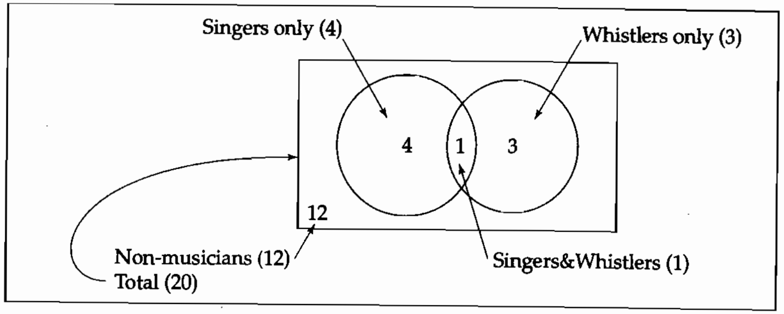
nonmusical people, resulting in a group of 20 people. Imagine we were interested in these two events:

Event A = a singer is selected at random from the whole group.
Event B = a whistler is selected at random from the whole group.

Here is a Venn diagram of the situation, where the entire box represents the room full of twenty people.

FIGURE 6.2



Notice the major change from the previous diagram: Figure 6.2 now has its circles enclosed in a rectangle. By convention, the area of the rectangle is set to 1. The areas of each of the circles correspond to the probability of occurrence of an event of the type that it represents: the area of circle A is 5/20, or 0.25, since there are 5 singers among 20 people. Likewise, the area of circle B is 4/20, or 0.2. The area of the region of overlap between A & B is 1/20, or 0.05.

These drawings can be used to illustrate the basic rules of probability.

(1) *Normality*: $0 \leq \Pr(A) \leq 1$.

This corresponds to the rectangle having an area of 1 unit: since all circles must lie within the rectangle, no circle, and hence no event can have a probability of greater than 1.

(2) *Certainty*: Pr(sure event) = 1. Pr(certain proposition) = 1.

With Venn diagrams, an event that is sure to happen, or a proposition that is certain, corresponds to a "circle" that fills the entire rectangle, which by convention has unit area 1.

(3) *Additivity*: If A and B are mutually exclusive, then:

$$\Pr(A \vee B) = \Pr(A) + \Pr(B).$$

If two groups are mutually exclusive they do not overlap, and the area covering members of either group is just the sum of the areas of each.

(4) *Overlap*:

To calculate the probability of AvB, determine how much of the rectangle is covered by circles A and B. This will be all the area in A, plus the area that appears *only* in B. The area only in B is the areas in B, less the area of overlap with A.

$$\Pr(A \vee B) = \Pr(A) + \Pr(B) - \Pr(A\&B)$$

(5) *Conditional*:

Given that event B has happened, what is the probability that event A will also happen? Look at Figure 6.2. If B has happened, you know that the person selected is a whistler. So we want the proportion of the area of B, that includes A. That is, the area of A&B divided by the area of B.

$$\Pr(A/B) = \Pr(A\&B) \div \Pr(B), \text{ so long as } \Pr(B) > 0.$$

So, in our numerical example, $\Pr(A/B) = 1/4$.
Conversely, $\Pr(B/A) = \Pr(A \& B)/\Pr(A) = 1/5 = 0.2$.

ODD QUESTION 2

Recall the Odd Question about Pia:

2. Pia is thirty-one years old, single, outspoken, and smart. She was a philosophy major. When a student, she was an ardent supporter of Native American rights, and she picketed a department store that had no facilities for nursing mothers. Rank the following statements in order of probability from 1 (most probable) to 6 (least probable). (Ties are allowed.)

_______(a) Pia is an active feminist.
_______(b) Pia is a bank teller.
_______(c) Pia works in a small bookstore.
_______(d) Pia is a bank teller and an active feminist.
_______(e) Pia is a bank teller and an active feminist who takes yoga classes.
_______(f) Pia works in a small bookstore and is an active feminist who takes yoga classes.

This is a famous example, first studied empirically by the psychologists Amos Tversky and Daniel Kahneman. They found that very many people think that, given the whole story:

The most probable description is (f) Pia works in a small bookstore and is an active feminist who takes yoga classes.

In fact, they rank the possibilities something like this, from most probable to least probable:

(f), (e), (d), (a), (c), (b).

But just look at the logical consequence rule on page 60. Since, for example, (f) logically entails (a) and (b), (a) and (b) must be more probable than (f).

In general:

Pr(A&B) ≤ Pr(B).

It follows that the probability rankings given by many people, with (f) most probable, are completely wrong. There are many ways of ranking (a)–(f), but any ranking should obey these inequalities:

Pr(a) ≥ Pr(d) ≥ Pr(e).
Pr(b) ≥ Pr(d) ≥ Pr(e).
Pr(a) ≥ Pr(f).
Pr(c) ≥ Pr (f).

ARE PEOPLE STUPID?

Some readers of Tversky and Kahneman conclude that we human beings are irrational, because so many of us come up with the wrong probability orderings. But perhaps people are merely *careless*!

Perhaps most of us do not attend closely to the exact wording of the question, "Which of statements (a)–(f) are more probable, that is have the highest probability."

Instead we think, "Which is the most useful, instructive, and likely to be true thing to say about Pia?"

When we are asked a question, most of us want to be informative, useful, or interesting. We don't necessarily want simply to say what is most probable, in the strict sense of having the highest probability.

For example, suppose I ask you whether you think the rate of inflation next year will be (a) less than 3%, (b) between 3% and 4%, or (c) greater than 4%.

You could reply, (a)-or-(b)-or-(c). You would certainly be right! That would be the answer with the highest probability. But it would be totally uninformative.

You could reply, (b)-or-(c). That is more probable than simply (b), or simply (c), assuming that both are possible (thanks to additivity). But that is a less interesting and less useful answer than (c), or (b), by itself.

Perhaps what many people do, when they look at Odd Question 2, is to form a character analysis of Pia, and then make an interesting guess about what she is doing nowadays.

If that is what is happening, then people who said it was most probable that Pia works in a small bookstore and is an active feminist who takes yoga classes, are not irrational.

They are just answering the wrong question—but maybe answering a more useful question than the one that was asked.

AXIOMS: HUYGENS

Probability can be axiomatized in many ways. The first axioms, or basic rules, were published in 1657 by the Dutch physicist Christiaan Huygens (1629–1695), famous for his wave theory of light. Strictly speaking, Huygens did not use the idea of probability at all. Instead, he used the idea of the fair price of something like a lottery ticket, or what we today would call the expected value of an event or proposition. We can still do that today. In fact, almost all approaches take probability as the idea to be axiomatized. But a few authors still take expected value as the primitive idea, in terms of which they define probability.

AXIOMS: KOLMOGOROV

The definitive axioms for probability theory were published in 1933 by the immensely influential Russian mathematician A. N. Kolmogorov (1903–1987). This theory is much more developed than our basic rules, for it applies to infinite sets and employs the full differential and integral calculus, as part of what is called measure theory.

EXERCISES

1 *Venn Diagrams.*

Let L: A person contracts a lung disease.
Let S: That person smokes.

Write each of the following probabilities using the Pr notation, and then explain it using a Venn diagram.
(a) The probability that a person either smokes or contracts lung disease (or both).
(b) The probability that a person contracts lung disease, given that he or she smokes.
(c) The probability that a person smokes, given that she or he contracts lung disease.

2 *Total probability.* Prove from the basic rules that Pr(A) + Pr(~A) = 1.

3 *Multiplying.* Prove from the definition of statistical independence that if 0 < Pr(A), and 0 < Pr(B), and A and B are statistically independent,

Pr(A&B) = Pr(A)Pr(B).

4 *Conventions.* In Chapter 4, page 40, we said that the rules for normality and certainty are just conventions. Can you think of any other plausible conventions for representing probability by numbers?

5 *Terrorists.* This is a story about a philosopher, the late Max Black.

One of Black's students was to go overseas to do some research on Kant. She was afraid that a terrorist would put a bomb on the plane. Black could not convince her that the risk was negligible. So he argued as follows:

BLACK: Well, at least you agree that it is almost impossible that *two* people should take bombs on your plane?
STUDENT: Sure.
BLACK: Then you should take a bomb on the plane. The risk that there would be another bomb on your plane is negligible.

What's the joke?