Probabilities of Conditionals and Conditional Probabilities — Revisited

Branden Fitelson

Northeastern University
branden@fitelson.org

- Lewis's Two Conceptions of "Rational Requirement"
  - 1976: "Probabilities of Conditionals & Conditional Probabilities"
    - Very strong conception, which assumes that all rational requirements hold (fully) *resiliently* — leads to trivialities.
  - 1980: "A Subjectivist's Guide to Objective Chance"
    - A weaker (*viz.*, more permissive) conception, which does *not* require (full) *resiliency*. It identifies "inadmissible" *exceptions* to resiliency, and thereby *avoids* triviality.
- The 1976 paper led many to *reject* The Equation.
- The 1980 paper led many to *accept* the Principal Principle.

☞ If we apply Lewis's 1980 conception to The Equation, we obtain a far more charitable (and plausible) way of modeling the epistemic probabilities of conditionals, Adams-style.

Consider the following constraint [1, 12] on a (prior/initial) credence function $\Pr(\cdot)$, over a language containing two factual atoms P and Q, and a third atom "$P \to Q$" (where "$P \to Q$" is interpreted *extra-systematically* as the indicative "if P, then Q").

The Equation. $\Pr(P \to Q) = \Pr(Q \mid P)$.

☞ Lewis [8] assumes that *if* **The Equation** is *rationally required*, *then* it must hold (fully) *resiliently* — *i.e.*, that the following *strengthening* of **The Equation** must be a constraint on $\Pr(\cdot)$.

The Resilient Equation.[1] For all x (Boolean-definable in terms of P, Q) such that $\Pr(P \,\&\, x) > 0$,

$$\Pr(P \to Q \mid x) = \Pr(Q \mid P \,\&\, x).$$

Various triviality results have been derived from **The Resilient Equation.** The strongest possible such triviality result [3] is this.

Triviality. If $\Pr(P \,\&\, Q) > 0$ and $\Pr(P \,\&\, \neg Q) > 0$, then

$$\Pr(P \,\&\, (Q \equiv (P \to Q))) = 1.$$

The following table illustrates the basic algebraic relationships between **The Equation** and **The Resilient Equation** [3].

P	Q	$P \to Q$	$\Pr(\cdot)$	$\Pr(\cdot)$ + **The Equation**	$\Pr(\cdot)$ + **R. Equation**
T	T	T	a	a	a
T	T	F	b	b	0
T	F	T	c	c	0
T	F	F	d	d	$1-a$
F	T	T	e	e	0
F	T	F	f	f	0
F	F	T	g	$\frac{a+b}{a+b+c+d} - a - c - e$	0
F	F	F	$1-\sum$	$1-\sum$	0

The Equation reduces the number of $\Pr(\cdot)$'s degrees of freedom (from 7) to 6, and **The Resilient Equation** reduces it to 1.

In 1976, Lewis assumed that any rational requirement on initial (*viz.*, prior) credence $\Pr(\cdot)$ must be *fully resilient*. What he *says* in connection with this *doesn't entail* the desired (full) resiliency:

> the ... class of all those probability functions that represent possible systems of beliefs ... is closed under conditionalizing. Rational change of belief never can take anyone to a subjective probability function outside the class; and ... the change of belief that results from coming to know an item of new evidence should take place by conditionalizing on what was learned.

Even if we grant Lewis all of these claims, they don't imply that all rational requirements on $\Pr(\cdot)$ must be *fully resilient*.

Of course, *some* constraints *do* satisfy even this very strong requirement. For instance, *probabilism itself* must satisfy it.

For *it is a theorem of the probability calculus* that if an initial credence function is a probability function $\Pr(\cdot)$, then so is $\Pr(\cdot \mid x)$, provided only that $\Pr(\cdot \mid x)$ well-defined. [The above quotation from Lewis (1976) articulates something close to this truism.]

My reconstruction: Lewis (1976) assumed (for *reductio*) that **The Equation** is a rational requirement on $\Pr(\cdot)$. Then, he used the full resiliency requirement to complete his *reductio*.

(1) **The Equation** is a rational requirement for $\Pr(\cdot)$.
(2) $\therefore$ **The Equation** must hold in a (fully) *resilient* way.
(3) $\therefore$ **The Resilient Equation** is a rational requirement for $\Pr(\cdot)$.
(4) But, **Triviality** is *not* a rational requirement for $\Pr(\cdot)$.
(5) Contradiction. [Since (3) entails $\neg$(4).]
(6) $\therefore$ **The Equation** is *not* a rational requirement for $\Pr(\cdot)$. □

Lewis (1976) presupposes that (1) *implies* (2) (I call this **Presupposition #1**). This is where the argument goes wrong.

Premise (4), which I call **Presupposition #2**, can be established directly, *via a knock-down counterexample* to **Triviality**.

☞ Lewis (1980) is *not* moved by an analogous "*reductio* of the Principal Principle" as a rational requirement for priors.

Lewis [9] maintains that the Principal Principle (PP) is a rational requirement on initial/prior credence functions $\Pr(\cdot)$.

(PP) $\Pr(p \mid \mathrm{Ch}(p) = c) = c$.

Lewis knows that if we require (PP) to hold (fully) *resiliently*, then we get something *trivial*. To wit, consider the following schema:

(PP_x) $\Pr(p \mid x \,\&\, \mathrm{Ch}(p) = c) = c$.

Resilient (PP) asserts that (PP_x) holds *for all x such that* (PP_x) *is well-defined*. *That* principle $[(\forall x)\,\mathrm{PP}_x]$ is *trivial*. Let P, $\mathrm{Ch}(P) = 1$ and $\mathrm{Ch}(P) = 0$ be our three atoms. Then, it can be shown that:

> **(PP)–Triviality.** $(\forall x)\mathrm{PP}_x$ implies *only two states can have non-zero probability*: $P \,\&\, \mathrm{Ch}(P) = 1$ and $\neg P \,\&\, \mathrm{Ch}(P) = 0$.

Strangely, Lewis is *not* swayed by a "*reductio*" of (PP), which is completely analogous the Lewisian *reductio* of **The Equation**.

The following table illustrates the basic algebraic relationships between (PP) and **Resilient (PP)**.

P	$\mathrm{Ch}(P) = 0$	$\mathrm{Ch}(P) = 1$	$\Pr(\cdot)$	$\Pr(\cdot)$ + (PP)	$\Pr(\cdot)$ + **Resilient (PP)**
T	T	T	0	0	0
T	T	F	a	0	0
T	F	T	b	b	b
T	F	F	c	c	0
F	T	T	0	0	0
F	T	F	d	d	$1-b$
F	F	T	e	0	0
F	F	F	$1-\sum$	$1-\sum$	0

(PP) reduces the number of $\Pr(\cdot)$'s degrees of freedom (from 5) to 3, and **Resilient (PP)** reduces it to 1.

(1) (PP) is a rational requirement for $\Pr(\cdot)$.
(2) $\therefore$ (PP) must hold in a (fully) *resilient* way.
(3) $\therefore$ **Resilient (PP)** is a rational requirement for $\Pr(\cdot)$.
(4) But, **(PP)-Triviality** is *not* a rational requirement for $\Pr(\cdot)$.
(5) Contradiction. [Since (3) entails $\neg$(4).]
(6) $\therefore$ (PP) is *not* a rational requirement for $\Pr(\cdot)$. □

This time, Lewis *rejects* the presupposition that (1) *implies* (2).

He introduces the notion of "admissibility" with the aim of demarcating those x's for which $(\forall x)\mathrm{PP}_x$ *is* a rational requirement. Lewisian admissibility is a rather subtle concept.

(PP)-Triviality suggests p and $\neg p$ are *in*admissible (with respect to PP). Analogously, **Triviality** suggests $\neg Q$, $P \supset Q$, and their negations are inadmissible (with respect to **The Equation**) [3].

In both of these cases, the simple restrictions inspired by *triviality* merely scratch the surface of (in)admissibility.

Lewis requires that *admissible* x's (with respect to PP) must satisfy **Chance Screening**, *i.e.*, x must be s.t., *for all* $c \in [0,1]$,

$$\Pr(p \mid x \,\&\, \mathrm{Ch}(p) = c) = \Pr(p \mid \mathrm{Ch}(p) = c).$$

The analogous quantifier restriction for **The Resilient Equation** is that x satisfy **Antecedent Screening**, *i.e.*, that x be s.t. *both*

$$\Pr(Q \mid x \,\&\, P) = \Pr(Q \mid P) \text{ and } \Pr(Q \mid x \,\&\, \neg P) = \Pr(Q \mid \neg P).$$

Why? Consider *indicative conditionals with chance antecedents*, *e.g.*, $P := \mathrm{Ch}(Q) = 1/2$. Then, **The Equation** & (PP) jointly imply

$$\Pr(P \to Q) = \Pr(Q \mid P) = 1/2$$

Now, the only way to bring **The Resilient Equation** into alignment with **The Quasi-Resilient Principal Principle** is to impose **Antecedent Screening** (*viz.*, $Q \perp\!\!\!\perp x \mid P$).

☞ This simple idea leads to a more plausible rational requirement.

Once we restrict **The Resilient Equation** to *admissible* x's, we end-up with a far more plausible requirement on initial $\Pr(\cdot)$.

The Quasi-Resilient Equation (TQRE). All rational initial credence functions $\Pr(\cdot)$ should satisfy the following *restricted* version of **The Resilient Equation.**

For all factual propositions p, q, and x:

$$\Pr(p \to q \mid x) = \Pr(q \mid p \,\&\, x),$$

provided that x satisfies **Antecedent Screening**.

Antecedent Screening restricts **The Resilient Equation** to x's *which don't interfere with/trump the (a priori) informational connection between antecedent and consequent.*

☞ Intuitively, when we advise a certain level of confidence in $P \to Q$, we are presupposing the advisee *doesn't know anything which trumps the informational connection between* P *and* Q.

Here is a *knock-down* counterexample to **The Resilient Equation** (thanks to Paolo Santorio). A fair die (Die) was tossed.

(P) Die landed on either 1, 3, 5, or 6.
(Q) Die landed on 6.
(X) Die landed even.

P	Q	$\Pr(\cdot)$
T	**T**	$1/6$
T	**F**	$1/2$
F	**T**	0
F	**F**	$1/3$

The Equation $\Rightarrow \Pr(P \to Q) = \Pr(Q \mid P) = 1/4$. $\therefore \Pr((P \to Q) \,\&\, X) \leqslant 1/4$.

$$\therefore \Pr(P \to Q \mid X) = \frac{\Pr((P \to Q) \,\&\, X)}{\Pr(X)} \leqslant \frac{1/4}{1/2} = 1/2 < 1 = \Pr(Q \mid P \,\&\, X).$$

Note, however, that X is *inadmissible* (by our criterion), since

$$\Pr(Q \mid X \,\&\, P) = 1 \neq 1/4 = \Pr(Q \mid P).$$

☞ *Every* knock-down counterexample to **The Resilient Equation** seems to involve *inadmissible* x's (in our sense).

The problem of formulating a *logic* for the conditional — which respects (TQRE) — is the next phase of my project.

Many existing $\rightarrow$-logics are *incompatible* with **The Equation.**

Stalnaker [13] shows that **The Equation** is incompatible with *his* preferred conditional logic (**C2**). His argument can be generalized to *all* Lewis-Stalnaker conditional logics.

Logics validating *import-export* (in general) are a *no-go* [4, 14].

van Fraassen [14] develops a logic that respects **The Equation** + *a limited amount of* resiliency (*i.e., a limited amount of Imp-Exp*).

My approach is similar to (but more modest and focused than) van Fraassen's. I begin with a **Minimal Probabilistic Logic** (**MPL**).

The basic idea behind **MPL** is to start with a single-premise entailment relation ($\Vdash$), which I define in the following way.

(**MPL**$_0$) $p \Vdash q \overset{\text{def}}{=} \Pr(p) \leqslant \Pr(q)$, $\forall$ Pr-functions satisfying (TQRE).

This is the most conservative/flat-footed way to extend $\vDash$ to $\Vdash$.

Consider the following entailment relation for **The Equation**:

$$p \Vvdash q \overset{\text{def}}{=} \Pr(p) \leqslant \Pr(q),\ \forall \text{ Pr's satisfying } \textbf{The Equation.}$$

$p \Vvdash q \Longrightarrow p \Vdash q$, since every probability function satisfying **The Quasi-Resilient Equation** also satisfies **The Equation**.

Therefore, the following chain of **MPL** validites holds

(**MPL**$_1$) $p \,\&\, q \Vdash p \rightarrow q \Vdash p \supset q$.

This chain holds for $\Vvdash$ because $\Pr(p \supset q) \geqslant \Pr(q \mid p) \geqslant \Pr(p \,\&\, q)$ is a theorem of Pr-calculus. So, $\rightarrow$ is "intermediate" between & and $\supset$, in terms of logical strength (according to **MPL**).

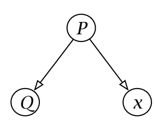
It remains open whether $p \Vdash q \Longrightarrow p \Vvdash q$.

Ideally, we would like to have an (elegant) *axiomatization* of **MPL**.

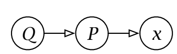
The good news is that **MPL** is *decidable*. I have written *Mathematica* functions (building on `PrSAT` [5]) for testing validities involving $\Vdash$ (and $\Vvdash$). Hopefully, these will help...

Because (TQRE) involves a screening-off relation $Q \perp\!\!\!\perp x \mid P$, there are natural connections between this project an the Bayes Nets/Causal Modeling/Probabilistic Causality literature [7].

The two most common network structures in which **Antecedent Screening** will arise are (conjunctive) *forks* and *chains*.



Fork



Chain

Chain example: P := John contracts HIV, Q := John has unprotected sex, and x := John contracts AIDS.

Fork example: P := the barometric pressure drops, Q := there is a storm, and x := the barometer's mercury column drops.

☞ In both cases, (TQRE) implies (sensibly): $\Pr(P \rightarrow Q \mid x) = \Pr(P \rightarrow Q)$.

[1] E. Adams, *The Logic of Conditionals*, 1965.

[2] R. Bradley, *Decision Theory with a Human Face*, 2017.

[3] B. Fitelson, *The Strongest Possible Lewisian Triviality Result*, 2015.

[4] ———, Gibbard's Collapse Theorem for the Indicative Conditional: An Axiomatic Approach, 2013.

[5] ———, *A Decision Procedure for Probability Calculus with Applications*, 2008.

[6] A. Hájek, *Probabilities of Conditionals — Revisited*, 1989.

[7] C. Hitchcock, *Probabilistic Causation*, 2018.

[8] D. Lewis, *Probabilities of Conditionals and Conditional Probabilities*, 1976.

[9] ———, *A Subjectivist's Guide to Objective Chance*, 1980.

[10] I. Nissan-Rozen, *Jeffrey Conditionalization, the Principal Principle, the Desire as Belief Thesis, and Adams's Thesis*, 2013.

[11] B. Skyrms, *Resiliency, Propensities, and Causal Necessity*, 1977.

[12] R. Stalnaker, *A Theory of Conditionals*, 1968.

[13] ———, *Letter to van Fraassen*, 1976.

[14] B. van Fraassen, *Probabilities of Conditionals*, 1976.