Probabilities of Conditionals and Conditional Probabilities — Revisited

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1 Lewisian Triviality: The Resilient Equation

Consider a simple, classical, propositional language containing two factual (*i.e.*, devoid of indicative conditionals) atomic sentences P and Q, together with a third atomic sentence, which we will label "$P \rightarrow Q$." This third atom[1] will be interpreted (extra-systematically) as the indicative conditional "if P, then Q." Adams [1], Stalnaker [20], and others have defended the following as a rational requirement on priors, *viz.*, initial credence functions, $\Pr(\cdot)$.

The Equation (EQ). $\Pr(P \rightarrow Q) = \Pr(Q \mid P)$

*It is an honor and a pleasure to contribute this note to our Festschrift for Alan Hájek. He's been such a good mentor, colleague, and friend over the past 25 years. And, we've had countless edifying discussions about conditionals and conditional probability. In addition to Alan, I have discussed the material in this paper with *many* other people over the past decade or so, including: Andrew Bacon, Alexandru Baltag, Justin Bledin, Fabrizio Cariani, Kevin Dorst, Igor Douven, Kenny Easwaran, Paul Egré, Ben Eva, Hartry Field, Melissa Fusco, Jeremy Goodman, Stephan Hartmann, Ben Holguín, Wes Holliday, Jim Joyce, Justin Khoo, Hannes Leitgeb, Matt Mandelkern, Vann McGee, Calum McNamara, Sarah Moss, Sven Neth, Adrian Ommundsen, Richard Pettigrew, Graham Priest, Paolo Santorio, Ted Shear, Jack Spencer, Bob Stalnaker, Reuben Stern, Mike Titelbaum, Bas van Fraassen, Steve Yablo, and Snow Zhang. My apologies to anyone I've omitted from this list.

[1]It is crucial that $P \rightarrow Q$ be treated as *atomic* — *i.e.*, as *logically independent* of P and Q. Otherwise, we would run afoul of Hájek's combinatorial triviality result — *a.k.a.*, the "wallflower argument" [9].

Lewis [12] supposes that if The Equation is a *bona fide* rational requirement on priors, then it must hold not merely initially but also after updating on (almost any) new evidence. That is, Lewis assumes that if The Equation is rationally required, then it must hold *resiliently*.

I will use the term *structural* to refer to rational requirements that hold resiliently — *i.e.*, requirements that hold not only for reasonable initial credence functions (*viz.*, priors), but also for any credence function obtained from such a prior *by indicatively supposing any proposition* x (so long as supposing x doesn't lead to any conditional probabilities being undefined, according to the ratio formula). I will use the term *substantive* to refer to rational requirements that are not structural in this sense (*i.e.*, constraints on the prior which only continue to be rational requirements under *some* consistent indicative suppositions x).[2] The assumption that The Equation is a structural rational requirement leads to the following vast strengthening.

The Resilient Equation (REQ). For all propositions x that are (a) Boolean functions of P and Q, and (b) such that $\Pr(P \,\&\, x) > 0$,

$$\Pr(P \to Q \mid x) = \Pr(Q \mid P \,\&\, x).$$

The Resilient Equation corresponds to the assumption that The Equation hold not only for prior credence functions $\Pr(\cdot)$, but also for all conditional credence functions $\Pr(\cdot \mid x)$ obtained by indicatively supposing arbitrary propositions x — subject to the minimal consistency requirement that $\Pr(P \,\&\, x) > 0$.

Lewis's key observation was that The Resilient Equation leads to triviality. The strongest such result [4] is the following.[3]

(REQ)-Triviality. $\Pr(P \,\&\, (Q \equiv (P \to Q))) = 1$.

To see the relationship between The Equation and The Resilient Equation more clearly, consider Table 1, which displays how these constraints affect the degrees of freedom in a probability model with three

[2]Here, I am adapting the terminology of Worsnip [25] to the present context.

[3]Actually, it can be shown that The Resilient Equation *is unsatisfiable* by *any* (Kolmogorov) probability function, if it is meant to hold for *all* factual P, Q. See the Appendix for a proof and further discussion.

atoms P, Q, and $P \to Q$. In the first column, we see that an unconstrained prior $\Pr(\cdot)$ over three atoms has seven degrees of freedom. The Equation merely reduces this (by one) to six degrees of freedom (which is still non-trivial). However, The Resilient Equation reduces the probability model *to a single degree of freedom*.

P	Q	$P \to Q$	$\Pr(\cdot)$	$\Pr(\cdot)$ + (EQ)	$\Pr(\cdot)$ + (REQ)
T	T	T	a	a	a
T	T	F	b	b	0
T	F	T	c	c	0
T	F	F	d	d	$1-a$
F	T	T	e	e	0
F	T	F	f	f	0
F	F	T	g	$\frac{a+b}{a+b+c+d} - (a+c+e)$	0
F	F	F	$1-\sum$	$1-\sum$	0

Table 1: The impact of (EQ) and (REQ) on probability models.

We can reconstruct Lewis's argument against The Equation as follows. Lewis assumes, for *reductio*, that The Equation is a rational requirement on prior credence functions $\Pr(\cdot)$. He then invokes a resiliency/structural rationality requirement to complete the *reductio*.

(1) The Equation is a rational requirement for $\Pr(\cdot)$.

(2) Therefore, The Equation must hold in a (fully) resilient way.

(3) Therefore, The Resilient Equation is a rational requirement for $\Pr(\cdot)$.

(4) But, (REQ)-Triviality is not a rational requirement for $\Pr(\cdot)$.

(5) Contradiction, since (3) entails $\neg$(4).

(6) Therefore, The Equation is not a rational requirement for $\Pr(\cdot)$.

The crucial step here is the transition from (1) to (2). Lewis presupposes that if The Equation is a rational requirement on initial credences, then it must hold fully resiliently (*i.e.*, it must be a structural rational requirement). This is the key presupposition; and, as Khoo [11] notes, it is also (plausibly) the weakest step in the argument.

It is worth noting that some rational requirements clearly *do* satisfy this very strong resiliency condition. That is, some rational requirements clearly *are* structural. For instance, *probabilism* itself — the requirement that credence functions be probability functions — satisfies full resiliency. This is just a theorem of the probability calculus: if an initial credence function $\Pr(\cdot)$ is a probability function, then so is $\Pr(\cdot \mid x)$, provided only that the latter is well-defined.

It is also worth noting that some rational requirements do *not* satisfy the resiliency condition. For instance, *regularity* — the requirement that rational agents only assign (*a priori*) credence 1 to tautologies. Lewis defends standard conditionalization. As a result, he must *reject* the resiliency of the regularity requirement. Nonetheless, he [13] *accepts* regularity as a rational constraint *on priors*. That is, he takes regularity to be a (merely) substantive rational requirement.

There is another important constraint on priors which Lewis defends, while explicitly rejecting its resiliency. He famously defends **The Principal Principle** as a (merely) substantive rational requirement.

2 The Principal Principle and Resiliency

Lewis [13] articulated and defended what he called the Principal Principle. This principle concerns the relationship between rational priors and objective chance. Let $\mathrm{Ch}(p) = c$ express the proposition that the objective chance of p equals c. Then the Principal Principle states:

(PP) $\Pr(p \mid \mathrm{Ch}(p) = c) = c$.

The Principal Principle tells us that rational credence should — *a priori* — defer to objective chance. One might think that if the Principal Principle is a genuine rational requirement, then it, too, should hold resiliently. That is, one might think of The Principal Principle as a structural rational requirement [17]. This would yield the following.

(PP$_x$) $\Pr(p \mid x \,\&\, \mathrm{Ch}(p) = c) = c$.

If we require (PP$_x$) to hold for *all* x (compatible with $\mathrm{Ch}(p) = c$), we get what I will call the **Resilient Principal Principle** (RPP). The Resilient Principal Principle leads to triviality in a manner very similar to The Resilient Equation. Consider a language with three atoms: P (where P is factual — it contains no statements about chance), $\mathrm{Ch}(P) = 0$, and $\mathrm{Ch}(P) = 1$. Table 2 shows the impact of (PP) and (RPP) on probability models in this setting.

P	$\mathrm{Ch}(P) = 0$	$\mathrm{Ch}(P) = 1$	$\Pr(\cdot)$	$\Pr(\cdot)$ + (PP)	$\Pr(\cdot)$ + (RPP)
T	T	T	0	0	0
T	T	F	a	0	0
T	F	T	b	b	b
T	F	F	c	c	0
F	T	T	0	0	0
F	T	F	d	d	$1 - b$
F	F	T	e	0	0
F	F	F	$1 - \sum$	$1 - \sum$	0

Table 2: The impact of (PP) and (RPP) on probability models.

The Principal Principle reduces the number of degrees of freedom from five to three, while the Resilient Principal Principle reduces this number down to one. This is the analogue of (REQ)-Triviality for the Principal Principle. We will refer to this as **(RPP)-Triviality**.

Given this similarity between The Resilient Equation and The Resilient Principal Principles, one might have expected Lewis to run an analogous *reductio against* the Principal Principle. To wit:

(1) (PP) is a rational requirement for $\Pr(\cdot)$.

(2) Therefore, (PP) must hold in a (fully) resilient way.

(3) Therefore, Resilient (PP) is a rational requirement for $\Pr(\cdot)$.

(4) But, (RPP)-Triviality is not a rational requirement for $\Pr(\cdot)$.

(5) Contradiction, since (3) entails $\neg$(4).

(6) Therefore, (PP) is not a rational requirement for $\Pr(\cdot)$.

Lewis explicitly rejects this reasoning. He maintains that the Principal Principle is indeed a genuine (substantive) rational requirement, despite the triviality of its fully resilient/structural counterpart. The key move Lewis makes is to reject the inference from (1) to (2). Not all rational requirements, Lewis rightly maintains, need to hold with full resiliency. Some are substantive, as opposed to structural, in nature.

To make this precise, Lewis introduces the notion of "admissibility," in order to restrict the universal quantifier in (RPP) to those suppositions x for which The Principal Principle should be expected to continue to hold. Among other things, he required admissible propositions x (with respect to PP) to satisfy what I will refer to as **Chance Screening**: x must be such that, for all values $c \in [0,1]$,

$$\Pr(p \mid x \,\&\, \mathrm{Ch}(p) = c) = \Pr(p \mid \mathrm{Ch}(p) = c).$$

In other words, Chance Screening says that — once we know the objective chance of p — learning x provides no further information relevant to the truth of p: *the p-chances screen off x from p*.

If we restrict the universal quantifier in (RPP) to only those x's satisfying Chance Screening, we get what I will call the **Quasi-Resilient Principal Principle** (QRPP). Unfortunately, (QRPP) turns out to be *equivalent* to (PP) itself — the Chance Screening restriction does no substantive work.[4] The details of Lewis's notion of "admissibility" are not crucial here. The important point is that Lewis explicitly rejected full resiliency for the Principal Principle, instead requiring only a restricted form of resiliency, which is limited to the *admissible* x's.

This suggests a strategy for defending The Equation: develop an analogous notion of admissibility for conditionals, and require The Equation to hold resiliently *only for these admissible x's*.

[4]Proof of the claim that (QRPP) $\Leftrightarrow$ (PP). ($\Rightarrow$) Let x be a tautology ($\top$). Then Chance Screening obtains: $\top$ is screened from every q by every p, on every $\Pr(\cdot)$. Since tautologies are admissible, (QRPP) implies (PP). ($\Leftarrow$) Let $\alpha \stackrel{\text{def}}{=} \Pr(p \mid \mathrm{Ch}(p) = c)$ and $\beta \stackrel{\text{def}}{=} \Pr(p \mid x \,\&\, \mathrm{Ch}(p) = c)$. Then Chance Screening is $\alpha = \beta$, (PP) is $\alpha = c$, and (QRPP) is $\alpha = \beta \Rightarrow \beta = c$ (where all three claims are universally quantified over p, x, and c). By logic, (QRPP) is equivalent to $\alpha = \beta \Rightarrow \alpha = c$, which is implied by (PP) $\alpha = c$.

3 Reconciliation: The Quasi-Resilient Equation

3.1 Antecedent Screening

Following Lewis's treatment of the Principal Principle, we can develop a restricted version of The Resilient Equation by introducing an appropriate admissibility condition. The natural analogue to Chance Screening for conditionals is what we will call **Antecedent Screening**.

Antecedent Screening. The antecedent P of the conditional $P \rightarrow Q$ *screens off its consequent* Q *from* x. More precisely,

$$\Pr(Q \mid P \,\&\, x) = \Pr(Q \mid P).$$

This condition ensures that — once we suppose P — learning x provides no additional information about Q. For propositions x that are Boolean functions of P and Q, Antecedent Screening can be characterized in several equivalent ways. One can show that P screens Q from x if and only if $P \,\&\, x$ neither entails Q nor entails $\neg Q$. Equivalently, Antecedent Screening holds just in case there exists some probability function $\Pr(\cdot)$ such that $0 < \Pr(Q \mid P \,\&\, x) = \Pr(Q \mid P) < 1$.

An important consequence of this characterization is that when x is a Boolean function of P and Q, *exactly three* propositions in the Boolean algebra generated by the three atoms $\{P, Q, P \rightarrow Q\}$ will satisfy Antecedent Screening: P, $P \vee Q$, and $P \vee \neg Q$. These are precisely the propositions that can be true while P is true, without determining the truth value of Q.

3.2 The Quasi-Resilient Equation

We can now state our restricted version of The Resilient Equation:

Quasi-Resilient Equation (QREQ). All rational priors $\Pr(\cdot)$ should satisfy the following restricted version of The Resilient Equation. For all propositions x that are Boolean functions of P and Q,

$$\Pr(P \rightarrow Q \mid x) = \Pr(Q \mid P \,\&\, x),$$

provided that x *satisfies Antecedent Screening.*

Given our simple characterization of Antecedent Screening for Boolean combinations of P and Q (in the Boolean algebra generated by the three atoms $\{P, Q, P \rightarrow Q\}$), we can state QREQ more explicitly. All rational initial credence functions $\Pr(\cdot)$ should be such that:

$$\begin{aligned} \Pr(P \rightarrow Q \mid P) &= \Pr(Q \mid P), \\ \Pr(P \rightarrow Q \mid P \vee Q) &= \Pr(Q \mid P), \text{ and} \\ \Pr(P \rightarrow Q \mid P \vee \neg Q) &= \Pr(Q \mid P). \end{aligned}$$

Unlike (QRPP), which turned out to be equivalent to (PP), the Quasi-Resilient Equation *is strictly logically stronger than* The Equation. One can construct probability models that satisfy The Equation but violate QREQ (we provide such a model in the Appendix). This means that QREQ genuinely strengthens The Equation by imposing some degree of resiliency; but, not enough resiliency to trigger triviality.

4 Convergence with McGee/Goldstein&Santorio

On the simplest $\{P, Q, P \rightarrow Q\}$ algebra, there is strong agreement between our approach and the theory developed by Vann McGee [15]. McGee also sought to restrict The Resilient Equation, but he did so indirectly, through a pair of principles governing how conditionals interact with factual propositions. Specifically, McGee proposed the following constraint on rational priors.

Independence (IND). For all factual propositions x, if $x \vDash \neg P$, then

$$\Pr(x \,\&\, (P \rightarrow Q)) = \Pr(x) \cdot \Pr(P \rightarrow Q).$$

Independence says that if factual proposition x entails that the antecedent P is false, then x and the conditional $P \rightarrow Q$ should be probabilistically independent. McGee also assumed a principle that has been defended by nearly everyone working on conditionals:

Modus Ponens (MP). For all factual propositions x, if $x \vDash P$, then

$$\Pr(x \,\&\, (P \rightarrow Q)) = \Pr(x \,\&\, P \,\&\, Q).$$

Remarkably, McGee's indirect restrictions on The Resilient Equation are *equivalent* to our direct restriction (Antecedent Screening).

> **Fact.** Assuming Modus Ponens, McGee's Independence is equivalent to our QREQ.[5]

This is significant for a couple of reasons. First, it shows that two apparently different approaches to restricting The Resilient Equation — one working indirectly through structural constraints on how conditionals interact with factual propositions, the other working directly through admissibility restrictions on conditionalization — converge on the same formal theory. Second, as Snow Zhang [26] has shown, McGee's theory is equivalent to the recent theory of Goldstein and Santorio [8], which means that all three of these contemporary theories converge on our QREQ. This provides some independent support for our approach to restricting The Resilient Equation.

To see the precise formal impact of QREQ and McGee's principles (in our simplest algebra), consider Table 3, which shows how they constrain the class of probability models. After imposing Modus Ponens, both McGee's theory and QREQ leave us with three degrees of freedom, represented by the adjustable parameters a, e, and g.

5 Santorio's Die Example

Consider the following example, due to Paolo Santorio. Suppose a fair six-sided die was tossed. Let:

P: The die landed on either 1, 3, 5, or 6.

Q: The die landed on 6.

X: The die landed even.

The rational prior for this scenario is straightforward (and determined by the Principal Principle). We have $\Pr(P \,\&\, Q) = 1/6$, $\Pr(P \,\&\, \neg Q) = 1/2$, $\Pr(\neg P \,\&\, Q) = 0$, and $\Pr(\neg P \,\&\, \neg Q) = 1/3$.

[5] Thanks to Snow Zhang for providing me with an independent proof and explanation of this result.

P	Q	$P \to Q$	$\Pr(\cdot)$	$\Pr(\cdot)$ + (MP) + (IND) = $\Pr(\cdot)$ + (MP) + **QREQ**
T	T	T	a	a
T	T	F	b	0
T	F	T	c	0
T	F	F	d	$\dfrac{a \cdot (1-(a+e+g))}{a+e+g}$
F	T	T	e	e
F	T	F	f	$e \cdot \left(\dfrac{1}{a+e+g} - 1\right)$
F	F	T	g	g
F	F	F	$1-\sum$	$1-\sum$

Table 3: Assuming MP, McGee's theory and QREQ are equivalent.

Now, The Equation tells us that $\Pr(P \to Q) = \Pr(Q \mid P) = 1/4$. From this, it follows that $\Pr((P \to Q)\,\&\,X) \leq 1/4$, since a conjunction can be no more probable than one of its conjuncts. Therefore,

$$\Pr(P \to Q \mid X) = \frac{\Pr((P \to Q)\,\&\,X)}{\Pr(X)} \leq \frac{1/4}{1/2} = \frac{1}{2}.$$

However, the conditional probability $\Pr(Q \mid P\,\&\,X)$ equals 1. This is because $P\,\&\,X$ determines that the die shows 6 (it must be even, and it must be in $\{1,3,5,6\}$, so it must be 6). Thus we have:

$$\Pr(P \to Q \mid X) \leq \frac{1}{2} < 1 = \Pr(Q \mid P\,\&\,X).$$

This is a clear counterexample to The Resilient Equation. Crucially, notice that X violates Antecedent Screening. We have:

$$\Pr(Q \mid X\,\&\,P) = \Pr(Q \mid (P \supset Q)\,\&\,P) = 1 \neq \frac{1}{4} = \Pr(Q \mid P).$$

The proposition X is inadmissible because conditioning on X changes the conditional probability $\Pr(Q \mid P)$. This is precisely what Antecedent Screening prohibits. An important empirical observation emerges from examining this and other counterexamples: every known knock-down counterexample to The Resilient Equation involves propositions x that are inadmissible in our sense. This suggests the following conjecture: all counterexamples to REQ violate Antecedent Screening. If this conjecture is correct, then QREQ successfully captures all the cases where The Equation can legitimately be extended to conditional contexts.

McGee, Goldstein & Santorio, and myself all agree that there is a unique rational prior (over the Boolean algebra generated by the three atoms $\{P, Q, P \to Q\}$) in Santorio's die case — depicted in Table 4.

P	Q	$P \to Q$	$\Pr(\cdot)$	$\Pr(\cdot)$ + (MP) + (IND) = $\Pr(\cdot)$ + (MP) + **QREQ**
T	T	T	a	$1/6$
T	T	F	b	0
T	F	T	c	0
T	F	F	d	$1/2$
F	T	T	e	0
F	T	F	f	0
F	F	T	g	$1/12$
F	F	F	$1 - \sum$	$1/4$

Table 4: McGee's theory and QREQ applied to Santorio's Die case.

6 Lewis's Revenge: Nested Conditionals

So far, we seem to have made some progress. We have identified a way to restrict The Resilient Equation, so as to avoid Lewisian triviality. And, our QREQ-approach turns out to be equivalent to the theories of McGee and Goldstein and Santorio — when applied to the simple Boolean algebra generated by the three atoms $\{P, Q, P \to Q\}$. Unfortunately, all is not beer and skittles.

More challenging issues arise when we consider nested conditionals. For instance, if we add $X \rightarrow (P \rightarrow Q)$ to our language (as a fourth atomic sentence), then QREQ runs into serious difficulties. On any screening-off-based notion of admissibility, all tautologies $\top$ will be admissible. This would lead us to expect that

$$\Pr(X \rightarrow (P \rightarrow Q) \mid \top) = \Pr(P \rightarrow Q \mid X).$$

In the Die case, we have: $\Pr(P \rightarrow Q \mid X) \leq 1/2$. Intuitively, however, the nested conditional $X \rightarrow (P \rightarrow Q)$ *should be more probable than not*. After all, $X \rightarrow (P \rightarrow Q)$ says "if the die landed even, then if it's in $\{1, 3, 5, 6\}$, it's 6," which seems probable (even certain) given the setup.

It seems we will need to revise the standard ratio formula when computing indicative suppositional probabilities $\Pr(p \rightarrow q \parallel x)$.[6] One natural revision, inspired by discussions with Wes Holliday and Jim Joyce, is based on the following principle for factual p, q, and x:

(†) $$\Pr(p \rightarrow q \parallel x) = \Pr[(x \,\&\, p) \rightarrow q \parallel x].$$[7]

The motivating intuition behind (†) is that — on the indicative supposition that x — we should be equally confident in the conditionals $p \rightarrow q$ and $(x \,\&\, p) \rightarrow q$. This leads to the following alternative to the ratio formula for the indicative suppositional credence $\Pr(p \rightarrow q \parallel x)$.

Ratio*. For factual propositions p, q, and x,

$$\Pr(p \rightarrow q \parallel x) = \Pr[(x \,\&\, p) \rightarrow q \mid x].$$

Finally, note that $x \,\&\, p$ screens off x from q, so QREQ can be applied to $\Pr[(x \,\&\, p) \rightarrow q \mid x]$, yielding the following revision of QREQ.[8]

QREQ*. Rational indicative suppositional credence functions $\Pr(\cdot \parallel \cdot)$ should be such that, for all factual p, q, and x,

$$\Pr(p \rightarrow q \parallel x) = \Pr(q \mid p \,\&\, x).$$

[6] See section 7.1, below, for a more direct way of reasoning to this conclusion.

[7] Holliday [10] proposes $\Pr(p \rightarrow q \parallel x) = \Pr[x \rightarrow (p \rightarrow q) \mid x]$, which is equivalent to **Ratio*** — if one assumes import-export as a (resilient) rational requirement.

[8] QREQ* is defended by Santorio & Goldstein. They call it "The Update Thesis."

This principle seems sensible in both admissible and inadmissible cases. For example, in the Die case, QREQ* tells us that $\Pr(P \rightarrow Q \parallel X) = \Pr(Q \mid P \,\&\, X) = 1$, which matches our intuition about the nested conditional. Moreover, if we require that $\Pr[x \rightarrow (p \rightarrow q)] = \Pr(p \rightarrow q \parallel x)$ [22, 21], then import-export — which is also defended by McGee and Goldstein & Santorio — follows from QREQ*, since

$$\Pr[x \rightarrow (p \rightarrow q)] = \Pr(p \rightarrow q \parallel x) = \Pr(q \mid p \,\&\, x) = \Pr[(x \,\&\, p) \rightarrow q].$$

7 Miscellaneous Related Arguments and Results

7.1 The Ratio Formula Conflicts Directly with The Resilient Equation

If we add The Resilient Equation to the following three fundamental principles of classical conditional probability.

(C1) $0 \leq \Pr(p \mid q) \leq 1$

(C2) $\Pr(p \mid p \,\&\, q) = 1$

(Ratio) $\Pr(p \,\&\, q \mid r) = \Pr(p \mid q \,\&\, r) \cdot \Pr(q \mid r)$

then we can derive the following *confirmational triviality* [16, 23].[9]

No Disconfirmation. For all factual propositions e, h, and k,

$$\Pr(h \mid k \,\&\, e) \geq \Pr(h \mid k).$$

This says that no factual evidence can ever disconfirm any factual hypothesis relative to any factual background knowledge. Since (C1) and (C2) seem non-negotiable, this strongly suggests — *pace* van Fraassen [22, 21] and my former self [4] — that The Resilient Equation *conflicts directly with* Ratio. Here, once again, we are in agreement with conclusions drawn independently by McGee and Goldstein & Santorio.

[9] Technically, the proof of No Disconfirmation also requires that conjunction is both associative and commutative.

7.2 Gibbard's Collapse Argument

But wait, didn't Gibbard [7, 5] show that any theory that accepts import-export for conditionals must reject The Equation, since such theories entail the equivalence of $\rightarrow$ and the material conditional $\supset$, and (at least sometimes) $\Pr(p \supset q) \neq \Pr(q \mid p)$? That is to say, doesn't Gibbard's collapse argument undermine what we've been saying here?

Gibbard's argument relies on the assumption that $p \,\&\, (p \rightarrow q) \vDash q$ holds for *all* conditionals, even nested ones. The version of Modus Ponens assumed above only applies to simple, non-nested conditionals.

We have already seen a counterexample to Gibbard's more general version of Modus Ponens (assuming import-export and The Equation). In Santorio's Die case, let $p \stackrel{\text{def}}{=} X$ and $q \stackrel{\text{def}}{=} P \rightarrow Q$. Import-export entails that $\Pr[X \rightarrow (P \rightarrow Q)] = \Pr[(X \,\&\, P) \rightarrow Q] = 1$.[10] But The Equation gives us $\Pr(P \rightarrow Q) = 1/4$. Therefore

$$\Pr[p \,\&\, (p \rightarrow q)] = \Pr[X \,\&\, (X \rightarrow (P \rightarrow Q))] = \frac{1}{2} > \frac{1}{4} = \Pr(q).$$

Note: if $\alpha \vDash \beta$, then $\Pr(\alpha) \leq \Pr(\beta)$. Therefore, $p \,\&\, (p \rightarrow q) \nvDash q$ in this case. Seen from this perspective, Gibbard's collapse argument is *unsound* when applied to cases like Santorio's Die.

7.3 Mandelkern's Collapse Argument

Mandelkern [14] proved that any conditional satisfying import-export, together with reflexivity ($p \rightarrow p$) and ad falsum (if $p \rightarrow q$ and $p \rightarrow \neg q$, then $\neg p$), must collapse to the material conditional.

But, Mandelkern's result makes essential use of *left-nested conditionals* (*i.e.*, conditionals of the form $\ulcorner (x \rightarrow p) \rightarrow q \urcorner$). McGee's theory does not even include such conditionals as well-formed formulae. Similarly, we have not said anything about such conditionals here. Goldstein and Santorio [8] restrict import-export to non-left-nested conditionals, which is how they manage to escape collapse to the material conditional. Our approach is compatible with various strategies for

[10] We don't really need import-export here. So long as $\Pr[X \& (X \rightarrow (P \rightarrow Q))] > 1/4$, this will be a counterexample to Gibbard's strong form of modus ponens. Intuitively, this really just means that $\Pr[X \rightarrow (P \rightarrow Q)]$ needs to be *greater than* $1/2$, which is quite plausible in Santorio's Die case — *independently of import-export*.

handling left-nested conditionals, but we leave the details of such an extension of our approach for future work.

7.4 No Antecedent Confirmation

The following principle is a consequence of our approach, along with *many* others, including trivalent theories and material conditional analyses of the indicative conditional [15, 8, 3, 24].[11]

No Antecedent Confirmation (NAC). For all factual p and q,

$$\Pr(p \to q \parallel p) \leq \Pr(p \to q).$$

In other words, supposing the antecedent of an indicative conditional should never increase our confidence in that conditional.

Some authors have argued that there are cases in which p confirms (is positively relevant to) $p \to q$.[12] If such cases exist, then all of these various analyses of the indicative conditional must be incorrect. More precisely, one of the following three principles (also accepted by all of these theories) must be rejected, for some *factual* p and q.

(1) $\Pr(p \,\&\, (p \to q)) \leq \Pr(p \,\&\, q)$.

(2) $\Pr(p \to q \parallel p) = \Pr(p \to q \mid p) = \dfrac{\Pr(p \,\&\, (p \to q))}{\Pr(p)}$.

(3) $\Pr(p \to q) \geq \Pr(q \mid p)$.

If (1) fails, then there are counterexamples to even the simplest form of modus ponens (*i.e.*, some counterexample with a simple conditional). I don't know of any arguments for *that*. So, if there are counterexamples to (NAC), then it would seem they must either be counterexamples to (2) or (3). It's hard to see why one would want to reject (2). The problematic cases we have seen do not seem to threaten it. Santorio's Die seems to be a counterexample to the more general principle

$$(2^\star) \qquad \Pr(p \to q \parallel x) = \Pr(p \to q \mid x).$$

[11] Our theory implies the stronger principle that p is *independent* of $p \to q$.

[12] This possibility has been raised by Igor Douven [2] and Steve Yablo (p.c.).

But, the $x := p$ instance of $(2^\star)$ seems far more plausible (I would explain this difference by appealing to QREQ: *p screens off p from q*).

If all of this is right, then any counterexample to (NAC) would have to be a case in which $\Pr(p \rightarrow q) < \Pr(q \mid p)$. It would be interesting to examine purported counterexamples to (NAC) more closely — with these considerations in mind. I'll have to leave that for another paper.

8 Appendix

8.1 REQ is *Unsatisfiable* (in Kolmogorov's theory)

Consider the following three instances of The Resilient Equation (where $\top$ is an arbitrary tautology).

(I_1) $\Pr(P \rightarrow Q \mid \neg Q) = \Pr(Q \mid P \,\&\, \neg Q)$.

(I_2) $\Pr(P \rightarrow Q \mid P \supset Q) = \Pr(Q \mid P \,\&\, (P \supset Q))$.

(I_3) $\Pr(P \rightarrow Q \mid \top) = \Pr(Q \mid P \,\&\, \top)$.[13]

In [4], I showed that I_1–I_3 jointly entail

(REQ)-Triviality. $\Pr(P \,\&\, (Q \equiv (P \rightarrow Q))) = 1$.

That proof presupposes that we are working in Kolmogorov's probability calculus. And, in that setting, it can actually be shown that The Resilient Equation is *unsatisfiable by any probability function*, if it is meant to hold for all factual P, Q.

The easiest way to see this is to add a fourth atomic sentence '$Q \rightarrow P$' to our base language (with the obvious extra-systematic interpretation). Then, the following will also be an instance of REQ.

(I_4) $\Pr(Q \rightarrow P \mid \neg P) = \Pr(P \mid Q \,\&\, \neg P)$.

In Kolmogorov's theory, I_1–I_4 *are jointly unsatisfiable*. After all, I_1–I_3 entail **(REQ)-Triviality**. And, **(REQ)-Triviality** entails that $\Pr(P) = 1$. Hence, **(REQ)-Triviality** entails that $\Pr(\neg P) = 0$. But, then, both sides

[13] Of course, I_3 is just The Equation itself.

of I_4 will be *undefined* in Kolmogorov's theory. So, no Kolmogorov probability function can satisfy I_1–I_4. Hence, *no Kolmogorov probability function can satisfy The Resilient Equation* (if it is meant to hold for *all* factual P and Q). Hence, I was wrong to call my [4] "The Strongest" Lewisian triviality result — nothing is stronger than *unsatisfiability*!

Popper's theory [18], which takes conditional probability as primitive, does avoid this unsatisfiability result. But, not by much. If we add The Resilient Equation to Popper's theory, then we get the result that the conditional probability function $\Pr(x \mid y)$ can only take the values 0 and 1 (for all statements x and y), which effectively collapses probability calculus back to truth-functional logic [19, p. 47].

8.2 (QREQ) *is Strictly Stronger than* (EQ)

Fact. (QREQ) $\Rightarrow$ (EQ), but (QREQ) $\not\Leftarrow$ (EQ).

Proof. ($\Rightarrow$) Let x be a tautology ($\top$). Antecedent Screening obtains: $\top$ is screened-off from every q by every p, on every $\Pr(\cdot)$. Since tautologies are admissible, (QREQ) implies (EQ).

($\not\Leftarrow$) We exhibit a regular probability function $\Pr(\cdot)$ over the $\{P, Q, P \rightarrow Q\}$ algebra such that (1) $\Pr(P \rightarrow Q) = \Pr(Q \mid P)$, and (2) $\Pr(Q \mid P \,\&\, x) = \Pr(Q \mid P)$, but (3) $\Pr(P \rightarrow Q \mid x) \neq \Pr(Q \mid P \,\&\, x)$, where x is a Boolean function of P and Q (specifically, $x \stackrel{\text{def}}{=} P$). To wit, consider the following regular probability distribution (discovered with PrSAT [6]).

P	Q	$P \rightarrow Q$	$\Pr(\cdot)$
T	T	T	$1/32$
T	T	F	$1/8$
T	F	T	$1/16$
T	F	F	$3/32$
F	T	T	$13/64$
F	T	F	$1/4$
F	F	T	$13/64$
F	F	F	$1/32$

On this model:

(1) $\Pr(P \rightarrow Q) = \Pr(Q \mid P) = 1/2$

(2) $\Pr(Q \mid P \,\&\, P) = \Pr(Q \mid P) = {}^1/_2$

(3) $\Pr(P \to Q \mid P) = {}^3/_{10} \neq {}^1/_2 = \Pr(Q \mid P \,\&\, P) = \Pr(Q \mid P)$

This shows that (QREQ) is strictly stronger than (EQ). □

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