Philosophy 148 — Announcements & Such

- Administrative Stuff
  - Branden's Thursday office hours will be 2:30–3:30 this week.
  - Raul's office hours will be 10–12 Wed., and by appointment.
  - Section times have been determined. Sections will meet Tuesday, 10–11 and Wednesday, 9–10. You should have received an email assigning you to a section. Otherwise, please see Raul about this.
  - Section locations will be announced soon. Meanwhile, 301 Moses.
- Last Time: Finite Boolean Algebras & Some Overview Stuff
- Today's Agenda
  - Review of Key Facts About Finite Propositional Boolean Algebras
  - Some Additional "Big Picture" Stuff (on Logic & Epistemology)
  - An Algebraic Approach to Probability Calculus
  - Next: An Axiomatic Approach to Probability Calculus

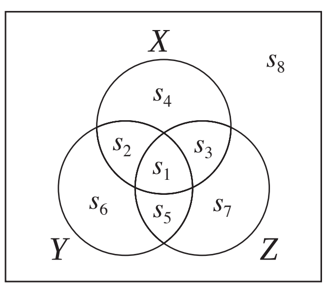
Overview of Finite Propositional Boolean Algebras I

- Consider a logical language $\mathcal{L}$ containing n atomic sentences. These may be sentence letters (X, Y, Z, *etc.*), or they may be atomic sentences of monadic or relational predicate calculus (Fa, Gb, Rab, Hcd, *etc.*).
- The Boolean Algebra $\mathcal{B}_{\mathcal{L}}$ set-up by such a language will be such that:
  - $\mathcal{B}_{\mathcal{L}}$ will have 2^n *states* (corresponding to the *state descriptions* of $\mathcal{L}$)
  - $\mathcal{B}_{\mathcal{L}}$ will contain 2^{2^n} *propositions*, in total.
    - ∗ This is because each proposition p in $\mathcal{B}_{\mathcal{L}}$ is equivalent to a disjunction of state descriptions. Thus, each subset of the set of state descriptions of $\mathcal{L}$ corresponds to a proposition of $\mathcal{B}_{\mathcal{L}}$.
    - ∗ Note: there are 2^{2^n} subsets of a set of size 2^n.
      - · The empty set $\varnothing$ of state descriptions corresponds to "the empty disjunction", which corresponds to *the logical falsehood*: $\perp$.
      - · Singelton sets of state descriptions correspond to "disjunctions with one member". [All other subsets are "normal" disjunctions.]

Overview of Finite Propositional Boolean Algebras II

- Example. Let $\mathcal{L}$ have three atomic sentences: X, Y, and Z. Then, $\mathcal{B}_{\mathcal{L}}$ is:

X	Y	Z	States
T	T	T	s_1
T	T	F	s_2
T	F	T	s_3
T	F	F	s_4
F	T	T	s_5
F	T	F	s_6
F	F	T	s_7
F	F	F	s_8



- Examples of reduction to disjunctions of state descriptions of $\mathcal{L}$:
  - '$X \,\&\, {\sim}X$' is equivalent to the *empty* disjunction: $\perp$.
  - '$X \,\&\, ({\sim}Y \,\&\, Z)$' is equivalent to the *singleton* disjunction: s_3.
  - '$X \leftrightarrow (Y \vee Z)$' is equivalent to: $s_1 \vee s_2 \vee s_3 \vee s_8$.
- In general: $p \dashv\vDash \bigvee\{s_i \mid s_i \vDash p\}$. And, if $\{s_i \mid s_i \vDash p\} = \varnothing$, then $p \dashv\vDash \perp$.

Inductive Logic — Basic Motivation and Ideas

- Intuitively, not all "logically good" arguments are deductively valid. Some invalid arguments seem (intuitively) logically *better than* others:

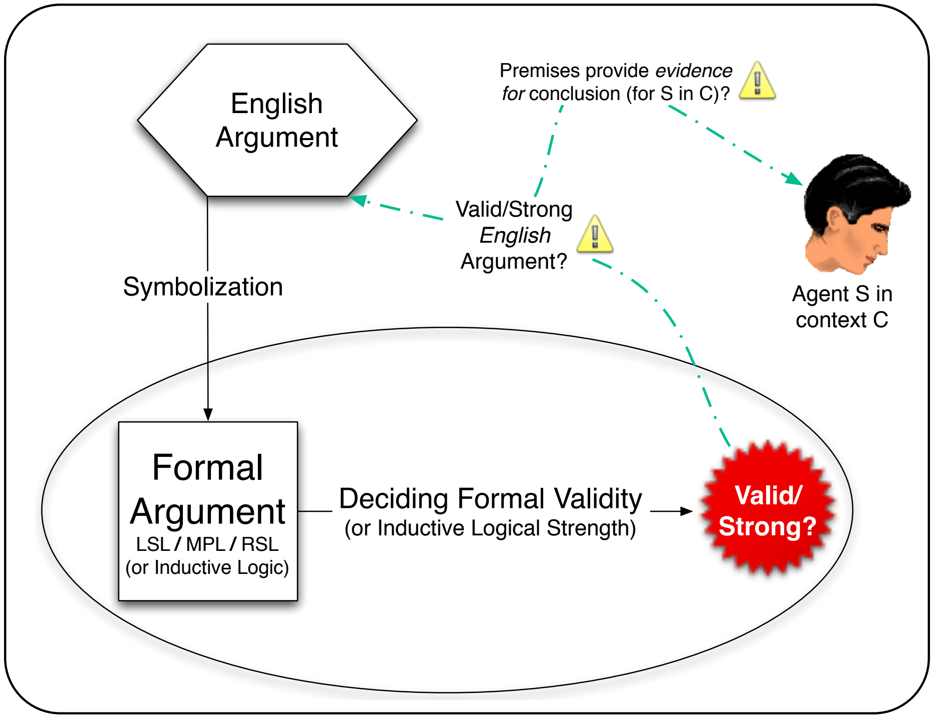
(6) p. Someone is wise. $\therefore q$. Socrates is wise.

(7) r. Someone is either wise or unwise. $\therefore q$. Socrates is wise.

- *Inductive* logic should *theoretically ground* our intuition that (6) is a *logically stronger* argument than (7) is. Neither argument is *valid*.
- More ambitiously, an inductive logician might aim for a theory of "the *degree* to which the premises of an argument *confirm* its conclusion".
- This ambitious project would aim to characterize a *function* $\mathfrak{c}(\mathscr{C}, \mathscr{P})$. And, an intuitive requirement would be that this function be such that:

$$\mathfrak{c}(q, p) > \mathfrak{c}(q, r)$$

- This course is (mainly) about *inductive logic*. We will examine how *probabilities* might be used to *quantitatively generalize* deductive logic.



Logic and Epistemology — A Prelude I

- As I mentioned, some have worried about the adequacy of classical logic as a formal explication of our informal "following-from" relation.
- Here's a fact about classical deductive logic that may seem "odd":

(†) If p and q are (classically) logically inconsistent, then the argument from p and q to r is (classically) valid — *for **any** r*.

- There's *something* "odd" about the fact that *everything follows-from inconsistent premises*, according to the classical formal explication of following-from. But, what, exactly, is supposed to be "odd" about it?
- Here's an *epistemological* principle that is downright *crazy*:

(‡) If one's beliefs are inconsistent (and one knows that they are), then one should believe everything (*i.e.*, every proposition).

- It is clear that (‡) is false. There are things I *know* to be false, and I shouldn't believe those things — no matter what else is true of me.

Logic and Epistemology — A Prelude II

- OK, (‡) is clearly false. So? What does that have to do with (†)?
- After all, (†) is a about *logic*, and (‡) is about *epistemology*.
- Perhaps those worried about (†) are assuming that logic and epistemology are connected, or bridged by something like:

(*) If an agent S's belief set B is such that $B \vDash p$ (and S knows that $B \vDash p$), then it would be reasonable for S to infer/believe p.

- If (*) were true, then (†) would imply (‡), and — as a result — classical logicians who accepted (*) would seem to be stuck with (‡) too.
- More precisely, classical logicians who believe (*) should find it reasonable to believe (‡). But, they don't (at least, they shouldn't!).

☞ But, *this* doesn't *force* classical logicians to give up (†). They could give up (*) instead. In such contexts, logic (alone) doesn't seem to tell us whether to infer something new, or reject something we already believe.

The Probability Calculus: An Algebraic Approach I

- Once we grasp the concept of a finite Boolean algebra of propositions, understanding the probability calculus *algebraically* is very easy.
- The central concept is a *finite probability model*. A finite probability model $\mathcal{M}$ is a finite Boolean algebra of propositions $\mathcal{B}$, together with a function $\Pr(\cdot)$ which maps elements of $\mathcal{B}$ to the unit interval $[0,1] \in \mathbb{R}$.
- This function $\Pr(\cdot)$ must be a *probability function*. It turns out that a probability function $\Pr(\cdot)$ on $\mathcal{B}$ is just a function that assigns a real number on $[0,1]$ to each state s_i of $\mathcal{B}$, such that $\sum_i \Pr(s_i) = 1$.
- Once we have $\Pr(\cdot)$'s *basic assignments* to the states of $\mathcal{B}$ (s.d.'s of $\mathcal{L}$), we define $\Pr(p)$ for *any* statement $\mathcal{L}$ of the language of $\mathcal{B}$, as follows:

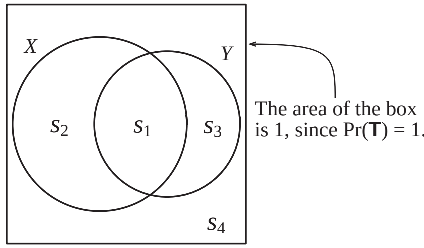
$$\Pr(p) = \sum_{s_i \vDash p} \Pr(s_i) \qquad \text{[note: if } p \dashv\vDash \bot \text{, then } \Pr(p) = 0]$$

- In other words, $\Pr(p)$ is the sum of the probabilities of the state descriptions in p's (equivalent) disjunction of state descriptions.

The Probability Calculus: An Algebraic Approach II

- Here's an example of a finite probability model $\mathcal{M}$, whose algebra $\mathcal{B}$ is characterized by a language $\mathcal{L}$ with two atomic letters "X" and "Y":

X	Y	States	$\Pr(s_i)$
T	T	s_1	$\frac{1}{6}$
T	F	s_2	$\frac{1}{4}$
F	T	s_3	$\frac{1}{8}$
F	F	s_4	$\frac{11}{24}$



- On the left, a *stochastic truth-table* (STT) representation of $\mathcal{M}$; on the right, a *stochastic Venn Diagram* (SVD) representation, in which *area is proportional to probability*. This is a *regular* model: $\Pr(s_i) > 0$, for all i.
- $\mathcal{M}$ determines a *numerical* probability for *each* p in $\mathcal{L}$. Examples?
- We can also use STTs to furnish an algebraic method for *proving* ***general*** *facts* about *all* probability models — *the algebraic method*.

The Probability Calculus: An Algebraic Approach III

- Let $a_i = \Pr(s_i)$ be the probability [under the probability assignment $\Pr(\cdot)$] of state s_i in $\mathcal{B}$ — *i.e.*, the area of region s_i in our SVD.
- Once we have real variables (a_i) for each of the basic probabilities, we can not only calculate probabilities relative to *specific* numerical models — *we can say* ***general*** *things, using only simple high-school algebra.*
- That is, we can *translate* any expression $\ulcorner\Pr(p)\urcorner$ into a *sum* of some of the a_i, and thus we can *reduce probabilistic* claims about the p's in $\mathcal{B}/\mathcal{L}$ into simple, high-school-*algebraic* claims about the real variables a_i.
- This allows us to be able to prove general claims about *probability functions*, by proving their corresponding *algebraic theorems*.
- Method: translate the probability claim into a claim involving sums of the a_i, and determine whether the corresponding claim is a theorem of algebra (assuming only that the a_i are on $[0,1]$ and that they sum to 1).

The Probability Calculus: An Algebraic Approach IV

- Here are two simple/obvious examples involving two atomic sentences:

  Theorem. $\Pr(X \vee Y) = \Pr(X) + \Pr(Y) - \Pr(X \,\&\, Y)$.

  Proof. $\Pr(X \vee Y) = a_1 + a_2 + a_3 = (a_1 + a_2) + (a_1 + a_3) - a_1$.

  Theorem. $\Pr(X) = \Pr(X \,\&\, Y) + \Pr(X \,\&\, {\sim}Y)$.

  Proof. $a_1 + a_2 = a_1 + a_2$.

- Here are two general facts that are also obvious from the set-up:

  Theorem. If $p \dashv\vDash q$, then $\Pr(p) = \Pr(q)$.

  Proof. Obvious, since the same regions always have the same areas, and the algebraic translation is *the same* for logically equivalent p/q.

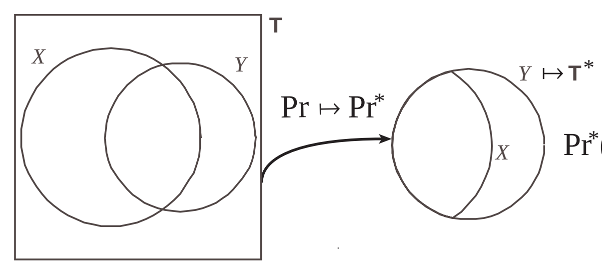
  Theorem. If $p \vDash q$, then $\Pr(p) \leq \Pr(q)$.

  Proof. Since $p \vDash q$, the set of state descriptions entailing p is a subset of the set of state descriptions entailing q. Thus, the set of a_i in the summation for $\Pr(p)$ will be a subset of the a_i in the summation for $\Pr(q)$. Thus, since all the $a_i \geq 0$, $\Pr(p) \leq \Pr(q)$.

The Probability Calculus: An Algebraic Approach V

- **Conditional Probability**. $\Pr(p \mid q) \overset{\text{def}}{=} \dfrac{\Pr(p \,\&\, q)}{\Pr(q)}$, *provided that* $\Pr(q) > 0$.
- Intuitively, $\Pr(p \mid q)$ is supposed to be the probability of p ***given that*** q *is true*. So, *conditionalizing* on q is like "supposing q to be true".
- Using Venn diagrams, we can explain: "Supposing Y to be true" is like "treating the Y-circle as if it is the bounding box of the Venn Diagram".
- This is like "moving to a new $\Pr^*(\cdot)$ such that $\Pr^*(Y) = 1$." Picture:

$\Pr(X)$



$$\Pr^*(X) = \Pr(X \mid Y) = \frac{\Pr(X \,\&\, Y)}{\Pr(Y)}$$

The Probability Calculus: An Algebraic Approach VI

- There may be other ways of defining conditional probability, which may also seem to capture the "supposing q to be true" intuition.
- But, any such definition must make $\Pr(\cdot \mid q)$ itself a *probability function*, *for all* q. We will look at this important constraint again (and in more generality), when we discuss the axiomatic approach to probability.
- But, algebraically, we can see that this is a strong constraint. Recall:

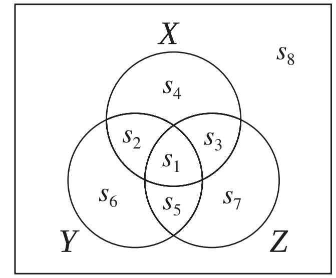
$$\Pr(X \vee Y) = \Pr(X) + \Pr(Y) - \Pr(X \,\&\, Y).$$

- Therefore, if $\Pr(\cdot \mid q)$ is to be a *probability* function *for all* q, then we must also have the following equality (in general), *for all* Z:

$$\Pr(X \vee Y \mid Z) = \Pr(X \mid Z) + \Pr(Y \mid Z) - \Pr(X \,\&\, Y \mid Z).$$

- Using our algebraic method, we can *prove* this. We just need to remind ourselves of what the 3-atomic sentence algebra looks like, and how the algebraic translation of this equation would go. Let's do that ...

X	Y	Z	States	$\Pr(s_i)$
T	T	T	s_1	a_1
T	T	F	s_2	a_2
T	F	T	s_3	a_3
T	F	F	s_4	a_4
F	T	T	s_5	a_5
F	T	F	s_6	a_6
F	F	T	s_7	a_7
F	F	F	s_8	a_8



- By our definition of conditional probability, we have:

$$\Pr(X \vee Y \mid Z) = \frac{\Pr((X \vee Y) \,\&\, Z)}{\Pr(Z)} = \frac{\Pr((X \,\&\, Z) \vee (Y \,\&\, Z))}{\Pr(Z)} = \frac{a_1 + a_3 + a_5}{a_1 + a_3 + a_5 + a_7}$$

and

$$\Pr(X \mid Z) + \Pr(Y \mid Z) - \Pr(X \& Y \mid Z) = \frac{\Pr(X \,\&\, Z)}{\Pr(Z)} + \frac{\Pr(Y \,\&\, Z)}{\Pr(Z)} - \frac{\Pr(X \,\&\, Y \,\&\, Z)}{\Pr(Z)}$$

$$= \frac{\Pr(X \,\&\, Z) + \Pr(Y \,\&\, Z) - \Pr(X \,\&\, Y \,\&\, Z)}{\Pr(Z)}$$

$$= \frac{(a_1 + a_3) + (a_1 + a_5) - a_1}{a_1 + a_3 + a_5 + a_7} = \frac{a_1 + a_3 + a_5}{a_1 + a_3 + a_5 + a_7}$$

The Probability Calculus: An Algebraic Approach VII

- We can use our algebraic method to demonstrate that our definition of $\Pr(\cdot \mid q)$ yields a probability function, for all q, in the following way.
- Intuitively, think about what an "unconditional" and a "conditional" stochastic truth-table must look like, for any pair of sentences p and q.

p	q	$\Pr(s_i)$
T	T	a_1
T	F	a_2
F	T	a_3
F	F	a_4

$$\xrightarrow{\cdot \mid q}$$

p	q	$\Pr(s_i \mid q)$
T	T	$\Pr(s_1 \mid q) \overset{\text{def}}{=} \frac{\Pr(s_1 \& q)}{\Pr(q)} = \frac{a_1}{a_1+a_3}$
T	F	$\Pr(s_2 \mid q) \overset{\text{def}}{=} \frac{\Pr(s_2 \& q)}{\Pr(q)} = 0$
F	T	$\Pr(s_3 \mid q) \overset{\text{def}}{=} \frac{\Pr(s_3 \& q)}{\Pr(q)} = \frac{a_3}{a_1+a_3}$
F	F	$\Pr(s_4 \mid q) \overset{\text{def}}{=} \frac{\Pr(s_4 \& q)}{\Pr(q)} = 0$

- Note: the new basic probabilities assigned to the state descriptions, under our "conditionalized" $\Pr(\cdot \mid q)$ satisfy the requirements for being a *probability* function, since $\frac{a_1}{a_1+a_3} + \frac{a_3}{a_1+a_3} = 1$, and $\frac{a_1}{a_1+a_3}, \frac{a_3}{a_1+a_3} \in [0,1]$.

The Probability Calculus: An Algebraic Approach VIII

- Here's a neat theorem of the probability calculus, proved algebraically.

Theorem. $\Pr(X \to Y) \geq \Pr(Y \mid X)$. [Provided that $\Pr(X) > 0$, of course.]

Proof. $\Pr(X \to Y) = \Pr({\sim}X \vee Y) = \Pr(s_1 \vee s_3 \vee s_4) = a_1 + a_3 + a_4$.

$$\Pr(Y \mid X) = \frac{\Pr(Y \,\&\, X)}{\Pr(X)} = \frac{\Pr(s_1)}{\Pr(s_1 \vee s_2)} = \frac{a_1}{a_1 + a_2}.$$

So, we need to prove that $\boxed{a_1 + a_3 + a_4 \geq \frac{a_1}{a_1 + a_2}}$.

- First, note that $a_4 = 1 - (a_1 + a_2 + a_3)$, since the a_i's must sum to 1.
- Thus, we need to show that $a_1 + a_3 + 1 - a_1 - a_2 - a_3 \geq \frac{a_1}{a_1 + a_2}$.
- By simple algebra, this reduces to showing that $\boxed{1 - a_2 \geq \frac{a_1}{a_1 + a_2}}$.
- If $a_1 + a_2 > 0$ and $a_i \in [0, 1]$, this must hold, since then we must have: $\boxed{a_2 \geq a_2 \cdot (a_1 + a_2)}$, and then the boxed formulas are equivalent. $\square$

The Probability Calculus: An Algebraic Approach IX

- Here are some further fundamental theorems of probability calculus, involving 2 or 3 atomic sentences and CP. Easy, given defn. of CP.
  - **The Law of Total Probability** (LTP):

$$\Pr(X \mid Y) = \Pr(X \mid Y \,\&\, Z) \cdot \Pr(Z \mid Y) + \Pr(X \mid Y \,\&\, {\sim}Z) \cdot \Pr({\sim}Z \mid Y)$$

  - Note: $\Pr(X \mid \top) = \Pr(X)$. Why? So, the LTP has a *special case*:

$$\begin{aligned}\Pr(X \mid \top) = \Pr(X) &= \Pr(X \mid \top \,\&\, Z) \cdot \Pr(Z \mid \top) + \Pr(X \mid \top \,\&\, {\sim}Z) \cdot \Pr({\sim}Z \mid \top)\\ &= \Pr(X \mid Z) \cdot \Pr(Z) + \Pr(X \mid {\sim}Z) \cdot \Pr({\sim}Z)\end{aligned}$$

  - Two forms of **Bayes's Theorem**. The second one *follows*, using (LTP):

$$\begin{aligned}\Pr(X \mid Y) &= \frac{\Pr(Y \mid X) \cdot \Pr(X)}{\Pr(Y)}\\ &= \frac{\Pr(Y \mid X) \cdot \Pr(X)}{\Pr(Y \mid Z) \cdot \Pr(Z) + \Pr(Y \mid {\sim}Z) \cdot \Pr({\sim}Z)}\end{aligned}$$

The Probability Calculus: An Algebraic Approach X

- One more interesting theorem (due to Popper & Miller), algebraically.
- Let $d(X,Y) \overset{\text{def}}{=} \Pr(X \mid Y) - \Pr(X)$. Then, we have the following theorem:

  Theorem (PM). $d(X,Y) = d(X \vee Y, Y) + d(X \vee {\sim}Y, Y)$.

  Proof (algebraic, using STT from X/Y language, above).

$$d(X,Y) \overset{\text{def}}{=} \Pr(X \mid Y) - \Pr(X) = \frac{a_1}{a_1 + a_3} - (a_1 + a_2)$$

$$d(X \vee Y, Y) \overset{\text{def}}{=} \Pr(X \vee Y \mid Y) - \Pr(X \vee Y) = 1 - a_1 - a_2 - a_3$$

$$d(X \vee {\sim}Y, Y) \overset{\text{def}}{=} \Pr(X \vee {\sim}Y \mid Y) - \Pr(X \vee {\sim}Y) = \frac{a_1}{a_1 + a_3} - (a_1 + a_2 + a_4)$$

$$\therefore\ d(X \vee Y, Y) + d(X \vee {\sim}Y, Y) = 1 - a_1 - a_2 - a_3 + \frac{a_1}{a_1 + a_3} - a_1 - a_2 - a_4$$

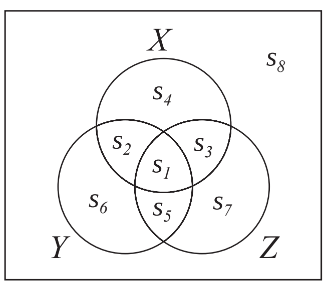
$$= \frac{a_1}{a_1 + a_3} + 1 - a_1 - a_2 - a_3 - a_1 - a_2 - (1 - (a_1 + a_2 + a_3))$$

$$= \frac{a_1}{a_1 + a_3} - (a_1 + a_2). \quad \square$$

The Probability Calculus: An Algebraic Approach XI

- The algebraic approach for *refuting* general claims involves two steps:
  1. Translate the claim from probability notation into algebraic terms.
  2. Find a (numerical) probability model on which the translation is *false*.
- Show that $\Pr(X \mid Y \,\&\, Z) = \Pr(X \mid Y \vee Z)$ can be *false*. Here's a model $\mathcal{M}$:

X	Y	Z	States	$\Pr(s_i)$
T	T	T	s_1	$a_1 = 1/6$
T	T	F	s_2	$a_2 = 1/6$
T	F	T	s_3	$a_3 = 1/4$
T	F	F	s_4	$a_4 = 1/16$
F	T	T	s_5	$a_5 = 1/6$
F	T	F	s_6	$a_6 = 1/12$
F	F	T	s_7	$a_7 = 1/24$
F	F	F	s_8	$a_8 = 1/16$



(1) Algebraic Translation: $\dfrac{a_1}{a_1 + a_5} = \dfrac{a_1 + a_2 + a_3}{a_1 + a_2 + a_3 + a_5 + a_6 + a_7}$.

(2) This claim is *false* on $\mathcal{M}$, since $1/2 \neq 2/3$. I used `PrSAT` to find $\mathcal{M}$.

The Probability Calculus: An Algebraic Approach XII

- There are *decision procedures* for Boolean propositional logic, based on truth-tables. These methods are *exponential* in the number of atomic sentences (n), because truth-tables grow exponentially in n (2^n).
- It would be nice if there were a decision procedure for probability calculus, too. In algebraic terms, this would require a decision procedure for the salient fragment of high-school (real) algebra.
- As it turns out, high-school (real) algebra (HSA) *is* a decidable theory. This was shown by Tarski in the 1920's. But, it's only been very recently that computationally feasible procedures have been developed.
- In my "A Decision Procedure for Probability Calculus with Applications", I describe a user-friendly decision procedure (called `PrSAT`) for probability calculus, based on recent HSA procedures.
- My implementation is written in *Mathematica* (a general-purpose mathematics computer programming framework). It is freely downloadable from my website, at: `http://fitelson.org/PrSAT/`.

The Probability Calculus: An Algebraic Approach XIII

- I encourage the use of PrSAT as a tool for finding counter-models and for establishing theorems of probability calculus. It is not a requirement of the course, but it is a useful tool that is worth learning.
- PrSAT doesn't give readable proofs of theorems. But, it will find concrete numerical counter-models for claims that are not theorems.
- PrSAT will also allow you to calculate probabilities that are determined by a *given* probability assignment. And, it will allow you to do algebraic and numerical "scratch work" without making errors.
- I have posted a *Mathematica* notebook which contains the examples from algebraic probability calculus that we have seen in this lecture. I will be posting further notebooks as the course goes along.
- Let's have a look at this first notebook (`examples_1.nb`). I will now go through the examples in this notebook, and demonstrate some of the features of PrSAT. I encourage you to play around with it.

Axiomatic Treatment of Probability Calculus I

- A probability model $\mathcal{M}$ is a Boolean algebra of propositions $\mathcal{B}$, together with a function $\Pr(\cdot) : \mathcal{B} \mapsto \mathbb{R}$ satisfying the following three *axioms*.
  1. For all $p \in \mathcal{B}$, $\Pr(p) \geq 0$. [non-negativity]
  2. $\Pr(\top) = 1$, where $\top$ is the tautological proposition. [normality]
  3. For all $p, q \in \mathcal{B}$, if p and q are mutually exclusive (inconsistent), then $\Pr(p \vee q) = \Pr(p) + \Pr(q)$. [additivity]
- Conditional probability is *defined* in terms of unconditional probability in the usual way: $\Pr(p \mid q) \stackrel{\text{def}}{=} \frac{\Pr(p \& q)}{\Pr(q)}$, provided that $\Pr(q) > 0$.
- We could also state everything in terms of a (propositional) *language* $\mathcal{L}$ with a finite number of atomic *sentences*. Then, we would talk about *sentences* rather than *propositions*, and the axioms would read:
  1. For all $p \in \mathcal{L}$, $\Pr(p) \geq 0$.
  2. For all $p \in \mathcal{L}$, if $p \dashv\vDash \top$, then $\Pr(p) = 1$.
  3. For all $p, q \in \mathcal{L}$, if $p \,\&\, q \dashv\vDash \bot$, then $\Pr(p \vee q) = \Pr(p) + \Pr(q)$.

Axiomatic Treatment of Probability Calculus II

- Instead of using the algebraic approach for proving theorems, we can also give *axiomatic* proofs. This is the standard way of proving claims in probability calculus (PrSAT doesn't give proofs, so we need axioms).
- Here are two examples of theorems and their *axiomatic* proofs (see the Eells *Appendix*). Note: these are *trivial* from an *algebraic* point of view!

  Theorem. $\Pr(\sim p) = 1 - \Pr(p)$.
  Proof. Since $p \vee \sim p$ is a tautology, (2) implies $\Pr(p \vee \sim p) = 1$; and since p and $\sim p$ are m.e., (3) implies $\Pr(p \vee \sim p) = \Pr(p) + \Pr(\sim p)$. Therefore, $1 = \Pr(p) + \Pr(\sim p)$, and thus $\Pr(\sim p) = 1 - \Pr(p)$, by simple algebra. □

  Theorem. If $p \dashv\vDash q$, then $\Pr(p) = \Pr(q)$. *Proof*. Assume $p \dashv\vDash q$. Then, p and $\sim q$ are mutually exclusive (inconsistent), and $p \vee \sim q \dashv\vDash \top$. So by axioms (2) and (3), and the previous theorem [$\Pr(\sim p) = 1 - \Pr(p)$]:

$$1 = \Pr(p \vee \sim q) = \Pr(p) + \Pr(\sim q) = \Pr(p) + 1 - \Pr(q)$$

  So, $1 = \Pr(p) + 1 - \Pr(q)$, and $0 = \Pr(p) - \Pr(q)$. $\therefore \Pr(p) = \Pr(q)$. □

Axiomatic Treatment of Probability Calculus III

- Here are two more axiomatic proofs:

  Theorem. If $p \dashv\vDash \bot$, then $\Pr(p) = 0$.
  Proof. Assume $p \dashv\vDash \bot$. Then, $\sim p \dashv\vDash \top$, and, by (2), $\Pr(\sim p) = 1$. Then, by the above theorem, $\Pr(\sim p) = 1 - \Pr(p) = 1$, and $\Pr(p) = 0$. □

  Theorem. If $p \vDash q$, then $\Pr(p) \leq \Pr(q)$.
  Proof. First, note the following two Boolean equivalences:

$$p \dashv\vDash (p \,\&\, q) \vee (p \,\&\, \sim q)$$
$$q \dashv\vDash (p \,\&\, q) \vee (\sim p \,\&\, q)$$

  Thus, by our theorem above, we must have the following two identities:

$$\Pr(p) = \Pr[(p \,\&\, q) \vee (p \,\&\, \sim q)]$$
$$\Pr(q) = \Pr[(p \,\&\, q) \vee (\sim p \,\&\, q)]$$

By axiom (3), this yields the following two identities:

$$\Pr(p) = \Pr(p \,\&\, q) + \Pr(p \,\&\, {\sim}q)$$
$$\Pr(q) = \Pr(p \,\&\, q) + \Pr({\sim}p \,\&\, q)$$

Now, assume $p \vDash q$. Then, $p \,\&\, {\sim}q \dashv\vDash \bot$. Hence, by our theorem above, $\Pr(p \,\&\, {\sim}q) = 0$. And, under these circumstances, we must have:

$$\Pr(p) = \Pr(p \,\&\, q)$$
$$\Pr(q) = \Pr(p \,\&\, q) + \Pr({\sim}p \,\&\, q)$$

That is to say, we must have the following:

$$\Pr(q) = \Pr(p) + \Pr({\sim}p \,\&\, q)$$

But, by axiom (1), $\Pr({\sim}p \,\&\, q) \geq 0$. So, by algebra, $\Pr(q) \geq \Pr(p)$. $\square$

- This gives us an alternative way to prove $p \dashv\vDash q \implies \Pr(p) = \Pr(q)$. We just apply the previous theorem, in both directions (plus algebra).
- You should now be able to prove that $\Pr(p) \in [0, 1]$, for all p.