PrSAT: First Examples

Philosophy 148

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■ First, load in the `PrSAT` package

See my **`PrSAT` website** for instructions on downloading and installing **`PrSAT`** (assuming you have *Mathematica* installed).

```
In[1]:= << PrSAT`
```

■ Example #1

The first example of a probability model that we saw was the following:

In[2]:= $\texttt{MODEL1 = PrSAT}\left[\left\{\Pr[X \land Y] == \frac{1}{6},\ \Pr[X \land \neg Y] == \frac{1}{4},\ \Pr[\neg X \land Y] == \frac{1}{8},\ \Pr[\neg X \land \neg Y] == \frac{11}{24}\right\}\right]$

Out[2]= $\left\{\{X \to \{a_2, a_4\},\ Y \to \{a_3, a_4\},\ \Omega \to \{a_1, a_2, a_3, a_4\}\},\ \left\{a_1 \to \frac{11}{24},\ a_2 \to \frac{1}{4},\ a_3 \to \frac{1}{8},\ a_4 \to \frac{1}{6}\right\}\right\}$

`PrSAT` will show us an STT representation of **`MODEL1`**:

```
In[3]:= TruthTable[MODEL1]
```

Out[3]//DisplayForm=

X	Y	var	Pr
T	T	a_4	$\frac{1}{6}$
T	F	a_2	$\frac{1}{4}$
F	T	a_3	$\frac{1}{8}$
F	F	a_1	$\frac{11}{24}$

We can use **`PrSAT`** to calculate probability, using **`MODEL1`**:

In[4]:= `EvaluateProbability[{Pr[X ∨ Y], Pr[X], Pr[Y]}, MODEL1]`

Out[4]= $\left\{\frac{13}{24}, \frac{5}{12}, \frac{7}{24}\right\}$

We can also check arbitrary claims to see if they are *true on* **`MODEL1`**:

```
In[5]:= EvaluateProbability[Pr[X | Y] > Pr[X], MODEL1]

Out[5]= True
```

■ Example #2

The second example we saw was an algebraic proof of the following theorem:

In[6]:= $\Pr(X \vee Y) = \Pr(X) + \Pr(Y) - \Pr(X \wedge Y)$

PrSAT easily verifies this theorem (note: it does not present a readable proof).

```
In[7]:= PrSAT[{Pr[X ∨ Y] ≠ Pr[X] + Pr[Y] - Pr[X ∧ Y]}]

        PrSAT::srchfail : Search phase failed; attempting FindInstance

Out[7]= {}
```

This output means there are no probability models on which **Pr[X ∨ Y] ≠ Pr[X] + Pr[Y] - Pr[X ∧ Y]**. That “proves“ that the above statement is a theorem of probability calculus.

■ Example #3

The second example we saw was an algebraic proof of the following theorem:

In[8]:= $\Pr(X) = \Pr(X \wedge Y) + \Pr(X \wedge \neg Y)$

PrSAT easily verifies this theorem (note: it does not present a readable proof).

```
In[8]:= PrSAT[{Pr[X] ≠ Pr[X ∧ Y] + Pr[X ∧ ¬ Y]}]

        PrSAT::srchfail : Search phase failed; attempting FindInstance

Out[8]= {}
```

This output means there are no probability models on which **Pr[X] ≠ Pr[X ∧ Y] + Pr[X ∧ ¬ Y]**. That “proves“ that the above statement is a theorem of probability calculus.

■ Example #4

The next example involves the following theorem:

In[10]:= $\Pr(X \to Y) \geq \Pr(Y \mid X)$

PrSAT easily verifies this theorem (note: it does not present a readable proof). First, we need to define the conditional operator.

```
In[9]:= X_ → Y_ := ¬ X ∨ Y;

In[10]:= PrSAT[{Pr[X → Y] < Pr[Y | X]}]

        PrSAT::srchfail : Search phase failed; attempting FindInstance

Out[10]= {}
```

This output means there are no probability models on which **Pr[X → Y] ≥ Pr[Y | X]**. That “proves“ that the above statement is a theorem of probability calculus.

■ Example #5

The next example involves the following theorem:

In[13]:= $d(X, Y) = d(X \vee Y, Y) + d(X \vee \neg Y, Y)$, where $d(X, Y) = \Pr(X \mid Y) - \Pr(X)$.

PrSAT easily verifies this theorem (note: it does not present a readable proof). First, we need to define **d(X, Y)**.

In[11]:=
```
d[X_, Y_] := Pr[X | Y] - Pr[X];
```

In[12]:=
```
PrSAT[{d[X, Y] ≠ d[X ∨ Y, Y] + d[X ∨ ¬ Y, Y]}]
```

PrSAT::srchfail : Search phase failed; attempting FindInstance

Out[12]= { }

This output means there are no probability models on which **d[X, Y] ≠ d[X ∨ Y, Y] + d[X ∨ ¬ Y, Y]**. That "proves" that the above statement is a theorem of probability calculus.

■ Example #6

The next example involves the fact that the following is NOT a theorem:

In[14]:= $\Pr(X \mid Y \vee Z) = \Pr(X \mid Y \wedge Z)$

PrSAT easily finds a counter-model to this claim.

In[13]:=
```
PrSAT[{Pr[X | Y ∨ Z] ≠ Pr[X | Y ∧ Z]}]
```

Out[13]= $\{\{X \to \{a_2, a_5, a_6, a_8\}, Y \to \{a_3, a_5, a_7, a_8\}, Z \to \{a_4, a_6, a_7, a_8\}, \Omega \to \{a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8\}\}, \left\{a_1 \to 0, a_2 \to 0, a_3 \to 0, a_4 \to 0, a_5 \to \frac{1}{28}, a_6 \to \frac{13}{28}, a_7 \to \frac{1}{2}, a_8 \to 0\right\}\}$

The model **PrSAT** finds by defualt is *non*-regular. We can force it to find a *regular* counter-model, as follows:

In[14]:=
```
MODEL2 = PrSAT[{Pr[X | Y ∨ Z] ≠ Pr[X | Y ∧ Z]}, Probabilities → Regular]
```

Out[14]= $\{\{X \to \{a_2, a_5, a_6, a_8\}, Y \to \{a_3, a_5, a_7, a_8\}, Z \to \{a_4, a_6, a_7, a_8\}, \Omega \to \{a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8\}\}, \left\{a_1 \to \frac{147181}{77577345}, a_2 \to \frac{1}{999}, a_3 \to \frac{46}{93}, a_4 \to \frac{1}{405}, a_5 \to \frac{1}{999}, a_6 \to \frac{1}{999}, a_7 \to \frac{1}{999}, a_8 \to \frac{83}{167}\right\}\}$

Here is an STT representation of **MODEL2**:

In[15]:= `TruthTable[MODEL2]`

Out[15]//DisplayForm=

X	Y	Z	var	Pr
T	T	T	a_8	$\frac{83}{167}$
T	T	F	a_5	$\frac{1}{999}$
T	F	T	a_6	$\frac{1}{999}$
T	F	F	a_2	$\frac{1}{999}$
F	T	T	a_7	$\frac{1}{999}$
F	T	F	a_3	$\frac{46}{93}$
F	F	T	a_4	$\frac{1}{405}$
F	F	F	a_1	$\frac{147181}{77577345}$

We can calculate the values of **Pr[X | Y ∧ Z]**, **Pr[X | Y ∨ Z]** on this model as follows:

In[16]:= `EvaluateProbability[{Pr[X | Y ∧ Z], Pr[X | Y ∨ Z]}, MODEL2]`

Out[16]= $\left\{\frac{82917}{83084}, \frac{38711715}{77352509}\right\}$

We can look at decimal representations of these exact real numbers, as follows:

In[17]:= `% // N`

Out[17]= {0.99799, 0.500458}

We gave a different model in the lecture notes. We can enter that model in by hand, and then verify it has the desired properties, as follows:

In[18]:= **MODEL3 = PrSAT**$\left[\left\{\Pr[X\land Y\land Z] == \frac{1}{6}, \Pr[X\land Y\land \neg Z] == \frac{1}{6}, \Pr[X\land \neg Y\land Z] == \frac{1}{4}, \Pr[X\land \neg Y\land \neg Z] == \frac{1}{16}, \Pr[\neg X\land Y\land Z] == \frac{1}{6}, \Pr[\neg X\land Y\land \neg Z] == \frac{1}{12}, \Pr[\neg X\land \neg Y\land Z] == \frac{1}{24}, \Pr[\neg X\land \neg Y\land \neg Z] == \frac{1}{16}\right\}\right]$

Out[18]= $\{\{X \to \{a_2, a_5, a_6, a_8\}, Y \to \{a_3, a_5, a_7, a_8\}, Z \to \{a_4, a_6, a_7, a_8\}, \Omega \to \{a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8\}\},$
$\left\{a_1 \to \frac{1}{16}, a_2 \to \frac{1}{16}, a_3 \to \frac{1}{12}, a_4 \to \frac{1}{24}, a_5 \to \frac{1}{6}, a_6 \to \frac{1}{4}, a_7 \to \frac{1}{6}, a_8 \to \frac{1}{6}\right\}\}$

In[19]:= `TruthTable[MODEL3]`

Out[19]//DisplayForm=

X	Y	Z	var	Pr
T	T	T	a_8	$\frac{1}{6}$
T	T	F	a_5	$\frac{1}{6}$
T	F	T	a_6	$\frac{1}{4}$
T	F	F	a_2	$\frac{1}{16}$
F	T	T	a_7	$\frac{1}{6}$
F	T	F	a_3	$\frac{1}{12}$
F	F	T	a_4	$\frac{1}{24}$
F	F	F	a_1	$\frac{1}{16}$

In[20]:= `EvaluateProbability[{Pr[X | Y ∧ Z], Pr[X | Y ∨ Z]}, MODEL3]`

Out[20]= $\left\{\frac{1}{2}, \frac{2}{3}\right\}$

We can also see the algebraic form of an expression, as follows:

In[21]:= `AlgebraicForm[Pr[X | Y ∧ Z] == Pr[X | Y ∨ Z], {X, Y, Z}]`

Out[21]= $\frac{a_8}{a_7 + a_8} == \frac{a_5 + a_6 + a_8}{a_3 + a_4 + a_5 + a_6 + a_7 + a_8}$

Note that **PrSAT** uses different conventions (*i.e.*, a different ordering in the truth-table) for the a_i than I use in the lecture notes.