Part I: The Liar Paradox

Roy T Cook

1 What is truth good for?

Truth is a centrally important notion in philosophy:

- *Truth* is an inherently interesting concept.
- *Truth* plays a central role in many important philosophical analyses (e.g. Plato's analysis of knowledge as justified *true* belief).
- More generally: the post-Fregean linguistic turn in philosophy involves investigating philosophically interesting or difficult notions via studying the language we use to talk about those notions. This often involves providing *truth* conditions for such talk.

Nevertheless, unrestricted talk about truth leads to paradoxes!

2 What is a paradox, and what do we do about them?

A *paradox* is an <u>argument</u> that:

- Begins with apparently unobjectionably true premises.
- Proceeds via apparently unobjectionable reasoning.
- Ends with a conclusion that is contradictory, false, or otherwise absurd or inappropriate.

Four options for 'solving' paradoxes:

1. Reject one or more premises.
2. Reject the reasoning.
3. Accept the conclusion.
4. Reject the central concept.

3 Why are contradictions so bad?

Theorem: (Explosion, or *Ex Falso Quolibet*): $\Phi \wedge \neg\Phi \vdash_{\mathsf{C}} \Psi$
Proof:

1	$\Phi \wedge \neg\Phi$	Assumption
2	Φ	1, $\wedge$ Elimination.
3	$\Phi \vee \Psi$	2, $\vee$ Introduction.
4	$\neg\Phi$	1, $\wedge$ Elimination.
5	Ψ	3, 4, Disjunctive Syllogism.

Note: This also holds for intuitionistic logic H, so that's no help here!

4 The Liar paradox

Premise$_1$: The Tarskian T-schema (informally):
Any meaningful, declarative sentence is *true*
if and only if *what it says is the case*.

Premise$_2$: The Liar sentence:
`This sentence is false.`
is a genuine, meaningful declarative sentence.

Conclusion: Contradiction.

5 The argument (informally)

1. By the *law of bivalence*, the Liar sentence is either true or false.
2. If the Liar sentence is true, then what it says must be the case. The Liar sentence says that it is false. So the Liar sentence must be false. But then the Liar sentence is both true and false. This violates the *law of non-contradiction*!
3. If the Liar sentence is false, then, since it says that it is false, what it say is the case. So the Liar sentence is true. Hence, again, the Liar sentence is both true and false. And again, this violates the *law of non-contradiction*!
4. Contradiction!

6 Logical presuppositions

The Law of Bivalence (Biv):

Every sentence is either true or false.

The Law of Non-Contradiction (LNC):

No sentence is both true and false.

- These two assumptions are the cornerstones of classical logic.
- Any strategy of type 2 (reject the reasoning) will require jettisoning at least one of Biv and LNC.

7 More formally

The Tarskian T-schema:

For every sentence Φ:

$$\mathsf{T}(\ulcorner\Phi\urcorner) \leftrightarrow \Phi$$

The Liar Sentence:

There exists a sentence λ such that:

$$\lambda \leftrightarrow \neg\mathsf{T}(\ulcorner\lambda\urcorner)$$

[Where $\ulcorner\Phi\urcorner$ is an appropriate name of the sentence Φ]

8 The argument formally

The Derivation:

1	$\mathsf{T}(\ulcorner\lambda\urcorner) \leftrightarrow \lambda$	T-schema.
2	$\lambda \leftrightarrow \neg\mathsf{T}(\ulcorner\lambda\urcorner)$	Liar sentence.
3	$\mathsf{T}(\ulcorner\lambda\urcorner) \leftrightarrow \neg\mathsf{T}(\ulcorner\lambda\urcorner)$	1, 2, logic.

Note on defining "contradiction"

- *Syntactic*: A contradiction is a formula of the form $\Phi \wedge \neg\Phi$.
- *Semantic*: A contradiction is a formula that cannot be true (as a matter of logic).

9 Other versions

- **Question:**

  Is the answer to this question “no”?

- **Command:**

  Do not obey this command!

10 The Curry Paradox

The Curry Sentence (C)

`If this sentence is true, then the moon is made of green cheese.`

Theorem 10.1. *Using the Curry sentence above, we can prove:*

`The moon is made of green cheese`

Proof. Assume that C is true. Then what it says must be the case. It says that if it is true, then the moon is made of green cheese. It is true. So the moon is made of green cheese.

The previous paragraph amounts to a proof that, if C is true, then the moon is made of green cheese. But this is just what C says. So C must be true. But C says that if it is true, then the moon is made of green cheese. Since C is true, it follows that the moon is made of green cheese! □

11 The Curry paradox formally

$$C \leftrightarrow (T(\ulcorner C \urcorner) \to \Phi)$$

1	$\quad C$	Assumption
2	$\quad T(\ulcorner C \urcorner) \to \Phi$	1, Df. of C
3	$\quad T(\ulcorner C \urcorner)$	1, T-schema
4	$\quad \Phi$	2, 3, Modus Ponens
5	$T(\ulcorner C \urcorner) \to \Phi$	1 - 4, Conditional Proof.
6	C	5, Df. of C
7	$T(\ulcorner C \urcorner)$	6, T-schema.
8	Φ	5, 7, Modus Ponens.

12 Generalizing

Note:

- There is nothing special about:

```
The moon is made of green cheese
```

- Along similar lines, we can use:

```
If this sentence is true, then 1 + 1 = 5.
If this sentence is true, then grass is pink.
If this sentence is true, then 1 + 1 = 2.
```

  to prove:

```
1 + 1 = 5.
Grass is pink.
1 + 1 = 2.
```

13 Bonus Question

Question: Is there anything wrong with using:

```
If this sentence is true, then 1 + 1 = 2.
```

to prove:

```
1 + 1 = 2.
```

After all, unlike:

```
The moon is made of green cheese.
```

it turns out that:

```
1 + 1 = 2.
```

is true!

14 The Curry Paradox and Conditionals

Some Thoughts:

- The Curry paradox involves neither falsity nor negation (i.e. “not”).
- The Curry paradox does, of course, still involve talk of truth (just like the Liar paradox).
- Instead, the Curry paradox seems to involve the conditional – that is, phrases of the form:

  `If ...then ...`

- Thus, a solution to the Curry paradox will likely involve giving up one or more of our basic intuitions about “If… then …”.

Clarification: Of course, our proof that the Curry sentence generates a paradox does involve talk of falsity. The point is that the Curry sentence itself contains no mention of falsity.

15 Defining Negation In Terms of the Conditional

Definition 15.1. *Let $\perp$ be an abbreviation of your favorite contradiction. Then:*

“It is not the case that Φ” $=_{df}$ “If Φ then $\perp$”.

So the Liar sentence:

`This sentence is false.`

is just a special case of the Curry paradox:

`If this sentence is true, then ⊥.`

16 Some (informal) definitions

Definition 16.1.

- *A sentence Φ (or set of sentences Σ) is* <u>*paradoxical*</u> *if and only if there is no way to coherently assign it a truth value (or to assign the sentences contained in it truth values).*
- *A sentence Φ (or set of sentences Σ) is* <u>*determinate*</u> *if and only if there is a unique way to coherently assign it a truth value (or to assign the sentences contained in it truth values).*
- *A sentence Φ (or set of sentences Σ) is* <u>*indeterminate*</u> *if and only if there is more than one way to coherently assign it a truth value (or to assign the sentences contained in it truth values).*

Note: This terminology violates the point I made earlier about paradoxes being arguments, not sentences. It's technical terminology, unfortunately, and there's nothing we can do about it.

17 Examples

Given a background of classical logic:

- The Liar sentence:

  `This sentence is false.`

  is paradoxical.
- The tautology-teller:

  `This sentence is either true or false.`

  is determinate.
- The (badly named!) truth-teller sentence:

  `This sentence is true.`

  is indeterminate.

Note: Whether or not a sentence falls into one of these categories is dependent on the background logic!

18 Strategy 4: Reject the concept

Idea: There is something wrong with the notion of *truth*. Hence, we need to either reject it altogether or replace it with a different concept. But:

- Rejecting truth altogether seems (to me at least) too extreme.
- Especially given the rich extant work along the lines of the other options!
- Nevertheless, in general this option should always be kept in mind:
    1. For example, we might think that the Russell paradox shows that there just aren't such things as *extensions*!
    2. Kevin Scharp's idea: Replace truth with two notions – *ascending truth* and *descending truth*, where:

$$\Phi \to \mathsf{T}_{\mathsf{A}}(\ulcorner \Phi \urcorner).$$
$$\mathsf{T}_{\mathsf{D}}(\ulcorner \Phi \urcorner) \to \Phi.$$

19 Strategy 3: Accept the conclusion

In other words, accept that the Liar sentence really is *both true and false* at the same time. But:

- This amounts to accepting a contradiction!
- Thus, we need some means by which to block *explosion*!
- As a result, we *still* need to formulate a non-classical logic!
- So, adopting strategy 3 in this case requires also adopting strategy 2!
- We'll talk more about strategy 3 in later lectures.

20 Strategy 1: Reject a premise: Attempt 1

Idea: Give up Tarski's T-schema (in its fully general form).

Hence, adopt some form of the restricted Tarskian T-Schema:

For every sentence Φ of some special sort:

$$\mathsf{T}(\ulcorner \Phi \urcorner) \leftrightarrow \Phi$$

Or, replace the T-schema with some other collection of axioms for truth.
We'll spend a lot of time on these sorts of approaches later today.

21 Strategy 1: Reject a premise: Attempt 2

Idea: Give up the claim that the Liar sentence is a meaningful sentence.

- Historically, this has been the traditional response to the problem.
- The thought is that the circularity present in the Liar sentence is somehow *bad*, and the sentence is thereby somehow faulty.
- But Gödel's theorems and recursion theory rely on exactly the same sort of self-reference as is found in (some versions of) the Liar paradox!
- In addition, there are variants of the Liar paradox that seem to be devoid of circularity (although they have other weird properties). For example, the Yablo paradox (two slides from now!)
- It has turned out to be extremely difficult to formulate a criterion for "goodness" that rules out the stuff we don't want (i.e. the paradoxes and other pathological constructioins) while retaining the stuff we do want (e.g. the Gödel sentence).

22 Kripke on rejecting the Liar

Self-referential sentences just *are* meaningful!

> A simpler, and more direct, form of self-reference uses demonstratives or proper names: Let "Jack" be a name of the sentence 'Jack is short', and we have a sentence that says of itself that it is short. I can see nothing wrong with "direct" self-reference of this type. If 'Jack' is not already a name in the language, why can we not introduce it as a name of any entity we please? In particular, why can it not be a name of the (uninterpreted) finite sequence of marks 'Jack is short'? (would it be permissible to call this sequence of marks "Harry", but not "Jack"? Surely prohibitions on naming are arbitrary here.) There is no vicious circle in our procedure, since we need not *interpret* the sequence of marks 'Jack is short' before we name it. Yet if we name it "Jack", it at once becomes meaningful and true. (Kripke 1975: 693)

23 The Yablo Paradox

Very New: Discovered by Stephen Yablo and Albert Visser in 1980s!

$$
\begin{aligned}
S_1 &: \text{For all } n > 1, S_n \text{ is false.}\\
S_2 &: \text{For all } n > 2, S_n \text{ is false.}\\
S_3 &: \text{For all } n > 3, S_n \text{ is false.}\\
&\vdots \quad \vdots \quad \vdots \quad \vdots\\
S_m &: \text{For all } n > m, S_n \text{ is false.}\\
S_{m+1} &: \text{For all } n > m+1, S_n \text{ is false.}\\
&\vdots \quad \vdots \quad \vdots \quad \vdots
\end{aligned}
$$

Note: I wrote an entire book on this paradox!

24 The Argument

Theorem 24.1. *The Yablo sequence of sentences (plus Bivalence, the Law of Non-contradiction, and the T-schema) entail a contradiction.*

Proof. Let S_n be any sentence in the Yablo sequence.

Assume S_n is true. Then, by the T-Schema, what it says must be the case. S_n says that, for every $m > n$, S_m is false. So, for every $m > n$, S_m is false. This entails that S_{n+1} is false. But this also entails that, for any $m > n+1$, S_m is false. This is exactly what S_{n+1} says. So, by the T-schema, S_{n+1} is true. Hence S_{n+1} is both true and false. This violates the Law of Non-Contradiction.

So S_n cannot be true. Thus, S_n is false. But n was arbitrary. So, for every $m \in \mathbb{N}$, S_m is false. This entails that S_1 is false. But this also entails that, for every $m > 1$, S_m is false. But this is exactly what S_1 says. So, by the T-schema, S_1 is true. Hence, S_1 is both true and false. This violates the Law of Non-Contradiction. □

25 No circularity!

- The Yablo paradox does not contain any sentence of the form:

```
This sentence is ...
```

- Instead, each sentence in the Yablo sequence is of the form:

```
Those sentences are ...
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- Of course, the Yablo construction does trade *circularity* for *ungroundedness*! (That is, there is no circularity, but there is an *infinitely descending* chain of reference!)

26 Strategy 2: Reject the reasoning

The Idea: Adopt a *non-classical logic* where some inference crucial to the Liar reasoning turns out to be invalid.

- *General approach* 1 (Gaps): Give up the law of bivalence. So some sentences are *neither true or false*.
- *General approach* 2 (Gluts): Give up the law of non-contradiction. So some sentences are *both truth and false*.
- *The Big Question*: How do we know that such logics will evaluate all sentences correctly (or even coherently)?
- *The Bigger Question*: What if there are sentences that can't be any of true, false, or neither (or true, false, or both)?

Part III: Many Valued Logics, Informally

Roy T Cook

1 The Plan:

1. Provide a brief introduction to many-valued logics.
2. Provide an introduction to the *revenge problem*.

2 Classical Logic

Bivalence:

Every sentence is either true or false.

Law of Non-Contradiction:

No sentence is both true and false.

Negation (not):

"not: Φ" is true if and only if "Φ" is false.

Conjunction (and):

"Φ and Ψ" is true if and only if "Φ" is true and "Ψ" is true.

Disjunction (or):

"Φ or Ψ" is true if and only if "Φ" is true or "Ψ" is true.

3 More Classical Logic:

Validity

An argument with $P_1, P_2, \ldots P_n$ as premises, and C as conclusion – that is:

$$\frac{P_1, P_2, \ldots P_n}{C}$$

is logically valid if and only if it is impossible for $P_1, P_2, \ldots P_n$ to all be true and C fail to be true.

Logical Truth:

A sentence is a logical truth if and only if it is impossible for it to fail to be true.

4 Classical Truth Tables

Negation ("not"):

Φ	$\neg\Phi$
T	F
F	T

Conjuction ("and"):

$\wedge$	T	F
T	T	F
F	F	F

Disjunction ("or"):

$\vee$	T	F
T	T	T
F	T	F

5 Classically Valid Arguments and Logical Truths

Disjunctive Syllogism:

$$\frac{\textit{not} : \Phi, \quad \Phi \text{ or } \Psi}{\Psi}$$

$$\frac{\neg\Phi, \quad \Phi \vee \Psi}{\Psi}$$

Explosion:

$$\frac{\textit{not} : \Phi \text{ and } \Phi}{\Psi}$$

$$\frac{\neg\Phi \wedge \Phi}{\Psi}$$

Excluded Middle:

$$\Phi \text{ or } \textit{not} : \Phi$$

$$\Phi \vee \neg\Phi$$

6 First-Degree Entailment (FDE)

Give Up Bivalence:

Some sentences can be neither true nor false.

Give Up the Law of Non-Contradiction:

Some sentences can be both true and false.

Historical Note:

The logic of First-Degree Entailment was introduced by Alan Anderson and Nuel Belnap.

7 More FDE

Reword Validity:

An argument with $P_1, P_2, \ldots P_n$ as premises, and C as conclusion – that is:

$$\frac{P_1, P_2, \ldots P_n}{C}$$

is logically valid if and only if it is impossible for $P_1, P_2, \ldots P_n$ to all be true (or both) and C fail to be true (or both).

Reword Logical Truth:

A sentence is a logical truth if and only if it is impossible for it to fail to be true (or both).

8 The Connectives:

Negation (not):

not: Φ (i.e. $\neg\Phi$) is true if and only if Φ is false.
not: Φ (i.e. $\neg\Phi$) is false if and only if Φ is true.

Conjunction (and):

Φ and Ψ (i.e. $\Phi \wedge \Psi$) is true if and only if Φ is true and Ψ is true.
Φ and Ψ (i.e. $\Phi \wedge \Psi$) is false if and only if Φ is false or Ψ is false.

Disjunction (or):

Φ or Ψ (i.e. $\Phi \vee \Psi$) is true if and only if Φ is true or Ψ is true.
Φ or Ψ (i.e. $\Phi \vee \Psi$) is false if and only if Φ is false and Ψ is false.

9 FDE Truth Tables

Negation:

Φ	$\neg\Phi$
T	F
B	B
N	N
F	T

Conjunction:

$\wedge$	T	B	N	F
T	T	B	N	F
B	B	B	F	F
N	N	F	N	F
F	F	F	F	F

Disjunction:

$\vee$	T	B	N	F
T	T	T	T	T
B	T	B	T	B
N	T	T	N	N
F	T	B	N	F

10 Validity and Logical Truth in FDE:

Disjunctive Syllogism:

$$\frac{\textit{not}:\Phi, \quad \Phi \text{ or } \Psi}{\Psi} \qquad\qquad \frac{\neg\Phi, \quad \Phi\vee\Psi}{\Psi}$$

This is not valid in FDE (just let Φ be both true and false, and Ψ be false).

Explosion:

$$\frac{\textit{not}:\Phi \text{ and } \Phi}{\Psi} \qquad\qquad \frac{\neg\Phi\wedge\Phi}{\Psi}$$

This is not valid in FDE (just let Φ be both true and false, and Ψ be false).

Excluded Middle:

$$\Phi \text{ or } \textit{not}:\Phi \qquad\qquad \Phi\vee\neg\Phi$$

This is not a logical truth in FDE (just let Φ be neither true nor false).

11 Logic of Paradox (LP)

Accept Bivalence:

Every sentence is either true or false.

Give up Law of Non-Contradiction:

Some sentences can be both true and false.

Note:

This sort of logic is often described as *glutty*.

Note:

This logic has been extensively studied by Graham Priest.

12 LP Truth Tables

Negation:

Φ	$\neg\Phi$
T	F
B	B
F	T

Conjunction:

$\wedge$	T	B	F
T	T	B	F
B	B	B	F
F	F	F	F

Disjunction:

$\vee$	T	B	F
T	T	T	T
B	T	B	B
F	T	B	F

13 Validity and Logical Truth in LP

Disjunctive Syllogism:

$$\frac{\textit{not} : \Phi, \quad \Phi \text{ or } \Psi}{\Psi} \qquad \frac{\neg\Phi, \quad \Phi \vee \Psi}{\Psi}$$

This is not valid in LP (just let Φ be both true and false, and Ψ be false).

Explosion:

$$\frac{\textit{not} : \Phi \text{ and } \Phi}{\Psi} \qquad\qquad \frac{\neg\Phi \wedge \Phi}{\Psi}$$

This is not valid in LP (just let Φ be both true and false, and Ψ be false).

Excluded Middle:

$$\Phi \text{ or } \textit{not} : \Phi \qquad\qquad \Phi \vee \neg\Phi$$

This is a logical truth in LP.

Note: In fact, every logical truth in classical logic is a logical truth in LP!

14 Strong Kleene Logic (K3)

Give Up Bivalence:

Some sentences are neither true nor false.

Accept Law of Non-Contradiction:

No sentence can be both true and false.

Note:

This sort of logic is often described as *gappy*.

Note:

This logic has been extensively studied by Stephen Kleene.

15 K3 Truth Tables

Negation:

Φ	$\neg\Phi$
T	F
N	N
F	T

Conjunction:

$\wedge$	T	N	F
T	T	N	F
N	N	N	F
F	F	F	F

Disjunction:

$\vee$	T	N	F
T	T	T	T
N	T	N	N
F	T	N	F

16 Validity and Logical Truth in K3

Disjunctive Syllogism:

$$\frac{\textit{not} : \Phi, \quad \Phi \text{ or } \Psi}{\Psi} \qquad\qquad \frac{\neg\Phi, \quad \Phi \vee \Psi}{\Psi}$$

This is valid in K3.

Explosion:

$$\frac{\textit{not} : \Phi \text{ and } \Phi}{\Psi} \qquad\qquad \frac{\neg\Phi \wedge \Phi}{\Psi}$$

This is valid in K3.

Excluded Middle:

$$\Phi \text{ or } \textit{not} : \Phi \qquad\qquad \Phi \vee \neg\Phi$$

This is not a logical truth in K3 (just let Φ be neither true nor false).

Note: In fact, no sentence is a logical truth in K3!

17 Do we have to choose?

Logical Monism:

The is exactly one logic that describes correct reasoning.

Logical Pluralism:

The is more than one logic that describes correct reasoning.

Logical Nihilism:

The is no logic that describes correct reasoning.

18 Summing up the Liar paradox: Tarski's Theorem (Informally)

Proposition 18.1. *Given a classical base language $\mathscr{L}_0$ (i.e. a language $\mathscr{L}_0$ interpretable in terms of the classical truth values $\{\top, \bot\}$), extend the language $\mathscr{L}_0$ to $\mathscr{L}_1$ by adding expressive resources sufficient to characterize $\top$ and $\bot$ (i.e. add "is true" and "is false"). Then:*

`This sentence is false.`

cannot receive either of the values $\{\top, \bot\}$.

19 Kripke's Solution

Add a third truth-value ρ_1!

Note: At the moment we don't care if this is a "glut" or a "gap".

20 Revenge

Proposition 20.1. *Extend the language $\mathscr{L}_1$ to $\mathscr{L}_2$ by adding expressive resources sufficient to characterize the pathological value ρ_1 (i.e. add "is pathological$_1$"). Then:*

`This sentence is false or pathological`$_1$`.`

cannot receive any of the values $\{\top, \bot, \rho_1\}$.

21 The Obvious Move?

Add a fourth truth-value ρ_2!

22 Super-revenge

Proposition 22.1. *Extend the language $\mathscr{L}_2$ to $\mathscr{L}_3$ by adding expressive resources sufficient to characterize the pathological value ρ_2 (i.e. add "is pathological$_2$"). Then:*

This sentence is false or pathological$_1$ or pathological$_2$.

cannot receive either of the values $\{\top, \bot, \rho_1, \rho_2\}$.

23 The Ultimate Revenge

Theorem 23.1. *Given any definite collection of exclusive semantic values Σ, there is a sentence λ_Σ that cannot receive any of the values in Σ.*

Proof. The required sentence λ_Σ is:

`This sentence has one of the values in Σ other than the true.`

Assume for *reductio* that the semantic value of λ_Σ is in Σ.

Assume further that the value of λ_Σ is $\top$. Then what λ_Σ says is not the case, so the semantic value of λ_Σ is $\bot$ (i.e. λ_Σ is false). Contradiction.

Hence, the semantic value of λ_Σ is not $\top$. So the semantic value of λ_Σ is in Σ and is not $\top$. But this is just what λ_Σ says. Hence the semantic value of λ_Σ is $\top$ (i.e λ_Σ is true). Contradiction.

So the value of λ_Σ cannot be in Σ. □

24 Negations strong, weak, and choice

Φ	$\neg_S\Phi$	$\neg_W\Phi$	$\neg_C\Phi$
$\top$	$\bot$	$\bot$	$\bot$
ρ	$\bot$	$\top$	ρ
$\bot$	$\top$	$\top$	$\top$

LP and K3 only work if we use *choice* negation.

25 The Conditional

Definition 25.1. $\Phi \to \Psi =_{df} (\neg_C\Phi) \vee \Psi$

What we get:

$\to_S$	$\top$	ρ	$\bot$
$\top$	$\top$	ρ	$\bot$
ρ	$\top$	ρ	ρ
$\bot$	$\top$	$\top$	$\top$

What we want:

$\to_S$	$\top$	ρ	$\bot$
$\top$	$\top$	ρ	$\bot$
ρ	$\top$	$\top$	ρ
$\bot$	$\top$	$\top$	$\top$

Again, for "revenge" reasons we can't have the second conditional.

26 Two Options

1. Cut our losses, and accept that at some point our linguistic resources fall short of expressing all of these 'pathological' semantic notions (and, in particular, accept that there is some definite set of exclusive semantic values Σ such that λ_{Σ} is not expressible).

2. Accept the picture just sketched, and *embrace revenge* as an inherent feature of semantic theorizing itself, and of the language(s) within which we carry out such endeavors.

Tutorial 1: Determinacy, Indeterminacy, and Paradoxicality

Roy T Cook

1 Useful Definitions

Definition 1.1. *A sentence Φ (or set of sentences Σ) is paradoxical if and only if there is no way to coherently assign it a truth value (or to assign the sentences contained in it truth values).*

Definition 1.2. *A sentence Φ (or set of sentences Σ) is determinate if and only if there is a unique way to coherently assign it a truth value (or to assign the sentences contained in it truth values).*

Definition 1.3. *A sentence Φ (or set of sentences Σ) is indeterminate if and only if there is more than one way to coherently assign it a truth value (or to assign the sentences contained in it truth values).*

Definition 1.4. *A subset X of the natural numbers (i.e. $X \subseteq \mathbb{N}$) is cofinite if and only if its complement (i.e. $\mathbb{N} \setminus X$) is finite.*

2 Instructions:

For each of the constructions below:

- Determine whether the sentence or set of sentences is *paradoxical*, *determinate*, or *indeterminate*.
- If the sentence of set of sentences is determinate or indeterminate, describe the different coherent truth-value assignment(s).
- **Hint:** For the finite lists below, work through a couple concrete examples.
- Discover anything else interesting there might be to say about the construction in question.
- Stars indicate the level of difficulty of the problem (in my, defeasible, opinion!)
- You **will not** be able to finish all these today. So jump around, try some different ones, and have fun!

3 Exercises

1. [*] Consider, for each $n \in \mathbb{N}$, the sequence of length n of the following form:

$$\begin{array}{ll} S_1: & S_2 \text{ is false.} \\ S_2: & S_3 \text{ is false.} \\ & \vdots \quad\quad \vdots \\ S_{n\text{-}1}: & S_n \text{ is false.} \\ S_n: & S_1 \text{ is false.} \end{array}$$

If $n = 1$, this is just the Liar paradox (this holds for a number of other constructions below). For each $n \in \mathbb{N}$, we will call this the *n-membered Liar cycle*. When $n = 2$, we also call this the *open pair* or the *no-no paradox*.

2. [**] Consider, for each $n \in \mathbb{N}$, the sequence of length n of the following form:

$$\begin{array}{ll} S_1: & \text{At least one of } S_1 \text{ through } S_n \text{ is false.} \\ S_2: & \text{At least two of } S_1 \text{ through } S_n \text{ is false.} \\ & \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \\ S_{n\text{-}1}: & \text{At least } n\text{-1 of } S_1 \text{ through } S_n \text{ is false.} \\ S_n: & \text{At least } n \text{ of } S_1 \text{ through } S_n \text{ is false.} \end{array}$$

3. [**] Consider, for each $n \in \mathbb{N}$, the sequence of length n of the following form:

$$\begin{array}{ll} S_1: & \text{At most one of } S_1 \text{ through } S_n \text{ is false.} \\ S_2: & \text{At most two of } S_1 \text{ through } S_n \text{ is false.} \\ & \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \\ S_{n\text{-}1}: & \text{At most } n\text{-1 of } S_1 \text{ through } S_n \text{ is false.} \\ S_n: & \text{At most } n \text{ of } S_1 \text{ through } S_n \text{ is false.} \end{array}$$

4. [**] Consider, for each $n \in \mathbb{N}$, the sequence of length n of the following form:

$$\begin{array}{ll} S_1: & \text{Exactly one of } S_1 \text{ through } S_n \text{ is false.} \\ S_2: & \text{Exactly two of } S_1 \text{ through } S_n \text{ is false.} \\ & \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \\ S_{n\text{-}1}: & \text{Exactly } n\text{-1 of } S_1 \text{ through } S_n \text{ is false.} \\ S_n: & \text{Exactly } n \text{ of } S_1 \text{ through } S_n \text{ is false.} \end{array}$$

5. [***] Consider the infinite list of sentences (one sentence for each $n \in \mathbb{N}$) of the following form:

$$\begin{array}{ll}
S_1: & \text{For all } m > 1,\ S_m \text{ is false.} \\
S_2: & \text{For all } m > 2,\ S_m \text{ is false} \\
 & \vdots \quad \vdots \quad \vdots \quad \vdots \\
S_n: & \text{For all } m > n,\ S_m \text{ is false} \\
S_{n+1}: & \text{For all } m > n+1,\ S_m \text{ is false.} \\
 & \vdots \quad \vdots \quad \vdots \quad \vdots
\end{array}$$

Intuitively, each sentence "says" that all of the sentences "below" it are false. This is the *Yablo paradox*.

6. [***] Consider the infinite list of sentences (one sentence for each $n \in \mathbb{N}$) of the following form:

$$\begin{array}{ll}
S_1: & \text{There exists an } m > 1 \text{ such that } S_m \text{ is false.} \\
S_2: & \text{There exists an } m > 2 \text{ such that } S_m \text{ is false} \\
 & \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \\
S_n: & \text{There exists an } m > n \text{ such that } S_m \text{ is false} \\
S_{n+1}: & \text{There exists an } m > n+1 \text{ such that } S_m \text{ is false.} \\
 & \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots
\end{array}$$

Intuitively, each sentence "says" that at least on of the sentences "below" it is false. This is (a version of) *Sorensen's queue paradox*, also known as the *dual of the Yablo paradox*.

7. [****] Consider the infinite list of sentences (one sentence for each $n \in \mathbb{N}$) of the following form:

$$\begin{array}{ll}
S_1: & \text{For infinitely many } m > 1,\ S_m \text{ is false.} \\
S_2: & \text{For infinitely many } m > 2,\ S_m \text{ is false} \\
 & \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \\
S_n: & \text{For infinitely many } m > n,\ S_m \text{ is false} \\
S_{n+1}: & \text{For infinitely many } m > n+1,\ S_m \text{ is false.} \\
 & \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots
\end{array}$$

Intuitively, each sentence "says" that infinitely many of the sentences "below" it are false. This is the *Schlenker unwinding*.

8. [****] Consider the infinite list of sentences (one sentence for each $n \in \mathbb{N}$) of the following form:

$$\begin{array}{ll}
S_1: & \text{For cofinitely many } m > 1,\ S_m \text{ is false.} \\
S_2: & \text{For cofinitely many } m > 2,\ S_m \text{ is false} \\
 & \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \\
S_n: & \text{For cofinitely many } m > n,\ S_m \text{ is false} \\
S_{n+1}: & \text{For cofinitely many } m > n+1,\ S_m \text{ is false.} \\
 & \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots
\end{array}$$

Intuitively, each sentence "says" that all but finitely many of the sentences "below" it are false. This is the *Yablo unwinding*.

9. [**] Consider the infinite list of sentences (one sentence for each $n \in \mathbb{N}$) of the following form:

$$\begin{array}{ll}
S_1: & S_2 \text{ is false.} \\
S_2: & S_3 \text{ is false.} \\
 & \vdots \quad \vdots \\
S_n: & S_{n+1} \text{ is false.} \\
S_{n+1}: & S_{n+2} \text{ is false.} \\
 & \vdots \quad \vdots
\end{array}$$

Intuitively, each sentence "says" that the sentence immediately following it is false.

10. [******] Consider the infinite list of sentences (one sentence for each $n \in \mathbb{N}$) of the following form:

$$\begin{array}{ll}
S_1: & \text{All sentences between } S_2 \text{ and } S_2 \text{ (inclusive) are false.} \\
S_2: & \text{All sentences between } S_3 \text{ and } S_4 \text{ (inclusive) are false.} \\
 & \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \\
S_n: & \text{All sentences between } S_{n+1} \text{ and } S_{n+n} \text{ (inclusive) are false.} \\
S_{n+1}: & \text{All sentences between } S_{(n+1)+1} \text{ and } S_{(n+1)+(n+1)} \text{ (inclusive) are false.} \\
 & \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots
\end{array}$$

Intuitively, S_1 "says" the next sentence is false, S_2 "says" the next two sentences are false, S_3 "says" the next three sentences are false, and in general, S_n "says" the next n sentences are false.

11. Bonus problem(s): Construct your own gnarly paradoxes and be like, totally rad!

Tutorial 2: Knights and Knaves Puzzles

Roy T Cook

1 Definition

Definition 1.1. *The* <u>*island of knights and knaves*</u> *contains two types of person:*

- Knights*: Knights always utter truths (and never allow themselves to be tricked or trapped into uttering paradoxical statements).*
- Knaves*: Knaves always utters falsehoods (and never allow themselves to be tricked or trapped into uttering paradoxical statements).*

There are absolutely no other people on the island other than you and the knights and knaves.

2 Exercises

1. You meet Alice and Betty. Alice says "Both of us are knaves." Who is a knight and who is a knave?
2. You meet Alice and Betty. Alice says "At least one of us is a knave." Who is a knight and who is a knave?
3. You meet three inhabitants of the island – Alice, Betty, and Carla. Alice says says "Both Betty and Carla are knaves." Betty says "Both Alice and Carla are knaves." . Carla either says "Alice is a knave" or "Betty is a knave", but you were distracted, and didn't hear which. How many of Alice, Betty, and Carla are knaves? Can you tell, of any of Alice, Betty, or Carla, whether they are a knight or a knave?
4. You meet Alice, who says "This is not the first time I have said this." Is Alice a knight or a knave?
5. You meet Alice, and Betty. Alice says "Betty just claimed to be a knave'.' Betty says "Alice is lying'.' Who is a knight, and who is a knave?
6. You meet Alice, who says "I don't know that I am a knight." Is Alice a knight or a knave? Does she know which she is?

7. You meet Alice, Betty, Carla, Debbie, Elizabeth, and Flo, who are lined up in front of you (in that order). Each of them says "There is exactly one knave standing adjacent to me." Who is a knight and who is a knave?

8. You meet a group of of Alices (Alice$_1$, Alice$_2$, Alice$_3$, and so on – it doesn't matter how many, as long as there are at least two). Each of the Alices says "Everyone but me is a knave." How many knights are there?

9. You meet Alice and Betty. You ask Alice whether she is a knight or a knave. You are distracted and don't hear her answer, but you then ask Betty whether she is a knight or a knave, and she says "My answer to that question is the same as Alice's." Is Betty a knight or a knave?

10. You meet an inhabitant of the island, but are unsure whether it is Alice or Betty. The islander says "Alice once said I was a knave", and then says "I am Alice". Is the islander Alice or Betty?

11. You encounter nine inhabitants of the island, standing in a 3×3 grid. Each of them says "Exactly one knave is standing adjacent to me" (adjacent here means in a north-south-east-west direction, not diagonal). You know at least one of the nine is a knight. How many knights are there? Is the person standing in the center a knight or a knave?

12. You meet Alice, and ask her whether she ever responds with "no" to a question. After her answer, you know whether she is a knight or a knave. What was her answer?

13. There is a nightclub on the island of Knights and Knaves, known as the Prime Club. The Prime Club has one strict rule: the number of occupants in the club must be a prime number at all times. The Prime Club also has strict bouncers (who stand outside the doors and do not count as occupants) enforcing this rule. In addition, a strange tradition has become customary at the Prime Club: Every so often the occupants form a conga line, and sing a song. The first lyric of the song is:

    At least one of us in the club is a knave.

    and is sung by the first person in the line. The second lyric of the song is:

    At least two of us in the club are knaves.

    and is sung by the second person in the line. The third person (if there is one) sings:

    At least three of us in the club are knaves.

    And so on down the line, until everyone has sung a verse. One day you walk by the club, and hear the song being sung. How many people are in the club?

14. Bonus problem(s): Construct your own totally tubular knights and knaves puzzles and be like, bodaciously boss!