BOOLEAN ALGEBRA

Boolean algebra, or the *algebra of logic*, was devised by the English mathematician *George Boole* (1815-64), and embodies the first successful application of algebraic methods to logic. Boole seems initially to have conceived of each of the basic symbols of his algebraic system as standing for the mental operation of selecting just the objects possessing some given attribute or included in some given class. Later he conceived of these symbols as standing for the attributes or classes themselves. He also recognized that the algebraic laws he proposed are essentially those of *binary arithmetic,* i.e., if the basic symbols are interpreted as taking just the number values 0 and 1. In each of these interpretations the basic symbols are conceived as being capable of combination under certain operations: *multiplication*, corresponding to conjunction of attributes or intersection of classes, *addition*, corresponding to (exclusive) disjunction or (disjoint) union, and *subtraction*, corresponding to "excepting" or diifference. Boole's ideas as outlined here have since undergone extensive development, and the resulting mathematical concept of *Boolean algebra* now plays a central role in mathematical logic, probability theory and computer design.

The algebraic structures implicit in Boole's analysis were first explicitly presented by Huntington in 1904 and termed "Boolean algebras" by Sheffer in 1913. As Huntington recognized, there are various equivalent ways of characterizing Boolean algebras. One of the most convenient definitions is the following.

A *Boolean algebra* is a structure $(B, +_B, \bullet_B, -_B, 0_B, 1_B)$, where B is a nonempty set, $+_B$ and $\bullet_B$ are binary operations on B, $-_B$ is a unary operation on B, and 0_B, 1_B are distinct elements of B satisfying the following laws: for all x, y, z in B,

associativity	$x + (y + z) = (x + y) + z$	$x \bullet (y \bullet z) = (x \bullet y) \bullet z$
commutativity	$x + y = y + x$	$x \bullet y = y \bullet x$
absorption	$x + (x \bullet y) = x$	$x \bullet (x + y) = x$
distributivity	$x + (y \bullet z) = (x + y) \bullet (x + z)$	$x \bullet (y + z) = (x \bullet y) + (x \bullet z)$
complementation	$x + (-x) = 1$	$x \bullet (-x) = 0.$

(Here and in the sequel we omit the subscript *"B"* when confusion is unlikely.) The operations

"+" (Boolean "addition"—here corresponding to *inclusive* disjunction), "$\bullet$" (Boolean "multiplication") and "–" (called Boolean *complementation*: note that the operation sending x, y to $x \bullet (-y)$ corresponds to Boole's "subtraction") are known as *Boolean operations*. The elements 0 and 1 are called the *zero* and *unit elements* of B, respectively. A Boolean algebra is customarily identified by means of its underlying set: thus, for example, the Boolean algebra just introduced will be denoted simply by by "B".

Two basic identities which are readily shown to hold in any Boolean algebra are

De Morgan's laws $-(x + y) = (-x)(-y) \quad -(x \bullet y) = (-x) + (-y)$
Law of double complementation $-(-x) = x$.

It follows immediately from these two laws that, in any Boolean algebra, + (respectively $\bullet$) is definable in terms of $\bullet$ and – (respectively + and –) by the equations

$$x + y = -[(-x) \bullet (-y)] \text{ and } x \bullet y = -[(-x) + (-y)].$$

A Boolean algebra may be regarded as a special kind of *Heyting algebra (q.v.).* In fact, if we define the relation $\leq$ on a Boolean algebra B by $x \leq y$ if and only if $x \bullet y = x$, and the binary operation $\Rightarrow$ on B by $x \Rightarrow y = (-x) + y$, then it is not hard to show that the structure $(B, +, \bullet, \Rightarrow, \leq, 0, 1)$ is a Heyting algebra. This Heyting algebra has the special property that its negation operation (which coincides with –) satisfies both the law of complementation and the law of double complementation, as stated above. It can be shown that any Heyting algebra satisfying either of these two (equivalent) laws arises from a Boolean algebra in the manner just described. Thus a Boolean algebra may be characterized as a Heyting algebra satisfying either law.

Two standard types of Boolean algebra (corresponding to Boole's original interpretations) arise in *set theory* and (classical) *logic*. To obtain the first of these, consider any nonempty set X and let $\mathbf{P}X$ be its power set, i.e. the set of all its subsets. Then the structure $(\mathbf{P}X, \cup, \cap, -, \varnothing, X)$ is a Boolean algebra—the *power set algebra of X*—in which $\cup$, $\cap$, – are the operations of set-theoretic union, intersection, and complementation with respect to X, respectively. When X has just one element, $\mathbf{P}X$ reduces to the set $\{0,1\}$ where $0 = \varnothing$ and $1 = X$. This algebra is called the *two element* or *initial* Boolean algebra and is denoted by 2. Its operations are displayed in the table below.

x	y	$x + y$	$x \cdot y$	$-x$
0	0	0	0	1
0	1	1	0	1
1	0	1	0	0
1	1	1	1	0

Power set algebras admit the following natural generalizations. We define a *subalgebra* of a Boolean algebra B to be a nonempty subset of B which is closed under the Boolean operations in B. Then a subalgebra of a power set algebra is called a *Boolean algebra of sets*. As an example, for any set X, let $Z(X)$ be the set of all subsets of X which are either finite or whose complement is finite. Then $(Z(X), \bigcup, \bigcap, -, \varnothing, X)$ is a Boolean algebra of sets called the *finite-cofinite algebra of X*.

The standard types of Boolean algebra arising in *logic* are the so-called *Lindenbaum-Tarski* algebras. To obtain these, we start with a consistent theory T in a classical propositional or first-order language **L**. Define the equivalence relation $\approx$ on the set of formulas of **L** by $\varphi \approx \psi$ if $T \vdash \varphi \leftrightarrow \psi$. For each formula φ write $[\varphi]$ for its $\approx$-equivalence class. On the set $A(T)$ of such equivalence classes define the operations $+$, $\cdot$, $-$ and the elements 1, 0 by $[\varphi] + [\psi] = [\varphi \vee \psi]$, $[\varphi \cdot \psi] = [\varphi \wedge \psi]$, $[-\varphi] = [\neg\varphi]$, $1 = [\alpha \Rightarrow \alpha]$, $0 = [\alpha \wedge \neg\alpha]$, where α is a fixed but arbitrary formula. Then the structure $(A(T), +, \cdot, -, 0, 1)$ is a Boolean algebra, called the *Lindenbaum-Tarski algebra* of T.

Boolean algebras of sets and Lindenbaum-Tarski algebras are *typical* Boolean algebras in the following sense. Call two Boolean algebras *isomorphic* if there is a bijection between them which preserves their respect Boolean operations: isomorphic Boolean algebras are *structurally indistinguishable.* Then it can be shown that any Boolean algebra is isomorphic both to an Boolean algebra of sets and to the Lindenbaum-Tarski algebra of some propositional theory. These facts —the first of which is the famous *Stone Representation Theorem* of 1936 — together show that Boolean algebras together just the common features of set theory and classical logic.

Boolean algebras arise naturally in *classical physics*. If **S** is a classical physical system and Σ its phase space, we may regard an observable on **S** as being a map $f: \Sigma \to \Omega$, where the codomain Ω, the observation space of f, is the set of "values" that f can assume. (Typically Ω will be a set of real numbers.) If $f_1,...,f_n$ are observables on **S** with observation spaces $\Omega_1,\ldots, \Omega_n$, the observation space associated with the n-tuple of obsevables $(f_1,...,f_n)$ is the Cartesian product

$\Omega_1 \times \ldots \times \Omega_n$. Each subset X of this Cartesian product is correlated with a proposition P_X concerning the state x of **S**, namely the assertion that the n-tuple of measured values of $f_1,\ldots,f_n$ lies in X when **S** in state x. X then has a *representative* $\widehat{X}$ in Σ defined by

$$\widehat{X} = \{x \in \Sigma: (f_1(x),\ldots,f_n(x)) \in X\}.$$

Thus $\widehat{X}$ is the set of states x of **S** such that P_X is verified when **S** is in state x. Accordingly we may also call $\widehat{X}$ the representative of the associated proposition P_X. The relation of *entailment* between propositions then corresponds to the relation of set-theoretical *inclusion* between their representatives; the representative of the *negation* of a proposition is the set-theoretical *complement* (in Σ) of its representative; and the representative of the *disjunction* of a pair of propositions is the set-theoretic *union* of their representatives. It follows that the logic of propositions concerning a classical physical system is *isomorphic to a Boolean algebra of subsets of its phase space.*

This is to be contrasted with the situation in *quantum mechanics*. If H is the Hilbert space associated with a quantum system **Q**, then propositions concerning **Q** are in fact correlated, not with arbitrary subsets of H, but with *closed subspaces*. Now the system $C(H)$ of closed subspaces of H does not form a Boolean algebra of sets because the "addition" operation in $C(H)$ is not set-theoretic union, but, instead, the operation of forming the closed subspace generated by the union, which in turn leads to the failure of the *distributive law* in $C(H)$. It is this failure which is presumed to distinguish from classical logic the *quantum logic (q.v.)* of propositions concerning a quantum system. This is analogous to the employment of the failure of the law of excluded middle or the law of double negation as a means of distinguishing intuitionistic (*q.v.*) from classical logic.

Bibliography

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