Assignment #2 Solutions and Common Mistakes

1 Problem #1

1.1 First Part

Bob's credence function violates the probability axioms in many ways. Here's one. The *generalized additivity theorem* requires that:

$$q(A \vee B) = q(A) + q(B) - q(A \,\&\, B)$$

However, putting in Bob's actual credences, we get

$$\frac{3}{4} \neq \frac{1}{8} + \frac{1}{2} - \frac{1}{4} = \frac{3}{8}$$

Thus Bob's credence function is not a probability function.

1.2 Second Part

Since there are many probability violations in Bob's credence function, there are many kinds of Dutch Books against him. In fact, there exists a Book consisting of only two bets, but we will look at a more complicated Book based on the probability violation identified above:

1. Set stakes at $-\$8$ on a bet regarding A (*i.e.*, make Bob bet against A at \$8).
2. Set stakes at $-\$8$ on a bet regarding B (*i.e.*, make Bob bet against B at \$8).
3. Set stakes at \$8 on a bet regarding $A \,\&\, B$ (*i.e.*, make Bob bet for $A \,\&\, B$ at \$8).
4. Set stakes at \$8 on a bet regarding $A \vee B$ (*i.e.*, make Bob bet for $A \vee B$ at \$8).

Here are the individual and total payoffs for Bob for each possible outcome:

A	B	A-bet	B-bet	$A \,\&\, B$-bet	$A \vee B$-bet	Total
T	T	$-\$7$	$-\$4$	\$6	\$2	$-\$3$
T	F	$-\$7$	\$4	$-\$2$	\$2	$-\$3$
F	T	\$1	$-\$4$	$-\$2$	\$2	$-\$3$
F	F	\$1	\$4	$-\$2$	$-\$6$	$-\$3$

The last column shows that Bob will lose \$3 here *come what may*. □

1.3 Third Part

The Book described above requires a "package principle": in calculating the total payoffs, we assume that if Bob is asked to make all the bets at once this will not change the odds he's willing to give on any one of them. In fact, it is *necessary* to use a package principle to make Book against Bob. The intuitive reason why is that no single credence of Bob's violates the probability axioms; we need to look at more than one of his credences to find a probabilistic inconsistency. Since susceptibility to Book results from a violation of the probability axioms, a Book against Bob must involve more than one of his credences, and therefore more than one bet—since each bet relates to exactly one of the credences involved. (For a more rigorous treatment of this question, see my handout "Conditions for a One-Bet Dutch Book" online. There I describe certain conditions and prove that if an agent's credence function meets these conditions, no one-bet Book can be made against the agent. As far as we are told in this problem, Bob's credences meet the conditions I describe there.)

1.4 Common Mistakes on Problem #1

- Some people got tangled up in the way stakes relate to bets in the Dutch Book, or got the payoffs backwards for certain bets. The easiest way to keep track is to remember the following payoff structure:

  $$\text{Mr. B's payoff (in \$) for a bet regarding } p = \begin{cases} \mathfrak{s} - q(p) \cdot \mathfrak{s} & \text{if } p \text{ is true.} \\ -q(p) \cdot \mathfrak{s} & \text{if } p \text{ is false.} \end{cases}$$

  That is, if an agent takes a bet with stakes \$$\mathfrak{s}$ regarding a proposition p for which he reports a credence q, then he *wins* \$$(\mathfrak{s} - q\mathfrak{s})$ if p is true and he *wins* \$$-q\mathfrak{s}$ if p is false. Losses are just *negative wins*. [If $\mathfrak{s} > 0$, then the bet is said to be *on* p, and if $\mathfrak{s} < 0$, then the bet is said to be *against* p.]

- On the third part, we wanted to know not just whether the Book you gave against Bob required a package principle, but also whether a package principle is *necessary in general* to make Book against Bob. In other words, is there *any* Book that can be made against Bob that doesn't require a package principle? Here, see the handout mentioned above.

2 Problem #2.1

On Problem #2.1, the most people lost points for doing a bunch of algebra, proving that a certain variable had to have a certain value (or that two quantities had to be equal), and then failing to state explicitly how this proved the result asked for in the problem. Throughout Problem #2.1, we will use this notation:

$$\begin{aligned}
&\text{let } q_1(A) - w(A) \overset{\text{def}}{=} x \\
&\text{let } q_2(A) - w(A) \overset{\text{def}}{=} x + a \\
&\text{let } q_1(B) - w(B) \overset{\text{def}}{=} y \\
&\text{let } q_2(B) - w(B) \overset{\text{def}}{=} y + b
\end{aligned}$$

Weak Convexity: Adding the first two equations above, we obtain

$$\begin{aligned} q_1(A) - w(A) + q_2(A) - w(A) &= x + x + a \\ \frac{1}{2}q_1(A) + \frac{1}{2}q_2(A) - w(A) &= x + \frac{1}{2}a \\ q_3(A) - w(A) &= x + \frac{1}{2}a \end{aligned}$$

Similarly, we can derive

$$q_3(B) - w(B) = y + \frac{1}{2}b$$

Now, since we are given

$$I(q_3, w) = I(q_1, w) = I(q_2, w)$$

we have

$$\begin{aligned} 2I(q_3, w) &= I(q_1, w) + I(q_2, w) \\ 2\left[\left(x + \frac{1}{2}a\right)^2 + \left(y + \frac{1}{2}b\right)^2\right] &= (x^2 + y^2) + \left[(x + a)^2 + (y + b)^2\right] \\ \ldots &= \ldots \\ \ldots &= \ldots \\ \frac{1}{2}a^2 + \frac{1}{2}b^2 &= a^2 + b^2 \\ 0 &= \frac{1}{2}a^2 + \frac{1}{2}b^2 \end{aligned}$$

(I have skipped a number of simplification steps in the middle.) Since a^2 and b^2 must be non-negative, the only way this last equation can be true is if a and b are both 0. But then $q_2(A) = q_1(A)$ and $q_2(B) = q_1(B)$, so $q_1 = q_2$. $\square$

Symmetry: We start with

$$\begin{aligned} I(q_1, w) &= I(q_2, w) \\ x^2 + y^2 &= (x + a)^2 + (y + b)^2 \\ x^2 + y^2 - (x + a)^2 - (y + b)^2 &= 0 \qquad (\star) \end{aligned}$$

We then take the equation we are attempting to prove and express it in terms of our variables:

$$\begin{aligned} I(\lambda \cdot q_1 + (1 - \lambda) \cdot q_2, w) &= I((1 - \lambda) \cdot q_1 + \lambda \cdot q_2, w) \\ [\lambda q_1(A) + (1 - \lambda)q_2(A) - w(A)]^2 &+ [\lambda q_1(B) + (1 - \lambda)q_2(B) - w(B)]^2 \\ = [(1 - \lambda)q_1(A) + \lambda q_2(A) - w(A)]^2 &+ [(1 - \lambda)q_1(B) + \lambda q_2(B) - w(B)]^2 \end{aligned}$$

The trick now is to multiply out and simplify this expression while leaving the $(1-\lambda)$ terms intact. If we do that, we eventually obtain

$$\lambda\left[x^2+y^2-(x+a)^2-(y+b)^2\right]=(1-\lambda)\left[x^2+y^2-(x+a)^2-(y+b)^2\right]$$

If we can show that this equation is true, our proof will be done. But we already know $(\star)$ that $x^2+y^2-(x+a)^2-(y+b)^2=0$, so we *are* done. $\square$

3 Problem #2.2

Again, there are many possible answers here (the counterexamples to WC and S in my lecture notes will do). Here's another example illustrating non-equivalence:

$$q(A)=\frac{1}{3}\qquad q'(A)=\frac{1}{6}$$
$$q(B)=\frac{1}{2}\qquad q'(B)=0$$

In a world where A is true and B is false, we have

$$I^{\dagger}(q,w)=\frac{25}{36}\qquad I^{\dagger}(q',w)=\frac{25}{36}$$
$$I^{*}(q,w)=\frac{7}{6}\qquad I^{*}(q',w)=\frac{5}{6}$$

So in this world, $I^{\dagger}(q,w)=I^{\dagger}(q',w)$ but $I^{*}(q,w)>I^{*}(q',w)$. This demonstrates that $I^{\dagger}$ and I^{*} are ordinally non-equivalent.