Exercise 1

Part 1.

(i) $E_1$ Hempel-Confirms $H$. ($E_1 = Raa & Rab & Rba & Rbb$)
$I = \{a, b\}, dev_I(H) = Raa & Rab & Rba & Rbb = E_1$.
Therefore, $E_1 \models dev_I(H)$, so $E_1$ directly Hempel-Confirms $H$, so it Hempel Confirms it.

(ii) $E_2$ does not Hempel-Confirm $H$. ($E_2 = Raa & Rab & Rba$)
$I = \{a, b\}, dev_I(H) = Raa & Rab & Rba & Rbb$.

Claim 1: $E_2$ does not directly Hempel-Confirm $H$, i.e. $E_2 \not\models dev_I(H)$. In solving this problem, some people thought that this claim can be supported by saying things like: ‘$dev_I(H)$ but not $E_2$ “contains” $Rbb$’. This is a sloppy way of putting things: if you want to show that $E_2 \not\models dev_I(H)$, you have to show that there are interpretations of our language $L = \{R, a, b\}$ that make $E_2$ true and $dev_I(H)$ false. This is, however, a relatively minor problem: we didn’t need to get this sophisticated, since the relevant entailments are really trivial!

(*) Here is a very important point!
Almost everybody thought that establishing Claim 1 was enough to show that $E_2$ does not Hempel-Confirm $H$. Now, Hempel Confirmation is a weaker notion than direct Hempel Confirmation. In order to show that $E_2$ does not Hempel-Confirm $dev_I(H)$ it was necessary to show something like this: take any $S$ such that $E_2$ directly Hempel-Confirms $S$, i.e. $Raa & Rab & Rba \models dev_I(S)$.
Now suppose $S \models H$, therefore $S \models Rbb$. Therefore it follows that whenever $b$ is in the class $I$, $dev_I(S) \models Rbb$. So $Raa & Rab & Rba \models Rbb$, but this is false. So there is no $S$ such that $E_2$ directly Hempel Confirms $S$ and $S \models H$. The same is true for the negative claims in Ex. 1 part 2.

(iii) Just analogous to (ii).
(iv) $E_4$ (i.e. $Raa$) Hempel Confirms $H$. $I=\{a\}$, hence $dev_I(H) = Raa = E_4$, so we have entailment of the development, that is, direct Hempel confirmation.

**Part 2**

Let $H = \forall x(Ex \rightarrow Gx)$, let $H' = \forall x(Ex \rightarrow (Ox \equiv Gx))$, and let $C = Ea\&Oa\&Ga$.

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(i) Now, $C$ Hempel Confirms $H$ and $H'$. Note $dev_I(H) = Ea \rightarrow Ga$ and $dev_I(H') = Ea \rightarrow (Ga \equiv Oa)$. Obviously $C \models Ea \rightarrow Ga$ and $C \models E \rightarrow (Ga \equiv Oa)$. In fact, we have just shown that $C$ directly Hempel Confirms $H$.

(ii) Moving to row 2, we deal with $C' = Ea\&Oa\& \sim Ga$. Here we have two choices: first, we could give an argument similar to the argument sketched in part 1, under (*). Otherwise, we could simply point out that $C'$ refutes $H$ and $H'$ (that is to say $C' \models \sim H$ and $C' \models \sim H'$). In general it is not enough to just observe that $C' \not\models dev_I(H)$ and $C' \not\models dev_I(H')$, for the same reasons mentioned in (*).

(iii) In row 3 we deal with $C'' = Ea\& \sim Oa\& Ga$. Again, $C''$ refutes $H'$. For what concerns $H$ we have: $I=\{a\}$. Therefore $dev_I(H) = Ea \rightarrow Ga$, and $C''$ is logically stronger than $Ea \rightarrow Ga$.

(iv) In row 4 we have $C''' = Ea\& \sim Oa\& \sim Ga$. Here $C'''$ refutes $H$. With the usual argument we can show that $C''' \models dev_I(H')$, i.e. $C''' \models Ea \rightarrow (Ga \equiv Oa)$

$\models H$, and $H \models H'$,
$S \models H'$, by the transitivity of logical implication. Therefore $E$ directly Hempel Confirms an $S$ that entails $H'$, so $E$ Hempel Confirms $H'$.

**Exercise 2.**

Let $H = \forall x (Rx \rightarrow Bx)$

Assume,

(i) $Pr(\sim Ba) > Pr(Ra)$

(ii) $Pr(Ra|H) = Pr(Ra)$

(iii) $Pr(Ba|H) = Pr(Ba)$

Show: $Pr(H|Ra&Ba) > Pr(H| \sim Ra & \sim Ba)$

Note:

(iv) $H & Ra \models Ba$

(v) $H & \sim Ba \models \sim Ra$

Both follow by logic, given the content of $H$.

Also note that (ii) and (iii), imply all the usual ways of expressing independence. In particular, given (ii) we have:

(vi) $Pr(H|Ra) = Pr(H)$ (because independence is symmetric)

(vii) $Pr(Ra|H) = Pr(H|Ra)$,

$Pr(\sim Ra|H) = Pr(\sim Ra),$ 

$Pr(H| \sim Ba) = Pr(H)$, etc. (cf. Homework 1!!!)

Also note that in all the proofs that follow, we implicitly use the fact that probabilities are always non-negative. In some cases we will make the stronger assumption that the probabilities we are dealing with are non-zero (I will explicitly signal the one step where this is really essential to the proof).

**Lemma 1.** $Pr(\sim Ra & \sim Ba) > Pr(Ra&Ba)$

**Proof** We show that:

(#) $Pr(\sim Ra & \sim Ba) - Pr(Ra&Ba) = Pr(\sim Ba) - Pr(Ra)$.

Together with assumption (i), (#) implies $Pr(\sim Ra & \sim Ba) > Pr(Ra&Ba)$.

Here is the proof of (#):

$Pr(\sim Ra & \sim Ba) - Pr(Ra&Ba) = Pr(\sim (Ra \lor Ba)) - Pr(Ra&Ba) =$

$= 1 - Pr(Ra \lor Ba) - Pr(Ra&Ba) =$

$= 1 - (Pr(Ra) + Pr(Ba) - Pr(Ra&Ba)) - Pr(Ra&Ba) =$

$= 1 - Pr(Ra) - Pr(Ba) = Pr(\sim Ba) - Pr(Ra).$
The first equality holds by logic, the second, third and fifth by the probability calculus (respectively: negation theorem, general disjunction rule, and negation again), the fourth just by simplifying.

Also note that, from this, it immediately follows:

\[
\frac{1}{\Pr(Ba\&Ra)} > \frac{1}{\Pr(\neg Ba \& \sim Ra)}
\]

**Lemma 2.**

\[\Pr(H \& \sim Ba) > \Pr(H \& Ra)\]

This follows at once from assumption (i), multiplying both sides by \(\Pr(H)\) (which we can do because multiplication is positive over the non-negative reals) and then appealing to the independence facts determined by (ii) and (iii).

**Lemma 3.**

\[\frac{\Pr(H \& Ra)}{\Pr(Ra \& Ba)} > \frac{\Pr(H \& \sim Ba)}{\Pr(\sim Ba \& \sim Ra)}\]

**Proof** Let: \(\Pr(H \& Ra) = a\), \(\Pr(H \& \sim Ba) = b\), \(\Pr(Ra \& Ba) = c\), \(\Pr(\sim Ra \& \sim Ba) = d\).

We know: \(b > a\) and \(d > c\). We want: \(\frac{a}{b} > \frac{c}{d}\).

We will prove equivalently that \(\frac{a}{b} > \frac{c}{d}\).

Notice:

\[
\frac{a}{b} = \frac{\Pr(H \& Ra)}{\Pr(H \& \sim Ba)} = \frac{\Pr(H) \cdot \Pr(Ra)}{\Pr(H) \cdot \Pr(\sim Ba)} = \frac{\Pr(Ra)}{\Pr(\sim Ba)} = \frac{\Pr(Ra \& Ba) + \Pr(Ra \& \sim Ba)}{\Pr(\sim Ba \& \sim Ra) + \Pr(Ra \& \sim Ba)}
\]

The second equality holds by the independence of \(H\) and \(Ra\) and of \(H\) and \(\sim Ba\), the third by canceling out \(\Pr(H)\), the fourth by the law of total probability.

Now, let: \(\Pr(Ra \& \sim Ba) = k\). We also assume that \(\Pr(Ra \& \sim Ba) \neq 0\). (Hence \(k \neq 0\)). Then we as a result of our equalities we can write

\[
\frac{a}{b} = \frac{\Pr(Ra \& Ba) + \Pr(Ra \& \sim Ba)}{\Pr(\sim Ba \& \sim Ra) + \Pr(Ra \& \sim Ba)} = \frac{\Pr(Ra \& Ba) + k}{\Pr(\sim Ba \& \sim Ra) + k} = \frac{c + k}{d + k}
\]

Now since \(k\) is positive, by simple algebra

\[
\frac{a}{b} = \frac{c + k}{d + k} > \frac{c}{d}
\]
as desired.

**Theorem** (i)&(ii)&(iii) ⇒ \( Pr(H|Ra&Ba) > Pr(H| Ra & ~ Ba) \)

**Proof** We have practically done all the work. From assumptions (i)-(iii) we have Lemma 3, i.e.

\[
\frac{Pr(H&Ra)}{Pr(Ra&Ba)} > \frac{Pr(H & & Ba)}{Pr(\sim Ba& \sim Ra)}
\]

By assumption (iv), \( Pr(H&Ra) = Pr(H&Ra&Ba) \).

Similarly by assumption (v), \( Pr(H & & Ba) = Pr(H & & Ra & & Ba) \).

Hence,

\[
\frac{Pr(H&Ra&Ba)}{Pr(Ra&Ba)} > \frac{Pr(H & & Ba & & Ra)}{Pr(\sim Ba& & Ra)}
\]

which is our goal.