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FAIR BETS AND INDUCTIVE PROBABILITIES[1]

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1. Introduction. The question of what constitutes fairness in betting quotients has been studied by Ramsey, deFinetti, and Shimony.[3] Thanks to their combined efforts we now have a satisfactory definition of fairness.

On the other hand, the explication of the concept of *degree of confirmation* (inductive probability) has progressed rapidly in recent years, thanks primarily to Carnap.[4] This explication has usually proceeded by laying down the axioms for frequency-probabilities, and elaborating on these. While in the case where a frequency interpretation is intended these axioms are clearly justified,[5] in our case they have been laid down without any justification. Carnap's presentation has been criticized for just this reason.[6]

The purpose of this paper is to show that the probability axioms are necessary and sufficient conditions to assure that the degrees of confirmation form a set of fair betting quotients. In addition it will be shown that one additional, highly controversial, axiom is precisely the condition needed to assure that not only deFinetti's weaker criterion but Shimony's criterion of fairness is also satisfied.

2. Definition of fairness. Let us first clarify what constitutes a bet. Under certain circumstances e, two people wager money on whether a certain event h will or will not take place. It may be an "even money" bet, but in general "odds" may be given. If a person offers to pay a sum qS if he is wrong, and is to receive $(1-q)S$ if he is right, then he is giving the odds $q:(1-q)$. We will say that q is the *betting quotient*, and S is *the stake*. A bet is determinate if we know: (1) e and h. (2) q and S. (3) whether the first person is betting on or against h's taking place.

We now suppose that we have a method for fixing the betting quotients

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[1] A summary of sections 1–5 was presented to the conference on induction, May 1953, at New York University, sponsored by the Institute for the Unity of Science. Section 6 has been added more recently. It has come to my attention since then that Sherman Lehman found a similar result independently of me. He will publish this in a forthcoming paper.

[2] The author wishes to thank S. Schanuel for his aid in the preparation of this manuscript. He is also indebted to Mr. Schanuel for proposing the problem solved in section 6.

[3] See [2], [3], and [9]. Shimony's improvement on the previous work was communicated to the author orally, and forms part of a paper by Shimony, this JOURNAL, vol. 20 (1955), pp. 1–28.

[4] See [1].

[5] A complete justification is given in [10].

[6] See [6].

for all pairs (e, h). How do we decide whether these are fair quotients? The idea due to Ramsey and deFinetti is that if the quotients are really fair, we may allow our opponent to choose everything else about the bets he cares to make, and he is still unable to *guarantee* a profit for himself. Let us make this more precise.

A *betting system* consists of the following decisions: (1) what circumstances to bet under, and what hypotheses to bet on in these circumstances, i.e., the choice of a set of (e, h)-pairs; (2) the choosing of a stake for each pair; (3) the decision as to whether the bet is on or against h, in each case.

A set of betting quotients (for all (e, h)-pairs) is *fair* if there is no betting system which will guarantee a profit.[7]

Shimony suggested a most intuitive strengthening of this requirement: A set of betting quotients is *strictly fair* if there is no betting system which will guarantee that there will be no loss and at the same time offer a possible gain.

The condition of fairness is violated if some betting system assures the gambler that, no matter what actually happens, he will profit. The condition of strict fairness is violated by a weaker guarantee, namely that under some possible outcomes the gambler will win, and under all others he will break even. Hence strict fairness implies fairness, and it certainly appears to be a most intuitive extension of it.

3. Relation of fairness to inductive probabilities.[8] The degree of confirmation (inductive probability) of a hypothesis h on evidence e, $c(h, e)$, has an important application to bets. If h is the hypothesis that a certain event will take place, and e describes certain circumstances, then $c(h, e)$ gives the "reasonable" betting quotient for a bet on this event under these circumstances. I.e., $c(h, e)$ is this so-called reasonable betting quotient for the pair (e, h).[9] Whatever else reasonableness may imply, it certainly implies fairness. And the author strongly believes that it implies strict fairness.

Hence we have the requirement that the function c thought of as defining a set of betting quotients should define a (strictly) fair set. We will show that the probability axioms are necessary conditions for the fulfillment of fairness.

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[7] The term 'coherent' is used by previous authors for what is here described as "fair" or "strictly fair." The present terminology is preferred because it is more suggestive, and it introduces a distinction between the weak and strong criteria. It must, of course, be remembered that 'fair' is used in a precisely defined, narrow sense.

[8] We will allow ourselves to use 'e' and 'h' to stand either for events, or for descriptions of events. Since nothing depends on the manner in which events are described, this ambiguity is permissible.

[9] More precisely, if we bet on h, we must offer c: $(1-c)$ odds. See [1], section 41B.

First of all, it is assumed in the description of a bet that qS and $(1-q)S$ are actual sums of money, hence each of them must be non-negative. Therefore q must be between 0 and 1.

(1) $0 \leq c(h, e) \leq 1$.

It is usually assumed that S is a positive amount, but this implies no restriction since only q concerns $c(h, e)$. Actually, it is very convenient to include 0 stakes to cover the case where no bet is made.

Again, the bets become nonsensical if the payments depend on the way events are described. Hence we have the rather obvious condition that c should depend only on the contents of h, e, not their form.

(2) If h and e are equivalent to h' and e', respectively, then $c(h, e) = c(h', e')$.

So far we have assured only that the bets make sense at all. Now let us consider the question of fairness.

Suppose that under circumstances e the event h is certain to take place. Then under circumstances e we adopt as a betting system: (1) Bet on h. (2) Bet a unit. (3) Bet on (rather than against) h. Then, as h is certain to take place, we are assured of a profit $1-q$. This profit must be nil if the bets are to be fair, hence q must be 1.

(3) If e implies h, then $c(h, e) = 1$.

Now consider two events h and h' which cannot both occur under circumstances e. Let the betting quotient of the first event be q_1, of the second q_2, and the quotient of the bet that one *or* the other takes place be q_3. Our system under these circumstances will be to make the three bets just described, each with a unit stake. If $q_3 > q_1+q_2$, we bet *on* the individual events, and *against* their disjunction. Otherwise we make the reversed bets. Let us suppose the former is the case.

There are three possible outcomes: (1) h takes place, but h' does not. We win the first bet gaining $1-q_1$, and lose the others with losses of q_2 and $1-q_3$. Our net gain is $q_3-q_1-q_2$. (2) h' takes place but h does not. Clearly, our gain is the same as before. (3) Neither event takes place. Then we suffer losses of q_1 and q_2 in the first two bets, but win q_3 in the third. Our net gain is again the same. We see that if $q_3 > q_1+q_2$ we have a guaranteed positive gain.

It is easily seen that in the reversed case we guarantee a gain of $q_1+q_2-q_3$. Hence q_3 cannot be less than q_1+q_2 either. So it must be equal to it.

(4) If e implies that h and h' cannot both be true, then $c(h \vee h', e) = c(h, e) + c(h', e)$.

This is the law of addition for probabilities.

Consider now two events h and h' which are supposed to occur in that order, if at all. Let q_1 be the quotient of the bet on both events occurring, q_2 on the first event alone. And let q_3 be the quotient on a bet on the second event which is placed after the first event has already taken place. Suppose $q_1 > q_2q_3$. We make all three bets (the third one being conditional on the occurrence of h), betting a unit *against* both events taking place, betting q_3 units *on* the first, and a unit *on* the second event. We again analyze the outcome into three possible cases: (1) The first event does not occur. Then we win bet number one, gaining q_1; we lose the second bet with a loss of q_2q_3; and the third bet is never made. The net profit is $q_1-q_2q_3$. (2) The first event takes place, but the second does not. We win the first two bets, gaining q_1 and $(1-q_2)q_3$. But we lose q_3 on the last bet. The net profit is the same as before. (3) Both h and h' occur. We lose $1-q_1$ on the first bet, and gain $(1-q_2)q_3$ and $1-q_3$ on the other two. The net profit is again the same. Hence q_1 cannot be greater than q_2q_3.

By reversing the bets we see that q_1 cannot be smaller than q_2q_3 either. Hence they must be equal. And it is important to note that the assumption about the order in which h and h' are to occur is irrelevant. By the obvious alteration in our system we can cover the reverse order as well.

(5) $c(h\&h', e) = c(h, e) \times c(h', h\&e)$.

This is the law of multiplication of probabilities.

(1)–(5), the usual laws of probability, have been shown to be necessary to assure that c defines fair bets.[10] If we require strict fairness, we can strengthen (3).

Suppose that h is *not* certain to take place under circumstances e, but $c(h, e)$ is nevertheless 1. We bet *against* h with a unit stake. If h does occur, our loss is $1-q = 0$. If h does not occur we gain $q = 1$. Hence we are assured of not losing, and we may win. This violates strict fairness. [Note that it does not violate fairness.] Hence for strict fairness the converse of (3) is also necessary.

(3′) $c(h, e) = 1$ if and only if e implies h.

We have now shown that (1)–(5) are necessary for fairness, and the stronger set is necessary for strict fairness. We will also show that these conditions suffice.

4. Betting systems. We will show that if c satisfies conditions (1)–(5), then it defines fair betting quotients. So far we have only made use of a few special betting systems. But now we must show that *no* betting system can guarantee a profit, and hence we need a general characterization of betting systems.

[10] Essentially this result (in a different formulation of the problem) was first found by deFinetti.

Let us suppose that the system is to go into effect when a certain initial information e^0 is available, and that the bets concern the outcome of N future events. Without loss of generality we may assume that there is a time-order for these events. A bet may concern the outcome of any hypothesis dependent on the N events, and we may either decide to place the bet initially or to wait for the outcome of the first n events and make our bet dependent on these. It will be convenient to think of the bets as all initially made. Some bets will be outright and some will be conditional on the outcome of n events.

Let e_i^n be a report of the outcome of the first n events added to e^0, where i can take as many values as there are possible outcomes. Let h_j^n be some hypothesis concerning the remaining $N-n$ events. (There are only a finite number of such hypotheses, at least if equivalent hypotheses are not to be counted as distinct.) So a bet may concern the pair (e_i^n, h_j^n). The odds are fixed by c; if we wish to bet on the hypothesis we must give $c(h_j^n, e_i^n) : 1-c(h_j^n, e_i^n)$ odds, while if we bet against it we receive the same odds. Our choice concerns the pairs, the stakes, and whether to bet on or against the hypothesis.

A very convenient standard form is found by choosing all pairs, and always betting both on and against the hypothesis — with stakes S_{ij}^{n1} and S_{ij}^{n2}, respectively. If both stakes are 0, the bet is in effect not made. If the first stake is positive and the second 0, we are betting on the hypothesis, while in the reverse case we are betting against it. The only novel feature of this characterization is that we are allowed to bet both on and against the hypothesis. But the more freedom we allow ourselves in laying bets, the stronger our theorem becomes. So our standard betting system is determined by the non-negative numbers S_{ij}^{n1} and S_{ij}^{n2}.

What will the payoff be? For any one bet we must consider three cases: (1) e_i^n does not take place, i.e., the first n events have some other outcome. Then the bet does not take effect. (2) e_i^n and h_j^n both take place. We win the bet on the hypothesis gaining $[1-c(h_j^n, e_i^n)]S_{ij}^{n1}$, and lose the bet against it with a loss of $[1-c(h_j^n, e_i^n)]S_{ij}^{n2}$. (Remember that in one case we gave the odds, and in the other we received them!) Our net profit is $[1-c(h_j^n, e_i^n)][S_{ij}^{n1}-S_{ij}^{n2}]$. (3) e_i^n takes place, but h_j^n does not. Then we win the second bet and lose the first one. Our net profit is $c(h_j^n, e_i^n)[S_{ij}^{n2}-S_{ij}^{n1}]$.

We note that the result depends in each case only on the difference $S_{ij}^n = S_{ij}^{n1} - S_{ij}^{n2}$, which is highly intuitive. A positive difference corresponds to betting on, a negative one against the hypothesis, while a 0 difference means that no bet was placed. So we see that the most general betting system is determined by the choice of the single set of numbers S_{ij}^n.

So far we have only considered the outcome of a single bet. But from the previous discussion it is immediately seen what the total net profit resulting from a betting system is. Of course this profit will depend on the outcome of the N events, but if this outcome is h_k^N, the profit is

(6) $prf(h_k^N) = \Sigma_1 [1-c(h_j^n, e_i^n)]S_{ij}^n + \Sigma_2 [-c(h_j^n, e_i^n)]S_{ij}^n$,

where the first sum is over all those bets for which h_k^N implies that both their e and their k took place, while the second sum is over bets where e did take place but h did not—according to h_k^N. These correspond to the last two cases discussed above. The first case contributes nothing to prf.

5. The converse theorems. It is now our task to show that the profit in (6) cannot always be positive, if c satisfies (1)–(5). Let us first define the c-mean estimate of profit, in accordance with the usual terminology:

(7) $Prf = \Sigma\, c(h_k^N, e^0) \times prf(h_k^N)$,

where the sum is taken over all possible values of k, i.e., all possible outcomes of the N events.

LEMMA. If c satisfies (1)–(5), then $Prf = 0$.

Proof. From (6) and (7) we see that Prf is a linear function of the S_{ij}^n, hence we may write it in the form

(8) $\Sigma\, a_{ij}^n S_{ij}^n$.

Let us find a_{ij}^n in terms of the c-values. The first sum in (6) will contribute $c(h_k^N, e^0)[1-c(h_j^n, e_i^n)]$ for all k such that h_k^N implies the occurrence of both e_i^n and h_j^n. The second sum will contribute $c(h_k^N, e^0)[-c(h_j^n, e_i^n)]$ for all k such that h_k^N implies the occurrence of e_i^n and the non-occurrence of h_j^n. Therefore,

(9) $a_{ij}^n = [1-c(h_j^n, e_i^n)]\, \Sigma_1\, c(h_k^N, e^0) - c(h_j^n, e_i^n)\, \Sigma_2\, c(h_k^N, e^0)$,

where the first sum is over outcomes that imply $e_j^n \& h_i^n$ while the second is over outcomes implying $e_i^n \& \sim h_j^n$.

The possible outcomes are mutually exclusive, and hence (4) may be applied to any pair of them. By repeated application each sum may be changed into a single c-value $c(h_{k_1}^N \vee \ldots \vee h_{k_r}^N, e^0)$ where the new hypothesis is the disjunction of the hypotheses occurring in the sum. Since the outcomes are exhaustive, the disjunction of all those implying a certain partial result is equivalent to the assertion of this partial result. Hence, by (2), the first sum may be replaced by $c(e_i^n \& h_j^n, e^0)$, and the second sum by $c(e_i^n \& \sim h_j^n)$. Putting these values into (9) and collecting terms we obtain

(10) $a_{ij}^n = c(e_i^n \& h_j^n, e^0) - c(h_j^n, e_i^n)[c(e_i^n \& h_j^n, e^0) + c(e_i^n \& \sim h_j^n, e^0)]$.

We can apply (4) to the sum in brackets. By (2), the result may be replaced by $c(e_i^n, e^0)$. We also note that since e_i^n implies e^0, it is equivalent to $e_i^n \& e^0$.

(11) $a_{ij}^n = c(e_i^n \& h_j^n, e^0) - c(h_j^n, e_i^n \& e^0) \times c(e_i^n, e^0)$.

But by (5) this must be 0. Since each coefficient in (8) is 0, the sum is 0. Hence Prf is 0. Q.E.D.

THEOREM 1. If c satisfies (1)–(5), then it defines fair betting quotients.

Proof. Suppose the set of quotients is not fair. Then for some betting system $prf(h_k^N) > 0$ for all possible outcomes. Consider the $c(h_k^N, e^0)$. Their sum may, by repeated uses of (4), be replaced by a single c-calue. In this the new hypothesis is a disjunction of all possible outcomes, and hence analytically true. Hence e^0 implies it. Therefore (3) tells us that the sum of the $c(h_k^N, e^0)$ is 1. By (1) we infer that all of them are non-negative and at least one is positive. But then $Prf > 0$, contrary to our lemma. Q.E.D.

THEOREM 2. If c satisfies (1), (2), (3′), (4), and (5), then it defines strictly fair betting quotients.

Proof. Suppose the set of quotients is not strictly fair. Then for some betting system all the $prf(h_k^N)$ are non-negative, and at least one is positive. $c(h_k^N, e^0)+c(\sim h_k^N, e^0) = c(h_k^N \vee \sim h_k^N, e^0) = 1$ by (4) and (3′). Since the outcome h_k^N is one of many possibilities according to e^0, its negation cannot be a consequence of e^0. Hence by (3′) $c(\sim h_k^N, e^0) \neq 1$. Hence by (1) it is less than 1. Hence $c(h_k^N) > 0$ for each outcome. Hence $Prf > 0$, contrary to our lemma. Q.E.D.

We have now shown that (1)–(5) are necessary and sufficient conditions for c to define fair betting quotients, and the stronger set of conditions is necessary and sufficient for strict fairness.

6. The choice of a c-function. In choosing a precise definition of inductive probabilities many factors enter besides considerations of fairness, factors that are far beyond the scope of this paper. But we can at least ask how much freedom of choice is left after we have assured that the betting quotients defined are fair. The answer is entirely different for fairness and for strict fairness.

To simplify the following discussion we will assume that there is a finite set of mutually exclusive and exhaustive strongest statements in our language.[11] (A statement is a strongest statement if it is self-consistent and there is no self-consistent statement which implies it and is not equivalent to it.) These are called the *state-descriptions* (*sds*) of the language. A *sd* has the property that for any statement it either implies the statement or its negation. And an exhaustive set enables us to write any statement as the disjunction of all those *sds* that imply it (rather than its negation). This is a convenient normal form for statements.[12]

[11] These results can be extended by employing the concept of a *model* in place of *sds*. This is described in [7], and developed in [8].

[12] A *sd* may be thought of as describing a possible state of the world. Every factual statement we make narrows down the possibilities. In its normal form we identify a statement with the assertion that one of the possible worlds in which it is true must be the real world.

Suppose we require strict fairness. Let t be an analytically true statement. From (5) we see that

(12) $c(h\&e, t) = c(e, t) \times c(h, e\&t)$.

The numbers $c(w, t)$ are the so-called *a priori* probabilities of statements, and the usual notation is

(13) $m(w) = c(w, t)$.

Using this in (12), and remembering that $e\&t$ is equivalent to e:

(14) $m(h\&e) = m(e) \times c(h, e)$.

Strict fairness helps us in assuring that if e is self-consistent (which has been taken for granted), then $c(e, t) = m(t) > 0$. We see this as in the proof of theorem 2. Hence we may write

(15) $c(h, e) = m(h\&e)/m(e)$.

And we see that the m-function determines the c-function. If we write w as a disjunction of *sds*, $z_1 \vee \ldots \vee z_s$, we have from (2) by repeated applications of (4):

(16) $m(w) = c(z_1 \vee \ldots \vee z_s, t) = c(z_1, t) + \ldots + c(z_s, t) =$
$= m(z_1) + \ldots + m(z_s)$.

Hence the choice of the $m(z)$ for the *sds* determines the m-function. These are known as the *weights* of the *sds*. By (1) the weights are non-negative, and by strict fairness they must be positive. Furthermore the analytically true statement is equivalent to a disjunction of all the *sds*, and since by (2) $m(t) = 1$, the sum of all the weights is 1.

We have shown that our only choice for a strictly fair c is the assignment of positive weights to the *sds*, whose sum is 1. It is a routine matter to check that any such assignment will define a strictly fair c-function.[13]

If we require only fairness, the situation is less simple. We still have (14); but if $m(e) = 0$, then $c(h, e)$ is not determined by this equation. We will introduce an iterative procedure, showing at each step what the maximum number of choices is. At the end we will prove that all these choices were actually available to us, by showing that any c-function determined by such choices is fair.

We already know that originally we may choose no more than the weights of the *sds*, which have to be non-negative and add up to 1. These define an m-function, call it $m_0(w)$, according to (16). That no more is open is seen by the fact that if all the weights are positive, then c is completely determined. Let us call this the 0th step. In the 1st step we are confronted with a

[13] In accordance with the intuitive description of *sds*, given in footnote 11, we see that all that we are allowed to do is to assign *a priori* probabilities to the possible worlds. No possible world may receive 0 probability, and the sum of the probabilities must be 1. The *a priori* probability of a statement is the sum of the probabilities of those worlds in which it would be true.

function $m_0(w)$, which defines $c(h, e)$ according to (15), whenever $m_0(e) > 0$.

Let Z_1 be the disjunction of all *sds* having received 0 weight.

(17) If e implies Z_1, $c(h\&e, Z_1) = c(e, Z_1) \times c(h, e)$.

And, clearly, the e's that imply Z_1 are precisely those which have $m_0(e) = 0$. So the doubtful c-values may all be determined by the $c(w, Z_1)$, where w implies Z_1.

(18) If w implies Z_1, $m_1(w) = c(w, Z_1)$.

At most the m_1-function is open in step 1, since choosing all positive values for the function fixes all the remaining c-values by (17). Since w can be written as a disjunction of *sds* forming part of Z_1, by (2) and (4) it suffices to fix the m_1-weights of these *sds*. By (1) these weights must be non-negative, and since by (3) $m_1(Z_1) = 1$, the sum of the weights must be 1. Hence in step 1 our task is the choosing of m_1-weights for all those *sds* having received 0 m_0-weights, where the weights are again non-negative and have sum 1.

We iterate. In the nth step we are confronted with an m_{n-1}-function. We can show, as we did above, that all that is open is the assignment of weights to those *sds* having received 0 m_{n-1}-weights. The new weights define the m_n-function for all statements which can be written as a disjunction of these *sds*. They in turn determine the c-values of some previously doubtful cases. Since the weights are non-negative and have sum 1, at least one *sd* receives a positive weight in each step, and the number of permissible choices decreases. Since there are only a finite number of *sds*, the process must terminate.

For any e there is a unique m-function such that $m_i(e) > 0$. This is the function constructed in the step in which for the first time one of the *sds* in e receives a positive weight.[14] In all earlier steps $m(e) = 0$, and in later steps m is not defined for e. Let us denote this m-function by 'm^e'. It is clear that each c-value is uniquely determined by the iterative process, and

(19) $c(h, e) = m^e(h\&e)/m^e(e)$.

We have shown that we have at most the following choices: We may choose weights for all the *sds*, and then in each iterative step we may choose a weight for each *sd* having received 0 weight previously. The weights are always non-negative, and all those chosen in a given step add up to 1. The process is finite, having no more steps than there are *sds*.[15]

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[14] By a *sd* 'in w' we mean a *sd* occurring in the normal form of w.

[15] We might give an intuitive interpretation of this result as follows (cf. footnote 13): We are allowed to label as "impossible" some of the logically permissible worlds. In each iterative step we are asked: "Suppose you know that what you have previously labeled as impossible did as a matter of fact take place, how would you modify your probabilities?" And in each step we may label some possibilities as still impossible. Hence we get a hierarchy of of "more and more impossible worlds."

To show that all these choices are actually free, we will show that any c-function defined by (19) is fair. For this we must show that all five conditions are satisfied.

(1) is given by the facts that $m^e(e) > 0$, that the weights are non-negative, and that $h\&e$ consists of some of the *sds* of e. (2) is trivial. If e implies h, $h\&e$ is equivalent to e, and hence (19) gives 1 — in accordance with (3). For (4) it suffices to show that if e implies that h and h' cannot both be true, then $m^e([h \vee h']\&e) = m^e(h\&e) + m^e(h'\&e)$. From the hypothesis we see that $[h \vee h']\&e$ is equivalent to the statement $[h\&e] \vee [h'\&e]$. And since the two alternatives are exclusive, their normal forms have no *sd* in common, and hence the m-value of the disjunction is the sum of the m-values. For (5) we have to show that

(20) $m^e(h\&h'\&e)/m^e(e) = [m^e(h\&e)/m^e(e)] \times [m^{h\&e}(h\&h'\&e)/m^{h\&e}(h\&e)]$.

Distinguish two cases. If $m^e(h\&e) > 0$, then some *sd* in $h\&e$ has positive m^e-weight, and hence m^e is the same as $m^{h\&e}$. (20) is then an identity. If $m^e(h\&e) = 0$, then $m^e(h\&h'\&e)$ must also be 0, hence both sides of (20) are 0. This shows that (19) defines a fair c-function, and that we really have all the described choices.

Let us conclude this section by constructing a c-function that is fair but not strictly fair. We know that it suffices to choose a single 0 weight. But to make the example as extreme as possible, we will choose — in each step — all but one of the weights as 0. Let us number the *sds*: $z_0, \ldots, z_n$. At the beginning we choose weight 1 for z_0, 0 for the others. And in each iterative step we assign 1 to the first open *sd*, and 0 to the rest. Let z_i be the first *sd* in the normal form of e. Then m^e is m_i. $m^e(h\&e)$ is 1 if z_i occurs in the normal form of h, 0 otherwise. From (19) we see:

(21) $c(h, e)$ is 1 if the first *sd* in e is also in h, 0 otherwise.

From our previous results we know that this c-function is fair but not strictly fair. It is certainly a most peculiar way of fixing betting quotients.[16]

7. Conclusion. By establishing the equivalence of (1)–(5) and the requirement of fairness we hope to have justified these five conditions as conditions of adequacy for a definition of inductive probability.

In addition we have shown that the requirement of strict fairness is equivalent to the stronger set gotten by replacing (3) by (3′). Strict fairness seems like a most intuitive requirement, and it serves to exclude such pe-

[16] This is a good example of the hierarchy of impossibles mentioned in foot note 15. Each *sd* is "more impossible" than the previous — i.e., even if we are forced to admit the first i *sds* as possible, we still refuse the remainder. But it must not be supposed that all definitions violating strict fairness, or even fairness, are so obviously unreasonable. For a definition which violates both (5) and (3′), and yet has considerable intuitive appeal, see [4] and [5].

culiar c-functions as (21). Hence, it seems to the author, we are justified in also requiring (3′) of a definition of inductive probability.

In addition we have shown what choice is left after fulfilling the requirement of fairness or strict fairness for a c-function. This serves as a guide in searching for additional requirements.

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