Assignment #1 Solutions and Common Mistakes

General grading comments: First, "-0" next to a mistake means that while I didn't penalize you for it this time, you should avoid it and may be penalized in the future. Second, if you're typing your assignments, you may want to consider learning TEX, a typesetting program commonly used by mathematicians and scientists. All handouts in this course (including Branden's lecture slides) are typeset in TEX. If you're interested, ask me or Branden for more information.

1 Problem #1

1.1 Part (*a*)

Rule (*i*): Identical to Axiom 2.
Rule (*iv*): Identical to Axiom 3.
Rule (*v*): Notice that p and $\sim p$ are mutually exclusive, and $p \vee \sim p$ is a tautology.

$$\begin{aligned} \Pr(p \vee \sim p) &= 1 && \text{(Axiom 2)} \\ \Pr(p) + \Pr(\sim p) &= 1 && (3) \\ \Pr(\sim p) &= 1 - \Pr(p) && (alg) \end{aligned}$$

Rule (*iii*): Given: p, q equivalent. Notice that given this equivalence, p and $\sim q$ are mutually exclusive, and $p \vee \sim q$ is a tautology.

$$\begin{aligned} \Pr(p \vee \sim q) &= 1 && (2) \\ \Pr(p) + \Pr(\sim q) &= 1 && (3) \\ \Pr(p) + (1 - \Pr(q)) &= 1 && (\text{Rule } v) \\ \Pr(p) &= \Pr(q) && (alg) \end{aligned}$$

Rule (*ii*): Notice that for any contradiction F, F is equivalent to $\sim\mathsf{T}$.

$$\begin{aligned} \Pr(\mathsf{F}) &= \Pr(\sim \mathsf{T}) && (iii) \\ &= 1 - \Pr(\mathsf{T}) && (v) \\ &= 1 - 1 && (2) \\ &= 0 && (alg) \end{aligned}$$

Rule (*vi*): Given: p, q. Notice that $((p \& q) \vee (p \& \sim q))$ and $(\sim p \& q)$ are mutually exclusive, as are $(\sim p \& q)$ and $(p \& q)$.

$$\begin{aligned} \Pr(p \vee q) &= \Pr(((p \& q) \vee (p \& \sim q)) \vee (\sim p \& q)) && (\text{logic and } iii) \\ &= \Pr((p \& q) \vee (p \& \sim q)) + \Pr(\sim p \& q) && (3) \\ &= \Pr(p) + \Pr(\sim p \& q) && (\text{logic and } iii) \\ &= \Pr(p) + \Pr(\sim p \& q) + \Pr(p \& q) - \Pr(p \& q) && (alg) \\ &= \Pr(p) + \Pr((\sim p \& q) \vee (p \& q)) - \Pr(p \& q) && (3) \\ &= \Pr(p) + (q) - \Pr(p \& q) && (\text{logic and } iii) \end{aligned}$$

1.2 Part (*b*)

In part (b), Kolmogorov's Axiom 2 is identical to Skyrms's Rule *i*, and Axiom 3 is identical to Skyrms's *iv*. It turns out Axiom 1 is impossible to prove from Skyrms's six rules (for more on why, see the handout online). So we told you you could skip part *(b)* of this problem.

1.3 Common Mistakes on Problem 1

- **State exclusivities and tautologies**: Axiom 3 applies only to mutually exclusive propositions, so before invoking Axiom 3, you must explicitly state that the two propositions in question are mutually exclusive. Similarly, before invoking Axiom 2 you must explicitly state that the proposition in question is a tautology. Note that the fact that two propositions are mutually exclusive does not imply that their disjunction is a tautology.
- Rule *v* tells you that $\Pr(\sim p) = 1 - \Pr(p)$. This does not allow you to make moves of the form $\Pr(q) = 1 - \Pr(\sim q)$. To make such a move, you have to substitute $\sim p$ in for q, then point out that $\sim\sim p$ is equivalent to p, then apply Rule *iii*.
- Even if p, q, and r are mutually exclusive, you cannot use Axiom 3 to set $\Pr(p \vee q \vee r)$ equal to $\Pr(p) + \Pr(q) + \Pr(r)$. Axiom 3 applies only to pairs of propositions, not triplets. You therefore have to use two applications of Axiom 3 (along with two accompanying exclusivity statements) to break down a disjunction of three propositions.
- **Generality issue**: Rule *ii* has to be shown to apply to *every* contradiction there is. Some people took a particularly convenient contradiction (e.g. $(p \,\&\, \sim p)$) and showed that its probability is 0. This is insufficient to establish the rule's universal claim.

2 Problem #2

2.1 Part (*a*)

This stochastic truth-table assigns our variables for the problem:

X	Y	Z	Pr
T	T	T	a
T	T	F	b
T	F	T	c
T	F	F	d
F	T	T	e
F	T	F	f
F	F	T	g
F	F	F	h

A counter-example to Argument 2 must meet the following constraints:

Constraint 2.0: Because Pr is a probabilitiy model,

$$a + b + c + d + e + f + g + h = 1$$

Constraint 2.1:

$$\begin{aligned} \Pr(Z \mid X) &= \Pr(Z \mid Y) \\ \frac{\Pr(Z \,\&\, X)}{\Pr(X)} &= \frac{\Pr(Z \,\&\, Y)}{\Pr(Y)} \\ \frac{a + c}{a + b + c + d} &= \frac{a + e}{a + b + e + f} \end{aligned}$$

Constraint 2.2:

$$\begin{aligned} \Pr(Z \mid X) &\neq \Pr(Z \mid (X \vee Y)) \\ \frac{\Pr(Z \,\&\, X)}{\Pr(X)} &\neq \frac{\Pr(Z \,\&\, (X \vee Y))}{\Pr(X \vee Y)} \\ \frac{a + c}{a + b + c + d} &\neq \frac{a + c + e}{a + b + c + d + e + f} \end{aligned}$$

Lemma: When X and Y are mutually exclusive, Argument 2 is valid. That is, when X and Y are mutually exclusive and Constraint 2.1 is met, Constraint 2.2 is violated. *Proof*: Notice that X and Y are mutually exclusive iff $a = b = 0$. With a bit of algebra, you can convince yourself that when a and b are equal, it follows from Constraint 2.1 that the two sides of the inequality in Constraint 2.2 are equal. QED. This lemma gives us another constraint:

Constraint 2.3:

$$a + b > 0$$

There are many ways to construct a model meeting these four constraints. Notice that the easiest way to satisfy Constraint 2.1 is to set $c = e$ and $d = f$. So this will be our model:

X	Y	Z	Pr
T	T	T	$a = 0.3$
T	T	F	$b = 0$
T	F	T	$c = 0.1$
T	F	F	$d = 0.2$
F	T	T	$e = 0.1$
F	T	F	$f = 0.2$
F	F	T	$g = 0$
F	F	F	$h = 0.1$

2.2 Part (*b*)

To show that our model is a counter-example, we need to show that it meets Constraints 2.0 through 2.2. (Constraint 2.0 is simply needed to show that it *is* a probability model. It's OK if you left this check out of your answer.)

Constraint 2.0: Because Pr is a probabilitiy model,

$$a + b + c + d + e + f + g + h = 1$$
$$0.3 + 0 + 0.1 + 0.2 + 0.1 + 0.2 + 0 + 0.1 = 1$$

Constraint 2.1:

$$\frac{a + c}{a + b + c + d} = \frac{a + e}{a + b + e + f}$$
$$\frac{0.4}{0.6} = \frac{0.4}{0.6}$$

Constraint 2.2:

$$\frac{a + c}{a + b + c + d} \neq \frac{a + c + e}{a + b + c + d + e + f}$$
$$\frac{0.4}{0.6} \neq \frac{0.5}{0.9}$$

3 Problem #3

This proof can be done either axiomatically or algebraically. Just for fun, we'll do both.

3.1 Axiomatic proof

To do this proof, it helps to have a few lemmas handy.

Lemma 3.1: If x and y are independent, so are x and $\sim y$. For a proof, see Lecture 4, Slide 17. Notice that this is a meta-theoretic theorem *schema*; it holds for any propositions x and y.

Lemma 3.2: If $\{x, y, z\}$ are mutually independent, x is independent of $y \vee z$. Proof can also be found on Lecture 4, Slide 17. This is also a theorem schema.

Lemma 3.3: If $\{X, Y, Z\}$ are mutually independent,

$$\Pr(X \,\&\, {\sim}Y \,\&\, {\sim}Z) = \Pr(X) \cdot \Pr({\sim}Y \,\&\, {\sim}Z)$$

Proof: By Lemma 3.2, X is independent of $Y \vee Z$. By Lemma 3.1, X is therefore independent of ${\sim}(Y \vee Z)$. We therefore have

$$\Pr(X \,\&\, {\sim}(Y \vee Z)) = \Pr(X) \cdot \Pr({\sim}(Y \vee Z))$$
$$\Pr(X \,\&\, {\sim}Y \,\&\, {\sim}Z) = \Pr(X) \cdot \Pr({\sim}Y \,\&\, {\sim}Z) \qquad (logic, iii)$$

Main Proof:
Notice that $(X \& Y \& Z)$ and $(X \& {\sim}Y \& {\sim}Z)$ are mutually exclusive, as are $(Y \& Z)$ and $({\sim}Y \& {\sim}Z)$.

$$\begin{aligned}
\Pr(X \& (Y \equiv Z)) &= \Pr((X \& Y \& Z) \vee (X \& {\sim}Y \& {\sim}Z)) && (logic, iii)\\
&= \Pr(X \& Y \& Z) + \Pr(X \& {\sim}Y \& {\sim}Z) && \text{(Axiom 3)}\\
&= \Pr(X) \cdot \Pr(Y) \cdot \Pr(Z) + \Pr(X \& {\sim}Y \& {\sim}Z) && \text{(Given \#4)}\\
&= \Pr(X) \cdot \Pr(Y \& Z) + \Pr(X \& {\sim}Y \& {\sim}Z) && \text{(Given \#3)}\\
&= \Pr(X) \cdot \Pr(Y \& Z) + \Pr(X) \cdot \Pr({\sim}Y \& {\sim}Z) && \text{(Lemma 3.3)}\\
&= \Pr(X) \cdot (\Pr(Y \& Z) + \Pr({\sim}Y \& {\sim}Z)) && (alg)\\
&= \Pr(X) \cdot \Pr((Y \& Z) \vee ({\sim}Y \& {\sim}Z)) && \text{(Axiom 3)}\\
&= \Pr(X) \cdot \Pr(Y \equiv Z) && (logic, iii)
\end{aligned}$$

(Thanks to Fabrizio Cariani for this simple and elegant proof.)

3.2 Algebraic proof

We will use variables as defined in the first table in the solution to Problem 2. With variables defined, our first step is to rewrite the given information:

(1) $a + b = (a + b + c + d)(a + b + e + f)$

(2) $a + c = (a + b + c + d)(a + c + e + g)$

(3) $a + e = (a + b + e + f)(a + c + e + g)$

(4) $a = (a + b + c + d)(a + b + e + f)(a + c + e + g)$

Lemma 3.4: $a = (a + b + c + d)(a + e)$. Follows from Givens (3) and (4).

Lemma 3.5: $h = 1 - a - b - c - d - e - f - g$. Because Pr is a probability function.

We will start our proof by writing out the formula to be proved in algebraic terms. We will then repeatedly invoke the givens, lemmas, and algebra to derive a tautology. Since each of our steps could run either way, this suffices to prove the formula from the givens.

$$\begin{aligned}
a + d &= (a + b + c + d)(a + d + e + h) && \text{(To be proved)}\\
a + d &= (a + b + c + d)(a + e) + (a + b + c + d)(d + h) && (alg)\\
a + d &= a + (a + b + c + d)(d + h) && \text{(Lemma 3.4)}\\
d &= (a + b + c + d)(d + h) && (alg)\\
d &= (a + b + c + d)(d + 1 - a - b - c - d - e - f - g) && \text{(Lemma 3.5)}\\
d &= (a + b + c + d)(1 + a + e - (a + b + e + f) - (a + c + e + g)) && (alg)\\
d &= (a + b + c + d) + (a + b + c + d)(a + e) - (a + b + c + d)(a + b + e + f) -\\
&\quad (a + b + c + d)(a + c + e + g) && (alg)\\
d &= (a + b + c + d) + a - (a + b) - (a + c) && \text{((1), (2), Lemma 3.4)}\\
d &= d && (alg)
\end{aligned}$$