Conditions Under Which the "Package Principle" is Required for a DBA

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Theorem: Suppose an agent's betting quotients $q(X)$ are defined over a set of propositions $\mathcal{B}$. If those betting quotients meet the following conditions, there does not exist a Dutch Book against the agent consisting of a single bet:

1. $q(\top) = 1$ for any tautology $\top \in \mathcal{B}$.
2. $q(\bot) = 0$ for any contradiction $\bot \in \mathcal{B}$.
3. $0 \leq q(X) \leq 1$ for all $X \in \mathcal{B}$.

Proof: Suppose for *reductio* that the agent's quotients meet the three conditions and that a single-bet Book against him exists. Let's call this bet "Bet B". A bet in a Dutch Book is based on the agent's betting quotient for a particular proposition and a stake. Let's call the proposition that Bet B is based on proposition "A", the agent's quotient for A "q", and the stake "s". Bet B then has the following payoffs for the agent:

$$\begin{array}{rl} s - qs & \text{if } A \\ -qs & \text{if not } A \end{array}$$

A could be a tautology, a contradiction, or a contingent proposition. So there are three cases we need to consider:

Case 1: A is a tautology. In this case, the payoff for the agent will be $s - qs$ no matter what. For Book to be made, there must be an s value such that $s - qs < 0$. There are three possibilities: s is positive, s is negative, or s is zero.
Case 1.1: s is positive. Then we have

$$\begin{aligned} s - qs &< 0 \\ s(1-q) &< 0 \\ 1 - q &< 0 \\ 1 &< q \end{aligned}$$

But this contradicts Condition 1.
Case 1.2: s is negative. Then

$$\begin{aligned} s - qs &< 0 \\ s(1-q) &< 0 \\ 1 - q &> 0 \\ 1 &> q \end{aligned}$$

which also contradicts Condition 1.
Case 1.3: s is zero. Then $s - qs = 0$, so no Book can be made.

So no single-bet Book can be made if A is a tautology.

Case 2: A is a contradiction. No matter what, the payoff for the agent will then be $-qs$, which must be negative for Book. We will consider the same three sub-cases:
Case 2.1: s positive.

$$-qs < 0$$
$$q > 0$$

This contradicts Condition 2.
Case 2.2: s negative.

$$-qs < 0$$
$$q < 0$$

Also contradicts Condition 2.
Case 2.3: s is zero. Then $-qs = 0$, so no Book.

So no single-bet Book can be made if A is a contradiction.

Case 3: A is a contingent proposition. For the agent to lose no matter what, both $s - qs$ and $-qs$ must be negative. Again, we consider three sub-cases:
Case 3.1: s is positive. As we saw in Case 1.1, for $s - qs$ to be negative when s is positive, q must be greater than 1. But this contradicts Condition 3.
Case 3.2: s is negative. As we saw in Case 2.2, for $-qs$ to be negative when s is negative, q must be less than 0. This also contradicts Condition 3.
Case 3.3: s is zero. As we saw in Cases 1.3 and 2.3, a negative payoff is impossible when s equals zero, so no Book is possible here.

So no single-bet Book can be made if A is a contingent proposition.

If a one-bet Dutch Book can be made against the agent, its single bet B must be made against some proposition A. But A must be either a tautology, a contradiction, or a contingent proposition, and we have found that a single-bet Book is possible in none of these three cases if the agent meets Conditions 1 through 3. Thus no single-bet Dutch Book is possible against an agent whose betting quotients meet Conditions 1 through 3. QED

Note that this result entails that for any agent whose betting quotients meet Conditions 1 through 3, if there exists a Dutch Book against the agent it must employ multiple bets, and so must invoke a “package principle”.