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# Examples from “A Decision Procedure for Probability Calculus with Applications”

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First, load the **PrSAT** package (freely downloadable from the **PrSAT** website: <http://fitelson.org/PrSAT/>).

<< **PrSAT**^

## ■ Example #1: A probability model in which three statements **A**, **B**, and **C** are *pariwise* independent but not *mutually* independent

Note: the additional equational side-constraints [**Pr**[**A**] ==  $\frac{1}{10}$ , **Pr**[**B**] ==  $\frac{1}{10}$ , **Pr**[**C**] ==  $\frac{1}{10}$ ] are added here to reduce the number of variables of the problem, so that the model is found *much* more quickly. This is a very useful heuristic for speeding-up model searches. The option **Probabilities→Regular** guarantees that the model generated is *regular* (i.e., that it assigns nonzero probability to all state descriptions of the minimal sentential language required for the expression of the problem given). Finally, the option **BypassSearch→True** tells *Mathematica* to skip Blum’s random search add-on, and send the problem straight to the decision procedure.

```

MODEL1 = PrSAT [
  {
    Pr[A ∧ B ∧ C] ≠ Pr[A] Pr[B] Pr[C],
    Pr[A ∧ B] == Pr[A] Pr[B],
    Pr[A ∧ C] == Pr[A] Pr[C],
    Pr[B ∧ C] == Pr[B] Pr[C],

    (* Heuristic -- add additional
      equational side-constraints *)
    Pr[A] ==  $\frac{1}{10}$ , Pr[B] ==  $\frac{1}{10}$ , Pr[C] ==  $\frac{1}{10}$ 
  },
  Probabilities → Regular,
  BypassSearch → True
]

{ {A → {a2, a5, a6, a8},
  B → {a3, a5, a7, a8}, C → {a4, a6, a7, a8},
  Ω → {a1, a2, a3, a4, a5, a6, a7, a8}},
  { a1 →  $\frac{1459}{2000}$ , a2 →  $\frac{161}{2000}$ , a3 →  $\frac{161}{2000}$ , a4 →  $\frac{161}{2000}$ ,
    a5 →  $\frac{19}{2000}$ , a6 →  $\frac{19}{2000}$ , a7 →  $\frac{19}{2000}$ , a8 →  $\frac{1}{2000}$  } }

```

**PrSAT** also includes a **TruthTable** function, which allows for the visualization of a model, as a stochastic truth-table:

**TruthTable [MODEL1]**

A	B	C	var	Pr
T	T	T	a <sub>8</sub>	$\frac{1}{2000}$
T	T	F	a <sub>5</sub>	$\frac{19}{2000}$
T	F	T	a <sub>6</sub>	$\frac{19}{2000}$
T	F	F	a <sub>2</sub>	$\frac{161}{2000}$
F	T	T	a <sub>7</sub>	$\frac{19}{2000}$
F	T	F	a <sub>3</sub>	$\frac{161}{2000}$
F	F	T	a <sub>4</sub>	$\frac{161}{2000}$
F	F	F	a <sub>1</sub>	$\frac{1459}{2000}$

If we set **BypassSearch**→**False**, then Blum's random-search add-on is consulted first, and a model is (usually) found (relatively) quickly *even without any side-constraints!*

```

MODEL11 = PrSAT[
  {
    Pr[A ∧ B ∧ C] ≠ Pr[A] Pr[B] Pr[C],
    Pr[A ∧ B] == Pr[A] Pr[B],
    Pr[A ∧ C] == Pr[A] Pr[C],
    Pr[B ∧ C] == Pr[B] Pr[C]
  },
  Probabilities → Regular,
  BypassSearch → False
]

```

$$\left\{ \begin{array}{l}
 \mathbf{A} \rightarrow \{a_2, a_5, a_6, a_8\}, \\
 \mathbf{B} \rightarrow \{a_3, a_5, a_7, a_8\}, \mathbf{C} \rightarrow \{a_4, a_6, a_7, a_8\}, \\
 \mathbf{\Omega} \rightarrow \{a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8\}, \\
 a_1 \rightarrow \frac{84\,418 - 39\sqrt{4\,676\,097}}{56\,277}, \\
 a_2 \rightarrow \frac{-42\,296 + 39\sqrt{4\,676\,097}}{168\,831}, \\
 a_3 \rightarrow \frac{-42\,296 + 39\sqrt{4\,676\,097}}{168\,831}, \\
 a_4 \rightarrow \frac{-42\,296 + 39\sqrt{4\,676\,097}}{168\,831}, a_5 \rightarrow \frac{1}{999}, \\
 a_6 \rightarrow \frac{1}{999}, a_7 \rightarrow \frac{1}{999}, a_8 \rightarrow \frac{42}{169} \}
 \end{array} \right.$$

### TruthTable [MODEL11]

A	B	C	var	Pr
T	T	T	a <sub>8</sub>	$\frac{42}{169}$
T	T	F	a <sub>5</sub>	$\frac{1}{999}$
T	F	T	a <sub>6</sub>	$\frac{1}{999}$
T	F	F	a <sub>2</sub>	$\frac{-42\,296+39\sqrt{4\,676\,097}}{168\,831}$
F	T	T	a <sub>7</sub>	$\frac{1}{999}$
F	T	F	a <sub>3</sub>	$\frac{-42\,296+39\sqrt{4\,676\,097}}{168\,831}$
F	F	T	a <sub>4</sub>	$\frac{-42\,296+39\sqrt{4\,676\,097}}{168\,831}$
F	F	F	a <sub>1</sub>	$\frac{84\,418-39\sqrt{4\,676\,097}}{56\,277}$

We can also Evaluate probabilities on given models, as follows:

**EvaluateProbability[**  
**{Pr[A | B], Pr[B | A ∧ ¬ C]}, MODEL11]**

$$\left\{ \frac{42\,127}{168\,831 \left( \frac{42\,296}{168\,831} + \frac{-42\,296+39\sqrt{4\,676\,097}}{168\,831} \right)}, \frac{1}{999 \left( \frac{1}{999} + \frac{-42\,296+39\sqrt{4\,676\,097}}{168\,831} \right)} \right\}$$

**% // N**

**{0.499521, 0.00400401}**

## ■ A Simultaneous Countermodel to the **S**-instances of both (\*) and (†)

Again, the additional equational side-constraints  $[\mathbf{Pr}[\mathbf{H}] == \frac{1}{2}, \mathbf{Pr}[\mathbf{E1}] == \frac{1}{4}, \mathbf{Pr}[\mathbf{E2}] == \frac{3}{4}]$  are added to speed the model search, and **Probabilities** → **Regular** indicates that we are asking **PrSAT** to find a *regular* probability model.

```

MODEL2 = PrSAT [
  {
    Pr[H | E1] > Pr[H],
    Pr[H | E2] > Pr[H],
    Pr[H | E1] > Pr[H | E2],
    Pr[H | E1] - Pr[H | ¬ E1] <
      Pr[H | E2] - Pr[H | ¬ E2],
    Pr[H | E1 ∧ E2] - Pr[H | ¬ E1 ∧ E2] ==
      Pr[H | E1] - Pr[H | ¬ E1],
    Pr[H | E2 ∧ E1] - Pr[H | ¬ E2 ∧ E1] ==
      Pr[H | E2] - Pr[H | ¬ E2],
    Pr[H | E1 ∧ E2] - Pr[H | ¬ (E1 ∧ E2)] <
      Pr[H | E2] - Pr[H | ¬ E2],

    (* Heuristic -- add additional
       equational side-constraints *)
    Pr[H] ==  $\frac{1}{2}$ , Pr[E1] ==  $\frac{1}{4}$ , Pr[E2] ==  $\frac{3}{4}$ 
  },
  Probabilities → Regular,
  BypassSearch → True
]

{ {E1 → {a2, a5, a6, a8},
  E2 → {a3, a5, a7, a8}, H → {a4, a6, a7, a8},
  Ω → {a1, a2, a3, a4, a5, a6, a7, a8}},
  { a1 →  $\frac{147}{1024}$ , a2 →  $\frac{193}{5120}$ , a3 →  $\frac{1341}{5120}$ , a4 →  $\frac{45}{1024}$ ,
    a5 →  $\frac{291}{5120}$ , a6 →  $\frac{127}{5120}$ , a7 →  $\frac{1539}{5120}$ , a8 →  $\frac{669}{5120}$  } }

```

### TruthTable [MODEL2]

E1	E2	H	var	Pr
T	T	T	a <sub>8</sub>	$\frac{669}{5120}$
T	T	F	a <sub>5</sub>	$\frac{291}{5120}$
T	F	T	a <sub>6</sub>	$\frac{127}{5120}$
T	F	F	a <sub>2</sub>	$\frac{193}{5120}$
F	T	T	a <sub>7</sub>	$\frac{1539}{5120}$
F	T	F	a <sub>3</sub>	$\frac{1341}{5120}$
F	F	T	a <sub>4</sub>	$\frac{45}{1024}$
F	F	F	a <sub>1</sub>	$\frac{147}{1024}$

### ■ A Simultaneous Countermodel to two claims concerning Hawthorne & Fitelson's new Bayesian approach to the raven paradox

The following single model shows that neither of the following two claims:

$$(6) \Pr(H \mid \sim R \ \& \ \sim B) > \Pr(H)$$

$$(7) \Pr(H \mid \sim R \ \& \ B) < \Pr(H)$$

follows from the following three claims:

$$(1) \Pr(R \mid H \ \& \ B) = 1$$

$$(2) \Pr(\sim B) > \Pr(R)$$

$$(C) \Pr(H \mid R) \geq \Pr(H \mid \sim B)$$

Here, a regular model is impossible (since one of the constraints requires a zero probability for one of the state descriptions). But, by adding the constraint  $\Pr [ (\neg H) \wedge B \wedge (\neg R) ] > 0$ , we can ensure that this is the only zero in the model. And, as usual, we add equational side-constraints  $[\Pr [ H ] = \frac{60}{100}, \Pr [ R ] = \frac{20}{100}, \Pr [ B ] = \frac{10}{100}]$  to speed the model-finding process by reducing the number of free variables in the problem.

```

MODEL3 = PrSAT [
  {
    Pr[H ∧ R ∧ (¬ B)] == 0,
    Pr[ (¬ H) ∧ B ∧ (¬ R)] > 0,
    Pr[¬ B] > Pr[R],
    Pr[H | R] == Pr[H | ¬ B],
    Pr[H | (¬ R) ∧ (¬ B)] < Pr[H],
    Pr[H | B ∧ (¬ R)] > Pr[H]
  },
  Constraints →
  { Pr[B] ==  $\frac{1}{10}$ , Pr[H] ==  $\frac{1}{2}$ , Pr[R] ==  $\frac{1}{10}$  },
  BypassSearch → True
]

```

```

{ {B → {a2, a5, a6, a8},
  H → {a3, a5, a7, a8}, R → {a4, a6, a7, a8},
  Ω → {a1, a2, a3, a4, a5, a6, a7, a8}},
  { a1 →  $\frac{141}{320}$ , a2 →  $\frac{21}{2560}$ , a3 →  $\frac{225}{512}$ , a4 →  $\frac{51}{2560}$ ,
    a5 →  $\frac{3}{256}$ , a6 →  $\frac{1}{32}$ , a7 → 0, a8 →  $\frac{25}{512}$  } }

```

**TruthTable** [**MODEL3**]

B	H	R	var	Pr
T	T	T	a <sub>8</sub>	$\frac{25}{512}$
T	T	F	a <sub>5</sub>	$\frac{3}{256}$
T	F	T	a <sub>6</sub>	$\frac{1}{32}$
T	F	F	a <sub>2</sub>	$\frac{21}{2560}$
F	T	T	a <sub>7</sub>	<b>0</b>
F	T	F	a <sub>3</sub>	$\frac{225}{512}$
F	F	T	a <sub>4</sub>	$\frac{51}{2560}$
F	F	F	a <sub>1</sub>	$\frac{141}{320}$

We can also solve this one (quickly) with Blum's random search add-on, and with no side-constraints:

```

PrSAT [
  {
    Pr[H  $\wedge$  R  $\wedge$  ( $\neg$  B)] == 0,
    Pr[( $\neg$  H)  $\wedge$  B  $\wedge$  ( $\neg$  R)] > 0,
    Pr[ $\neg$  B] > Pr[R],
    Pr[H | R] == Pr[H |  $\neg$  B],
    Pr[H | ( $\neg$  R)  $\wedge$  ( $\neg$  B)] < Pr[H],
    Pr[H | B  $\wedge$  ( $\neg$  R)] > Pr[H]
  },
  BypassSearch  $\rightarrow$  False
]

{ {B  $\rightarrow$  {a2, a5, a6, a8},
  H  $\rightarrow$  {a3, a5, a7, a8}, R  $\rightarrow$  {a4, a6, a7, a8},
   $\Omega$   $\rightarrow$  {a1, a2, a3, a4, a5, a6, a7, a8}},
  {a1  $\rightarrow$   $\frac{36\,693}{94\,240}$ , a2  $\rightarrow$   $\frac{2}{15}$ , a3  $\rightarrow$  0, a4  $\rightarrow$   $\frac{1}{93}$ ,
  a5  $\rightarrow$   $\frac{11}{32}$ , a6  $\rightarrow$   $\frac{7}{57}$ , a7  $\rightarrow$  0, a8  $\rightarrow$  0}} }

```

## ■ Theorems and Countermodels from Sobel's "Lotteries and Miracles"

Since Sobel's problems only involve two atomic sentences, no heuristics are needed to yield a fast solution by the decision procedure (and Blum's random search add-on is also not necessary, since the decision procedure is quite fast in such cases).

■ A PrSAT model showing that Sobel's (1)-(3) do *not* entail (4)

$$\begin{aligned}
 \text{MODEL4} = \text{PrSAT} [ & \\
 & \left\{ \text{Pr}[\mathbf{T}] < \frac{1}{2}, \right. \\
 & \quad \text{Pr}[\mathbf{T} \mid \mathbf{W}] > \frac{1}{2}, \\
 & \quad \text{Pr}[\mathbf{W} \mid \mathbf{T}] > \frac{1}{2}, \\
 & \quad \text{Pr}[\mathbf{T} \mid \neg \mathbf{W}] > \text{Pr}[\mathbf{W}] \\
 & \left. \right\}, \\
 & \text{Probabilities} \rightarrow \text{Regular}, \\
 & \text{BypassSearch} \rightarrow \text{True} \\
 & ] \\
 & \left\{ \left\{ \mathbf{T} \rightarrow \{a_2, a_4\}, \right. \right. \\
 & \quad \left. \left. \mathbf{W} \rightarrow \{a_3, a_4\}, \Omega \rightarrow \{a_1, a_2, a_3, a_4\} \right\}, \right. \\
 & \left. \left\{ a_1 \rightarrow \frac{6033}{8192}, a_2 \rightarrow \frac{1007}{8192}, a_3 \rightarrow \frac{1}{64}, a_4 \rightarrow \frac{1}{8} \right\} \right\}
 \end{aligned}$$

**TruthTable [MODEL4]**

T	W	var	Pr
T	T	$a_4$	$\frac{1}{8}$
T	F	$a_2$	$\frac{1007}{8192}$
F	T	$a_3$	$\frac{1}{64}$
F	F	$a_1$	$\frac{6033}{8192}$

■ A PrSAT model showing that Sobel's (1)-(3) do *not* entail (5)

$$\text{MODEL5} = \text{PrSAT} \left[ \begin{array}{l} \left\{ \text{Pr}[\mathbf{T}] < \frac{1}{2}, \right. \\ \text{Pr}[\mathbf{T} \mid \mathbf{W}] > \frac{1}{2}, \\ \text{Pr}[\mathbf{W} \mid \mathbf{T}] > \frac{1}{2}, \\ \left. \text{Pr}[\mathbf{T} \mid \mathbf{W}] - \text{Pr}[\mathbf{T} \mid \neg \mathbf{W}] < \text{Pr}[\neg \mathbf{W}] - \text{Pr}[\mathbf{W}] \right\}, \\ \text{Probabilities} \rightarrow \text{Regular}, \\ \text{BypassSearch} \rightarrow \text{True} \end{array} \right]$$

$$\left\{ \left\{ \mathbf{T} \rightarrow \{a_2, a_4\}, \right. \right. \\ \left. \left. \mathbf{W} \rightarrow \{a_3, a_4\}, \Omega \rightarrow \{a_1, a_2, a_3, a_4\} \right\}, \right. \\ \left. \left\{ a_1 \rightarrow \frac{51}{64}, a_2 \rightarrow \frac{1}{16}, a_3 \rightarrow \frac{1}{16}, a_4 \rightarrow \frac{5}{64} \right\} \right\}$$

**TruthTable [MODEL5]**

T	W	var	Pr
T	T	$a_4$	$\frac{5}{64}$
T	F	$a_2$	$\frac{1}{16}$
F	T	$a_3$	$\frac{1}{16}$
F	F	$a_1$	$\frac{51}{64}$

- A PrSAT “Proof” that Sobel’s (1)-(3) do entail the disjunction (4)  $\vee$  (5)

```

PrSAT [
  { Pr [T] < 1/2,
    Pr [T | W] > 1/2,
    Pr [W | T] > 1/2,
    Not [Or [Pr [T | ¬ W] < Pr [W],
            Pr [T | W] - Pr [T | ¬ W] > Pr [¬ W] - Pr [W] ] ]
  ],
  BypassSearch → True
]
{}

```