random sample of 10 million distributions

```
Simpsons = Parallelize[Select[DISTs, EvaluateProbability[
      And @@ {d[ℍ, 𝔼, 𝕂] > 0, d[ℍ, 𝔼, ¬ 𝕂] > 0, d[ℍ, 𝔼, T] < 0}, #] &]];
```

The raw data file (containing the 83,000 distributions from our random sample that exhibit Simpson's paradox) can be downloaded (in compressed form) from http://fitelson.org/Simpsons.nb.zip.
Note: this compressed version of the file is 8.5 MB. Uncompressed, the file is 107.3 MB.

```
ProbSimp = Length[Simpsons] / Length[DISTs]
```

Out[•]= $\frac{83\,223}{10\,000\,000}$

In[•]:=
```
ProbSimp // N
```

Out[•]= 0.0083223

So, the probability of getting a (qualitative) Simpson Paradox is approximately 1 in 125 (if sampling randomly/uniformly from all possible probability distributions over ℍ,𝔼,𝕂).

Then, we calculate the *average strength* of reversal among these Simpson Paradox cases, for each of our five measures.

In[∘]:=
```
ASPSd = Mean[Min[Abs[#]] & /@
    (EvaluateProbability[{d[ℍ, 𝔼, 𝕂], d[ℍ, 𝔼, ¬ 𝕂], d[ℍ, 𝔼, T]}, #] & /@ Simpsons)]
```

Out[∘]= 0.0138237

In[∘]:=
```
ASPSl = Mean[Min[Abs[#]] & /@
    (EvaluateProbability[{ℓ[ℍ, 𝔼, 𝕂], ℓ[ℍ, 𝔼, ¬ 𝕂], ℓ[ℍ, 𝔼, T]}, #] & /@ Simpsons)]
```

Out[∘]= 0.0404394

In[∘]:=
```
ASPSz = Mean[Min[Abs[#]] & /@
    (EvaluateProbability[{zc[ℍ, 𝔼, 𝕂], zc[ℍ, 𝔼, ¬ 𝕂], zdc[ℍ, 𝔼, T]}, #] & /@ Simpsons)]
```

Out[∘]= 0.0246045

In[∘]:=
```
ASPSr = Mean[Min[Abs[#]] & /@
    (EvaluateProbability[{r[ℍ, 𝔼, 𝕂], r[ℍ, 𝔼, ¬ 𝕂], r[ℍ, 𝔼, T]}, #] & /@ Simpsons)]
```

Out[∘]= 0.0192955

In[∘]:=
```
ASPSs = Mean[Min[Abs[#]] & /@
    (EvaluateProbability[{s[ℍ, 𝔼, 𝕂], s[ℍ, 𝔼, ¬ 𝕂], s[ℍ, 𝔼, T]}, #] & /@ Simpsons)]
```

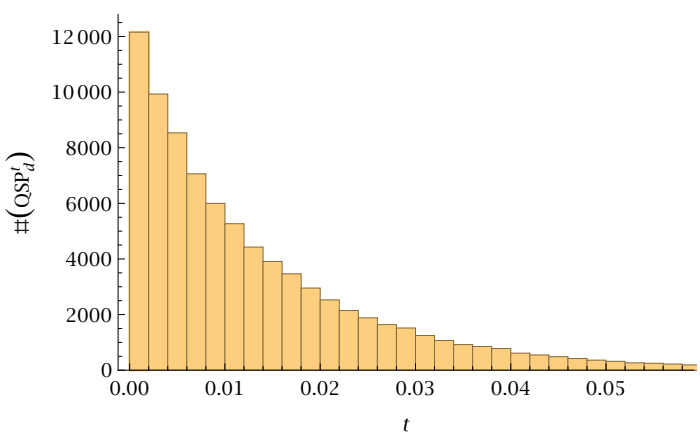
Out[∘]= 0.0345906

Finally, we compute *histograms of strengths* for each measure.

In[∘]:=
```
HistDatad = Min[Abs[#]] & /@
    (EvaluateProbability[{d[ℍ, 𝔼, 𝕂], d[ℍ, 𝔼, ¬ 𝕂], d[ℍ, 𝔼, T]}, #] & /@ Simpsons);
```

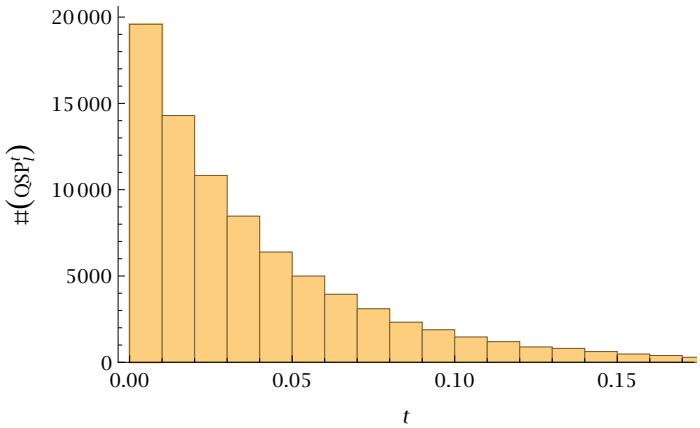
In[∘]:=
```
HSPSd = Histogram[HistDatad,
  FrameLabel → {"t", "#(QSP_d^t)"}, Frame → {{True, False}, {True, False}},
  LabelStyle → Directive[Black, FontFamily → "Lucida Bright", FontSize → 10]]
```

Out[∘]=



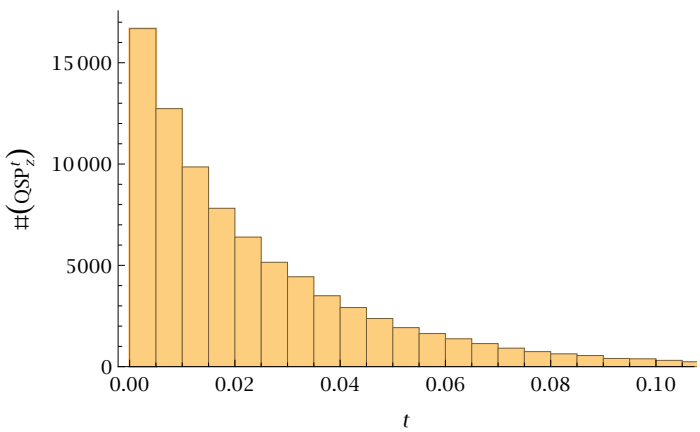
In[∘]:=
```
HistDatal = Min[Abs[#]] & /@
    (EvaluateProbability[{ℓ[ℍ, 𝔼, 𝕂], ℓ[ℍ, 𝔼, ¬ 𝕂], ℓ[ℍ, 𝔼, T]}, #] & /@ Simpsons);
```

In[∘]:=
```
HSPSl = Histogram[HistDatal,
  FrameLabel → {"t", "#(QSP_l^t)"}, Frame → {{True, False}, {True, False}},
  LabelStyle → Directive[Black, FontFamily → "Lucida Bright", FontSize → 10]]
```

Out[∘]=


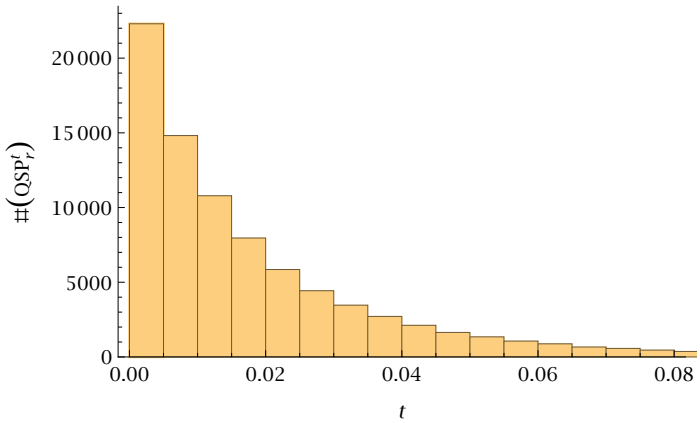
In[∘]:=
```
HistDataz = Min[Abs[#]] & /@
   (EvaluateProbability[{zc[ℍ, 𝔼, 𝕂], zc[ℍ, 𝔼, ¬ 𝕂], zdc[ℍ, 𝔼, T]}, #] & /@ Simpsons);
```

In[∘]:=
```
HSPSz = Histogram[HistDataz,
  FrameLabel → {"t", "#(QSP_z^t)"}, Frame → {{True, False}, {True, False}},
  LabelStyle → Directive[Black, FontFamily → "Lucida Bright", FontSize → 10]]
```

Out[∘]=


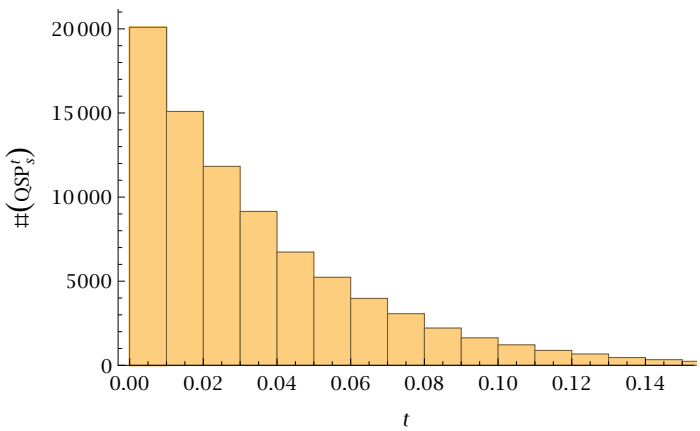
In[∘]:=
```
HistDatar = Min[Abs[#]] & /@
   (EvaluateProbability[{r[ℍ, 𝔼, 𝕂], r[ℍ, 𝔼, ¬ 𝕂], r[ℍ, 𝔼, T]}, #] & /@ Simpsons);
```

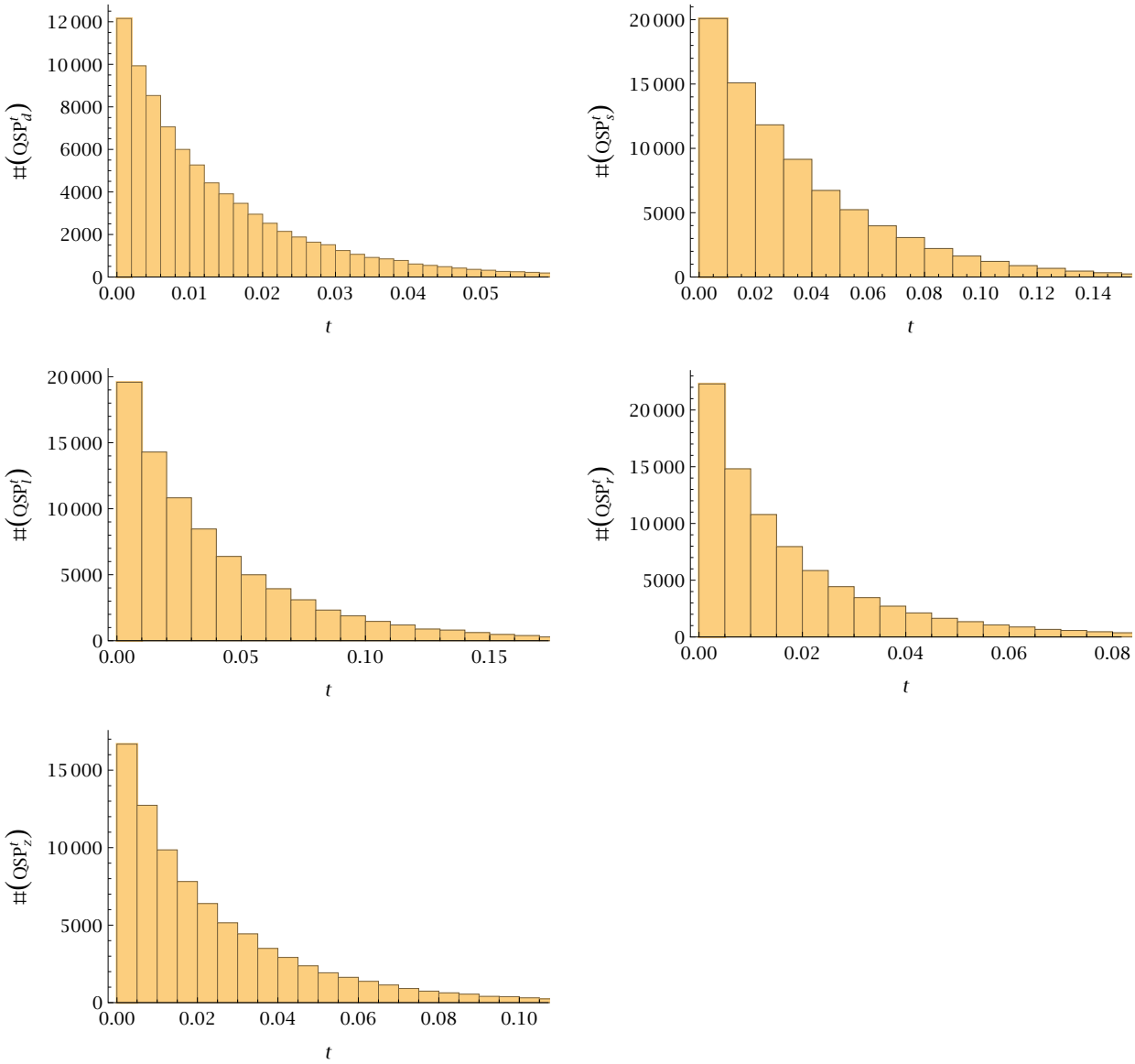
In[•]:=
```
HSPSr = Histogram[HistDatar,
  FrameLabel → {"t", "#(QSP_r^t)"}, Frame → {{True, False}, {True, False}},
  LabelStyle → Directive[Black, FontFamily → "Lucida Bright", FontSize → 10]]
```

Out[•]=


In[•]:=
```
HistDatas = Min[Abs[#]] & /@
   (EvaluateProbability[{s[ℍ, 𝔼, 𝕂], s[ℍ, 𝔼, ¬ 𝕂], s[ℍ, 𝔼, T]}, #] & /@ Simpsons);
```

In[•]:=
```
HSPSs = Histogram[HistDatas,
  FrameLabel → {"t", "#(QSP_s^t)"}, Frame → {{True, False}, {True, False}},
  LabelStyle → Directive[Black, FontFamily → "Lucida Bright", FontSize → 10]]
```

Out[•]=


In[∘]:=
```
Figure = GraphicsGrid[{{HSPSd, HSPSs}, {HSPSl, HSPSr}, {HSPSz}}, Alignment → Center]
```

Out[∘]=