Examples from “A Decision Procedure for Probability Caluclus with Applications“

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First, load the **PrSAT** package (freely downloadable from the **PrSAT** website: **http://fitelson.org/PrSAT/**).

```
<< PrSAT`
```

■ Example #1: A probability model in which three statements A, B, and C are *pariwise* independent but not *mutually* independent

Note: the additional equational side-constraints [**Pr[A]** $== \frac{1}{10}$**, Pr[B]** $== \frac{1}{10}$**, Pr[C]** $== \frac{1}{10}$] are added here to reduce the number of variables of the problem, so that the model is found *much* more quickly. This is a very useful heuristic for speeding-up model searches. The option **Probabilities→Regular** guarantees that the model generated is *regular* (*i.e.,* that it assigns nnonzero probability to all state descrip tions of the minimal sentential language required for the expression of the problem given). Finally, the option **BypassSearch→True** tells *Mathematica* to skip Blum‘s random search add-on, and send the problem straight to the decision procedure.

```
MODEL1 = PrSAT[
  {
   Pr[A ∧ B ∧ C] ≠ Pr[A] Pr[B] Pr[C],
   Pr[A ∧ B] == Pr[A] Pr[B],
   Pr[A ∧ C] == Pr[A] Pr[C],
   Pr[B ∧ C] == Pr[B] Pr[C],

   (* Heuristic -- add additional
     equational side-constraints *)
   Pr[A] == 1/10, Pr[B] == 1/10, Pr[C] == 1/10
  },
  Probabilities → Regular,
  BypassSearch → True
 ]
```

$$\{\{A \to \{a_2, a_5, a_6, a_8\}, B \to \{a_3, a_5, a_7, a_8\}, C \to \{a_4, a_6, a_7, a_8\}, \Omega \to \{a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8\}\},$$
$$\{a_1 \to \frac{1459}{2000}, a_2 \to \frac{161}{2000}, a_3 \to \frac{161}{2000}, a_4 \to \frac{161}{2000}, a_5 \to \frac{19}{2000}, a_6 \to \frac{19}{2000}, a_7 \to \frac{19}{2000}, a_8 \to \frac{1}{2000}\}\}$$

`PrSAT` also includes a **`TruthTable`** function, which allows for the visualization of a model, as a stochastic truth-table:

TruthTable[MODEL1]

A	B	C	var	Pr
T	T	T	a_8	$\frac{1}{2000}$
T	T	F	a_5	$\frac{19}{2000}$
T	F	T	a_6	$\frac{19}{2000}$
T	F	F	a_2	$\frac{161}{2000}$
F	T	T	a_7	$\frac{19}{2000}$
F	T	F	a_3	$\frac{161}{2000}$
F	F	T	a_4	$\frac{161}{2000}$
F	F	F	a_1	$\frac{1459}{2000}$

If we set **BypassSearch→False**, then Blum‘s random-search add-on is consulted first, and a model is (usually) found (relatively) quickly *even without any side-constraints*!

```
MODEL11 = PrSAT[
  {
   Pr[A ∧ B ∧ C] ≠ Pr[A] Pr[B] Pr[C],
   Pr[A ∧ B] == Pr[A] Pr[B],
   Pr[A ∧ C] == Pr[A] Pr[C],
   Pr[B ∧ C] == Pr[B] Pr[C]
  },
  Probabilities → Regular,
  BypassSearch → False
 ]
```

$$\Big\{\{A \to \{a_2, a_5, a_6, a_8\},\ B \to \{a_3, a_5, a_7, a_8\},\ C \to \{a_4, a_6, a_7, a_8\},\ \Omega \to \{a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8\}\},$$

$$\Big\{a_1 \to \frac{84\,418 - 39\sqrt{4\,676\,097}}{56\,277},$$

$$a_2 \to \frac{-42\,296 + 39\sqrt{4\,676\,097}}{168\,831},$$

$$a_3 \to \frac{-42\,296 + 39\sqrt{4\,676\,097}}{168\,831},$$

$$a_4 \to \frac{-42\,296 + 39\sqrt{4\,676\,097}}{168\,831},\ a_5 \to \frac{1}{999},$$

$$a_6 \to \frac{1}{999},\ a_7 \to \frac{1}{999},\ a_8 \to \frac{42}{169}\Big\}\Big\}$$

```
TruthTable[MODEL11]
```

A	B	C	var	Pr
T	T	T	$\mathbb{a}_8$	$\frac{42}{169}$
T	T	F	$\mathbb{a}_5$	$\frac{1}{999}$
T	F	T	$\mathbb{a}_6$	$\frac{1}{999}$
T	F	F	$\mathbb{a}_2$	$\frac{-42\,296+39\,\sqrt{4\,676\,097}}{168\,831}$
F	T	T	$\mathbb{a}_7$	$\frac{1}{999}$
F	T	F	$\mathbb{a}_3$	$\frac{-42\,296+39\,\sqrt{4\,676\,097}}{168\,831}$
F	F	T	$\mathbb{a}_4$	$\frac{-42\,296+39\,\sqrt{4\,676\,097}}{168\,831}$
F	F	F	$\mathbb{a}_1$	$\frac{84\,418-39\,\sqrt{4\,676\,097}}{56\,277}$

We can also Evaluate probabilities on given models, as follows:

```
EvaluateProbability[
 {Pr[A | B], Pr[B | A ∧ ¬ C]}, MODEL11]
```

$$\left\{\frac{42\,127}{168\,831\left(\frac{42\,296}{168\,831}+\frac{-42\,296+39\,\sqrt{4\,676\,097}}{168\,831}\right)},\ \frac{1}{999\left(\frac{1}{999}+\frac{-42\,296+39\,\sqrt{4\,676\,097}}{168\,831}\right)}\right\}$$

```
% // N
```

{0.499521, 0.00400401}

■ A Simultaneous Countermodel to the **S**-instances of both (*) and (†)

Again, the additional equational side-constraints [**Pr[H]**$==\frac{1}{2}$**,Pr[E1]**$==\frac{1}{4}$**,Pr[E2]**$==\frac{3}{4}$] are added to speed the model search, and **Probabilities→Regular** indicates that we are asking **PrSAT** to find a *regular* probability model.

```
MODEL2 = PrSAT[
  {
   Pr[H | E1] > Pr[H],
   Pr[H | E2] > Pr[H],
   Pr[H | E1] > Pr[H | E2],
   Pr[H | E1] - Pr[H | ¬ E1] <
    Pr[H | E2] - Pr[H | ¬ E2],
   Pr[H | E1 ∧ E2] - Pr[H | ¬ E1 ∧ E2] ==
    Pr[H | E1] - Pr[H | ¬ E1],
   Pr[H | E2 ∧ E1] - Pr[H | ¬ E2 ∧ E1] ==
    Pr[H | E2] - Pr[H | ¬ E2],
   Pr[H | E1 ∧ E2] - Pr[H | ¬ (E1 ∧ E2)] <
    Pr[H | E2] - Pr[H | ¬ E2],

   (* Heuristic -- add additional
     equational side-constraints *)
   Pr[H] == 1/2, Pr[E1] == 1/4, Pr[E2] == 3/4
  },
  Probabilities → Regular,
  BypassSearch → True
 ]
```

$$\Big\{\{E1 \to \{a_2, a_5, a_6, a_8\},\ E2 \to \{a_3, a_5, a_7, a_8\},\ H \to \{a_4, a_6, a_7, a_8\},\ \Omega \to \{a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8\}\},$$

$$\Big\{a_1 \to \frac{147}{1024},\ a_2 \to \frac{193}{5120},\ a_3 \to \frac{1341}{5120},\ a_4 \to \frac{45}{1024},\ a_5 \to \frac{291}{5120},\ a_6 \to \frac{127}{5120},\ a_7 \to \frac{1539}{5120},\ a_8 \to \frac{669}{5120}\Big\}\Big\}$$

`TruthTable[MODEL2]`

E1	E2	H	var	Pr
T	T	T	a_8	$\frac{669}{5120}$
T	T	F	a_5	$\frac{291}{5120}$
T	F	T	a_6	$\frac{127}{5120}$
T	F	F	a_2	$\frac{193}{5120}$
F	T	T	a_7	$\frac{1539}{5120}$
F	T	F	a_3	$\frac{1341}{5120}$
F	F	T	a_4	$\frac{45}{1024}$
F	F	F	a_1	$\frac{147}{1024}$

■ A Simultaneous Countermodel to two claims concerning Hawthorne & Fitelson's new Bayesian approach to the raven paradox

The following single model shows that neither of the following two claims:

(6) Pr(H | ~R & ~B) > Pr(H)
(7) Pr(H | ~R & B) < Pr(H)

follows from the following three claims:

(1) Pr(R | H & B) = 1
(2) Pr(~B) > Pr(R)
(C) Pr(H | R) ≥ Pr(H | ~B)

Here, a regular model is impossible (since one of the contraints requires a zero probability for one of the state descriptions). But, by adding the constraint **`Pr[(¬H)∧B∧(¬R)]>0`**, we can ensure that this is the only zero in the model. And, as usual, we add equational side-constraints [$\mathbf{Pr[H]} == \frac{60}{100}, \mathbf{Pr[R]} == \frac{20}{100}, \mathbf{Pr[B]} == \frac{10}{100}$] to speed the model-finding process by reducing the number of free variables in the problem.

```
MODEL3 = PrSAT[
  {
   Pr[H ∧ R ∧ (¬ B)] == 0,
   Pr[(¬ H) ∧ B ∧ (¬ R)] > 0,
   Pr[¬ B] > Pr[R],
   Pr[H | R] == Pr[H | ¬ B],
   Pr[H | (¬ R) ∧ (¬ B)] < Pr[H],
   Pr[H | B ∧ (¬ R)] > Pr[H]
  },
  Constraints →
   {Pr[B] == 1/10, Pr[H] == 1/2, Pr[R] == 1/10},
  BypassSearch → True
 ]
```

$$\left\{\{B \to \{a_2, a_5, a_6, a_8\}, H \to \{a_3, a_5, a_7, a_8\}, R \to \{a_4, a_6, a_7, a_8\}, \Omega \to \{a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8\}\}, \left\{a_1 \to \frac{141}{320}, a_2 \to \frac{21}{2560}, a_3 \to \frac{225}{512}, a_4 \to \frac{51}{2560}, a_5 \to \frac{3}{256}, a_6 \to \frac{1}{32}, a_7 \to 0, a_8 \to \frac{25}{512}\right\}\right\}$$

```
TruthTable[MODEL3]
```

B	H	R	var	Pr
T	T	T	a_8	$\frac{25}{512}$
T	T	F	a_5	$\frac{3}{256}$
T	F	T	a_6	$\frac{1}{32}$
T	F	F	a_2	$\frac{21}{2560}$
F	T	T	a_7	0
F	T	F	a_3	$\frac{225}{512}$
F	F	T	a_4	$\frac{51}{2560}$
F	F	F	a_1	$\frac{141}{320}$

We can also solve this one (quickly) with Blum‘s random search add-on, and with no side-constraints:

```
PrSAT[
 {
  Pr[H ⋀ R ⋀ (¬ B)] == 0,
  Pr[(¬ H) ⋀ B ⋀ (¬ R)] > 0,
  Pr[¬ B] > Pr[R],
  Pr[H | R] == Pr[H | ¬ B],
  Pr[H | (¬ R) ⋀ (¬ B)] < Pr[H],
  Pr[H | B ⋀ (¬ R)] > Pr[H]
 },
 BypassSearch → False
]
```

$$\left\{\{B \to \{a_2, a_5, a_6, a_8\}, H \to \{a_3, a_5, a_7, a_8\}, R \to \{a_4, a_6, a_7, a_8\}, \Omega \to \{a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8\}\}, \left\{a_1 \to \frac{36693}{94240}, a_2 \to \frac{2}{15}, a_3 \to 0, a_4 \to \frac{1}{93}, a_5 \to \frac{11}{32}, a_6 \to \frac{7}{57}, a_7 \to 0, a_8 \to 0\right\}\right\}$$

■ Theorems and Countermodels from Sobel‘s “Lotteries and Miracles“

Since Sobel‘s problems only involve two atomic sentences, no heuristics are needed to yield a fast solution by the decision procedure (and Blum‘s random search add-on is also not necessary, since the decision procedure is quite fast in such cases).

- **A `PrSAT` model showing that Sobel's (1)-(3) do *not* entail (4)**

```
MODEL4 = PrSAT[
  {Pr[T] < 1/2,
   Pr[T | W] > 1/2,
   Pr[W | T] > 1/2,
   Pr[T | ¬ W] > Pr[W]
  },
  Probabilities → Regular,
  BypassSearch → True
 ]
```

$$\left\{\{T \to \{a_2, a_4\},\ W \to \{a_3, a_4\},\ \Omega \to \{a_1, a_2, a_3, a_4\}\},\ \left\{a_1 \to \frac{6033}{8192},\ a_2 \to \frac{1007}{8192},\ a_3 \to \frac{1}{64},\ a_4 \to \frac{1}{8}\right\}\right\}$$

```
TruthTable[MODEL4]
```

T	W	var	Pr
T	T	a_4	$\frac{1}{8}$
T	F	a_2	$\frac{1007}{8192}$
F	T	a_3	$\frac{1}{64}$
F	F	a_1	$\frac{6033}{8192}$

- **A `PrSAT` model showing that Sobel's (1)-(3) do *not* entail (5)**

```
MODEL5 = PrSAT[
  {Pr[T] < 1/2,
   Pr[T | W] > 1/2,
   Pr[W | T] > 1/2,
   Pr[T | ¬ W] - Pr[T | ¬ W] < Pr[¬ W] - Pr[W]
  },
  Probabilities → Regular,
  BypassSearch → True
 ]
```

$$\left\{\{T \to \{a_2, a_4\}, W \to \{a_3, a_4\}, \Omega \to \{a_1, a_2, a_3, a_4\}\}, \left\{a_1 \to \frac{51}{64}, a_2 \to \frac{1}{16}, a_3 \to \frac{1}{16}, a_4 \to \frac{5}{64}\right\}\right\}$$

```
TruthTable[MODEL5]
```

T	W	var	Pr
T	T	a_4	$\frac{5}{64}$
T	F	a_2	$\frac{1}{16}$
F	T	a_3	$\frac{1}{16}$
F	F	a_1	$\frac{51}{64}$

- **A `PrSAT` ``Proof'' that Sobel's (1)-(3) *do* entail the *disjunction* (4) ∨ (5)**

```
PrSAT[
 {Pr[T] < 1/2,
  Pr[T | W] > 1/2,
  Pr[W | T] > 1/2,
  Not[Or[Pr[T | ¬ W] < Pr[W],
    Pr[T | W] - Pr[T | ¬ W] > Pr[¬ W] - Pr[W]]]
 },
 BypassSearch → True
]

{}
```