KEY WORDS FOR REVIEW

Thought experiments
Degrees of belief
Stake
Representing by numbers
Betting rate
Conditional bets
Conditional betting rate
"Fair" betting rate

14 Coherence

Consistent personal betting rates satisfy the basic rules of probability. Consistency in this sense is called coherence.

Personal probabilities and betting rates are all very well. But thus far they have no structure, no rules—in fact, not much meaning. Now we give one argument that betting rates ought to satisfy the basic rules for probability. We have already had three thought experiments in Chapter 13. Here are two more.

FOURTH THOUGHT EXPERIMENT: SETS OF BETTING RATES

A group of beliefs can be represented by a set of betting rates.

> Imagine yourself ***advertising*** a set of betting rates. For each of the propositions A, B, . . . K in the set, you offer betting rates p_a, p_b, p_c, . . . , p_k.

In this imaginary game, you are prepared to bet, say,

> on A at rate p_a, or
> against A at rate $(1-p_a)$.

FIFTH THOUGHT EXPERIMENT: SIMPLE INCONSISTENCY

These are *personal* betting rates. Couldn't you choose any fractions you like?

Of course. But you might be inconsistent.

For example, suppose you are concerned with just two possibilities, B (for "below zero") and ~B:

B: On the night of next March 21, the temperature will fall below 0°C at the Toronto International Airport meteorological station.

~B: On the night of next March 21, the temperature will not fall below 0°C at the Toronto International Airport meteorological station.

You go through a thought experiment. You feel gloomy. B makes you think of winter. Will winter never end? You think it is more likely than not to be cold. You advertise a betting rate:

(1) Your betting rate on B: 5/8.

You go through another thought experiment. Thinking about ~B cheers you up. Can spring be far behind? You advertise a betting rate:

(2) Your betting rate on ~B: 3/4. Hence your betting rate against ~B is (1 − 3/4) = 1/4. But that is the same as a betting rate on B of 1/4.

These two assessments are not consistent.

Now we generalize this idea of inconsistent betting rates.

SURE-LOSS CONTRACTS

A *betting contract* is a contract to settle a bet or a group of bets at certain agreed betting rates.

A *bookmaker* is someone who makes betting contracts. He pays you your prize if you win the bet. He takes your money if you lose the bet.

A clever bookmaker named *Sly* sees your advertised betting rates (1) and (2) about the weather. He offers you a contract at your rates. That's fair! But **he** chooses which way to bet. That's fair too! But your rates are bad news for you.

"You bet $5 **on** B, at your advertised odds (1)," says *Sly*.

So you stake $5. *Sly* stakes $3. You win $3 if it is below zero on March 21. Otherwise you lose $5.

"And you bet $6 **on** ~B, at your advertised odds (2)," says *Sly*. So you stake $6. He stakes $2. You win $2 if it is not below zero on March 21. Otherwise you lose $6.

You are in trouble. Look at the table. Your gains and losses are marked plus and minus:

	Bet on B	Bet against B	Payoff
Below zero	+$3	−$6	−$3
Not below zero	−$5	+$2	−$3

You will lose $3 no matter what happens next March 21!

This is a *sure-loss contract.*

A sure-loss contract, for a person X, is a contract with X, at X's betting rates, so that X will lose no matter what happens.

It would be foolish to advertise betting rates open to a sure-loss contract. It is a logical mistake. Your betting rates are not internally consistent. Because the word "inconsistency" has a standard sense in deductive logic, this kind of error in inductive logic is called incoherence.

A set of betting rates is coherent if and only if it is not open to a sure-loss contract.

We will soon establish a remarkable fact:

A set of betting rates is coherent
if and only if
it satisfies the basic rules of probability.

A THREE-STEP ARGUMENT

1 Personal degrees of belief can be represented by betting rates. (See Chapter 13.)
2 Personal betting rates should be coherent.
3 A set of betting rates is coherent if and only if it satisfies the basic rules of probability.

Therefore:

Personal degrees of belief should satisfy the basic rules of probability.

Now we check this out for each basic rule.

NORMALITY

We require that $0 \leq$ (betting rate on A) ≤ 1.

This holds because betting rates are defined as fractions between 0 and 1.

CERTAINTY

We require that $Pr(\Omega) = 1$. That means that the probability of something certain should be 1.

Imagine that A is certainly true, or bound to occur, but that your betting rate on A is $p < 1$. Then *Sly* will ask you to bet against A at rate $1-p$. If the stake is \$1, you lose $\$(1-p)$.

ADDITIVITY

Remember that the additivity rule applies to mutually exclusive events. Let A and B be mutually exclusive. Additivity requires that:

Betting rate on AvB = betting rate on A + betting rate on B.

Suppose that a certain person, *Hilary*, offers betting rates that do not satisfy the rule. Let *Hilary* advertise these betting rates.

On A: p.
On B: q.
On AvB: r.

Suppose that, for example, $r < p+q$, so that the additivity rule is violated.

Sly asks *Hilary* to make three bets at the advertised rates. His trick is to arrange the contracts so that they actually involve *Hilary*'s betting rates. The stake in each of these bets will be \$1.

Bet (i). Bet $\$p$ on A to win $\$(1-p)$ if A is true. *Hilary* loses $\$p$ if A is not true.
Bet (ii). Bet $\$q$ on B to win $\$(1-q)$ if B is true. *Hilary* loses $\$q$ if B is not true.
Bet (iii). Bet \$(1−r) against AvB, to win \$r if (AvB) is false—if neither A nor B is true. But if one or the other is true, then *Hilary* loses \$(1−r).

These are "unit" bets. To make them more realistic, make the stake, say, \$100, so that if, for example, $p = 0.3$, *Hilary* puts up \$30 to win \$70.

Note that since A and B are mutually exclusive, one or the other might be true, but both cannot be true together. Here is *Hilary*'s payoff table, omitting dollar signs:

	Payoff on (i)	Payoff on (ii)	Payoff on (iii)	Payoff
A&(~B)	$1-p$	$-q$	$-(1-r)$	$r-p-q$
(~A)&B	$-p$	$1-q$	$-(1-r)$	$r-p-q$
(~A)&(~B)	$-p$	$-q$	r	$r-p-q$

If $r < p+q$, $r-(p+q)$ is negative. So *Hilary* loses \$[r − (p+q)] no matter what happens.

If $r > p+q$, *Sly* asks *Hilary* to bet at the same advertised rates, but the opposite way. *Hilary* is asked to bet $1-p$ against A, $1-q$ against B, and r on AvB at these rates. His profit—*Hilary*'s loss—will be $\$[r-(p+q)]$ no matter what happens.

Therefore, coherent betting rates must be additive. We can also use the above table to point out that if they are additive—if $r = p+q$—then there is no way to make sure-loss contracts involving A, B, and AvB.

We have now checked the rules for normality, certainty, and additivity.

A necessary and sufficient condition that a set of betting rates is coherent is that it satisfies the basic rules of probability.

CONDITIONAL SURE-LOSS CONTRACTS

What about *conditional* bets?

A bet on A, conditional on B, is cancelled when B does not occur. Hence one cannot have a guaranteed sure-loss contract for conditional bets. If B does not occur, no one will win or lose anything.

Instead we can have a ***conditional sure-loss contract***. A contract is sure-loss, on condition B, if and only if you are bound to lose whenever B occurs.

We extend the idea of coherence. A set of betting rates is *coherent conditional on B* if and only if it is not open to a conditional sure-loss contract.

CONDITIONAL COHERENCE

We now need to show that conditional bets avoid conditional sure-loss contracts if and only if they satisfy the basic rules, and the definition of conditional probability. That is, we require that when a betting rate on B is not zero:

Rate for betting A, conditional on B = [betting rate on A&B] ÷ [betting rate on B].

This is not quite so easy as for the first three basic rules. Suppose that Hilary advertises the following betting rates:

On A&B: q.
On B: $r > 0$.
On A conditional on B: p.

Here bookmaker *Sly* does not choose the total sum staked to be \$1, but instead uses a function of the betting rates themselves.

Sly asks *Hilary* to make three bets, using her advertised betting rates:

(i) Bet $\$qr$ on A&B [to win $\$(1-q)r$]. The stake is $\$r$.
(ii) Bet $\$(1-r)q$ against B [to win $\$rq$]. The stake is $\$q$.
(iii) Bet $\$(1-p)r$ against A, conditional on B [to win $\$pr$]. The stake is $\$r$, as in (i).

Hilary's payoff table is:

	Payoff on (i)	Payoff on (ii)	Payoff on (iii)	Payoff
A&B	$(1-q)r$	$-(1-r)q$	$-(1-p)r$	$pr-q$
(~A)&B	$-qr$	$-(1-r)q$	pr	$pr-q$
~B	$-qr$	rq	0	0

Hence, if $p < q/r$, *Hilary* is guaranteed a loss.

For example, suppose that Hilary's betting rates are:

On A&B: 0.6.
On B: 0.8.
On A conditional on B: 0.5.

Sly asks Hilary to:

(i) Bet $48 *on* A&B to win $32 if A&B occurs.
(ii) Bet $12 *against* B, to win $48 if B does not occur.
(iii) Bet $40 *against* A, conditional on B, to win $40 if B occurs, but A does not occur.

If both A and B occur, *Hilary* loses $20
If B occurs but A does not, *Hilary* loses $20.
If B does not occur, the conditional bet is cancelled, and *Hilary* wins and loses $48 on the other two bets, for a net payoff of 0.

On the other hand, if $p > q/r$, bookmaker *Sly* asks *Hilary* to take the other side of all these bets (so that the table above is the table for *his* payoffs), and he makes a profit whatever happens.

When betting rates follow the definition of conditional probability ($p = q/r$), it is impossible to make a conditional sure-loss contract.

> A necessary and sufficient condition that a set of betting rates, including conditional betting rates, should be coherent is that they should satisfy the basic rules of probability.

FRANK RAMSEY AND BRUNO DE FINETTI

The first systematic theory of personal probability was presented in 1926 by F. P. Ramsey (page 144), in a talk he gave to a philosophy club in Cambridge, England. He mentioned that if your betting rates don't satisfy the basic rules of probability, then you are open to a sure-loss contract. But he had a much more profound—and difficult—argument that personal degrees of belief should satisfy

the probability rules. Where we have throughout this book taken the idea of utility for granted, Ramsey's approach developed the ideas of probability and utility in an interdependent way. Moreover, he allowed for the declining marginal utility of money in his definition.

In 1930, another young man, the Italian mathematician Bruno de Finetti (see page 144), independently pioneered the theory of personal probability. He invented the word "coherence," and did make considerable use of the sure-loss argument.

When Ramsey referred to the argument in passing, he called a sure-loss contract a *Dutch Book.* A "book" is a betting contract made by a bookmaker. But why "Dutch"? I think this was some English undergraduate betting slang of the day, but I don't know. Two Dutch students of mine once tried to find out the source of this ethnic slur, but failed. Nevertheless, the name "Dutch Book" is now standard in inductive logic. We prefer to speak of a *sure-loss contract.*

EXERCISES

1 *Diogenes* is a cynic. He thinks the Maple Leafs will come in last in their league next year. His betting rate that they will come in last (proposition B) is 0.9. His betting rate that they will not come in last (proposition ~B) is 0.2. Make a sure-loss contract against Diogenes.

2 *Epicurus* is an optimist. He thinks the Leafs will come in first in their league next year (T). His betting rate on T is 0.7. His betting rate on ~T is 0.2. Make a sure-loss contract against Epicurus.

3 *Optimistic Cinderella* is at a dance. She has been told that her Cadillac will turn into a pumpkin if she stays at the dance after midnight, but she doubts if that is true. She decides that she will stay at the dance if and only if, when she rolls a fair die at 11.59 P.M., it falls 3 or 4 uppermost.

She is, then, concerned with these possibilities:

S: She stays at the dance.
P: Her Cadillac turns into a pumpkin.

Her personal beliefs, represented as betting rates, are as follows:

On P&S: 0.2
On S: ⅓
On P, conditional on S: ½

likeMake a conditional sure-loss contract against Cinderella, with a guaranteed profit for you of $1.

4 *Pessimistic Cinderella.* Same as (3), but where her personal betting rate on P&S is 0.1.

5 *A mysterious gift.* A distant relative tells you she is going to give you a gift. You hope it is cash—in fact, you hope it is at least $100. The possibilities that interest you are:

C: She gives you a cash gift. H: The gift is cash of at least $100.

Your personal degrees of belief, represented by betting rates, are:

Betting rate on C&H: .3.
Betting rate on C: .8.
Betting rate on H, given C: .5.

Show how a sly bookmaker could arrange a sure-loss contract, where you lose $100 for sure (you'll need that gift from your relative!).

KEY WORDS FOR REVIEW

Sure-loss contract
Coherence
Conditional sure-loss
Conditional coherence

15 Learning from Experience

Bayes' Rule is central for applications of personal probability. It offers a way to represent rational change in belief, in the light of new evidence.

BAYES' RULE

Bayes' Rule is of very little interest when you are thinking about frequency-type probabilities. It is just a rule. On page 70 we derived it in a few lines from the definition of conditional probability.

For many problems—shock absorbers, tarantulas, children with toxic metals poisoning, taxicabs—it shortens some calculations. That's all.

But Bayes' Rule really does matter to personal probability, or to any other belief-type probability.

Today, belief-type probability approaches are often called "Bayesian." If you hear a statistician talking about a Bayesian analysis of a problem, it means some version of ideas that we discuss in this chapter. But there are many versions, ranging from personal to logical. An independent-minded Bayesian named I. J. Good (see page 184) figured out that, in theory, there are 46,656 ways to be a Bayesian!

HYPOTHESES

"Hypothesis" is a fancy word, but daily life is full of hypotheses. Most decisions depend upon comparing the evidence for different hypotheses.

Should Albert quit this course? The drop date is tomorrow. He is a marginal student and has done poorly so far. Will the course get harder, as courses often do? (Hypothesis 1) Or will it stay at its present level, where he can pass the course? (Hypothesis 2)

Should I park illegally? Refer to page 103. What were the hypotheses there? What were the probabilities?