On the ternary relation and conditionality*

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1 Introduction

Here is a familiar history: modal logics (see [13]) were around for some time before a semantic framework was found for them (by Kripke and others).[1] This framework did at least two Very Good Things for modal logics: 1) it connected the powerful mathematical tools of model theory to these logics, allowing a variety of technical results to be proven, and 2) it connected modal logics (more) firmly to philosophy, allowing their application to the understanding of metaphysics, tense, scientific laws, modal verbs, and so on. This application crucially depends not only on simply having Kripke-style semantic frameworks, but on interpreting the frameworks in various ways. The points of evaluation of the framework, in various interpretations, might be metaphysically possible worlds, say, or times, or morally allowable outcomes, or fictions of some sort, or . . . whathaveyou; these interpretations give modal logics an 'applied semantics' as opposed to merely 'pure semantics'.

Relevant logics have a similar history. The logics themselves were articulated and explored (see [1]) before semantic frameworks were found for them [11, 24, 25, 26, 28, 31]. Of these frameworks, the one that's had the most mileage put on it is the Routley-Meyer one, and that's the one we're most concerned to

*This paper emerged from some working sessions at the University of Melbourne in 2009. Beall, Brady, Hazen, Priest, Restall, Ripley, and Slaney were involved *on the ground*. The work of Dunn and Mares kept coming up during the sessions, and the paper greatly benefited from their joining the effort after the event. Looming even larger in the initial discussions were two others: despite their deaths, much of the early work of Sylvan and Meyer was heavily represented. While we can't speak for the current views of Bob (Meyer) or Richard (Sylvan), we wanted to honor them for their starting this ternary-relation idea in the first place. Hence, we include them as authors. While not all of us (authors) agree on all ideas herein, we do agree with the main thrust of this paper: namely, that, despite first appearances, the ternary-relation approach to conditionality is very much philosophically plausible as capturing an important aspect of conditionality.

[1] Here, and elsewhere in this paper, when we say 'modal logics' we mean 'unary normal modal logics'—those that have received the most philosophical attention.

discuss here (although see §3.2 for comments on Urquhart's and Fine's frameworks). You might at first think that, just as in the modal case, a semantic framework for relevant logics would do (at least) two Very Good Things; that is, that such a framework would 1) allow the exploitation of model-theoretic techniques in the exploration of relevant logics, and 2) give relevant logics a firmer connection to issues of broad philosophical import. As in the modal case, this second connection would depend crucially on interpreting the semantic framework, and especially the ternary relation which features so prominently.[2]

According to some, it's here that the parallels between modal and relevant logics break down, to relevant logic's detriment. That the ternary relation does 1), no one can doubt. The problem is with 2). The story goes like this: whereas the binary relation invoked by Kripke in the semantics of modal logics has several philosophically interesting and revealing interpretations (as relative possibility, or as a temporal ordering, or as the relation of being-morally-ideal-from-the-point-of-view-of, or . . .), the ternary relation invoked by Routley and Meyer has no such standardly accepted interpretations/applications. 'Sure,' the objector says (it helps here to imagine the hint of a sneer), 'there are mathematical structures of the sort described by Routley and Meyer, and those structures bear important and interesting relations to the logics described by Anderson and Belnap, but these logics were supposed to tell us something interesting about *conditionality*, or at least some important kind of conditionality, and it would take more than just abstract mathematical structures to tell us *that*. I want to know what it is that *instantiates* these structures that has anything to do with conditionals.'

Very well. In this paper we give an answer. In fact we give three answers. Conditionality can be—and often is—thought of in at least three ways. We will show that in whichever of these ways one thinks of conditionality, the ternary relation makes perfectly good sense. In section 2, we set out more carefully the problem to be faced. In section 3, we explain an appropriate understanding of the ternary relation for three ways of looking at conditionality. Section 4 draws some threads together and sums up.

2 Frame semantics

2.1 Points and relations

Routley-Meyer semantics and Kripke semantics are close cousins; both evaluate sentences at various *points*. These points have sometimes been more objectively/metaphysically interpreted as 'worlds' or times, and other times more subjectively/epistemically interpreted as (evidential) situations, or pieces/states of information. Sometimes they have been interpreted concretely as theories (set of sentences), and other times they have been given neutral names (indices, or Routley and Meyer's 'setups'). As far as we are concerned they could be

[2]The interpretation of the 'Routley star' is also an important issue. But our concern in this paper is with the ternary relation.

garden gnomes, or whatever. When we're not concerned with a specific interpretation, we'll use the word 'points' to be maximally generic. A model for a language, to a first approximation, can be taken to be a set of points together with an evaluation of the language's sentences at those points; a sentence A either holds at a point x in a model M (in which case we write $x \vDash_M A$),[3] or else it doesn't (in which case we write $x \nvDash_M A$).[4]

We can use these points to give semantics for the various connectives that occur in the language. This amounts to restricting the evaluations to those that respect certain dependencies among the language's sentences. Some connectives—here, conjunction ($\wedge$) and disjunction ($\vee$)—are *extensional*: whether a conjunction holds at a point in a model depends only on whether its conjuncts hold at *that very point* in the model, and the same goes for disjunction. Other connectives, like the modal necessity ($\Box$) and the relevant conditional ($\rightarrow$), are *intensional*: whether $\Box A$ holds at a point in a model can depend not just on whether A holds at that point in the model, but rather on whether A holds or not at some other point(s) in the model. The same goes for the value of $A \rightarrow B$ at a point in a model.

For each connective, extensional or intensional, a semantics must tell us *how* compound sentences built with it depend on the holding-or-not of their components, and for intensional connectives a semantics must tell us more: it must tell us *at which points* it matters whether the component sentences hold. It's here that relations on points prove vital: to a second approximation, a model is a set of points together with an evaluation of sentences at those points (as before), together with a *relation* on those points for each intensional connective in the language.

Consider the modal $\Box$. The Kripke semantics for it appeals to a binary relation $R_\Box$ on points: for any sentence A at any point x in any model M,

$$x \vDash_M \Box A \text{ iff } y \vDash_M A \text{ for all } y \text{ such that } R_\Box xy$$

Note that the unary connective $\Box$ is given its semantics by appealing to a binary relation $R_\Box$ on points. In fact, it turns out that n-ary connectives can often be given good and sensible semantics by appealing to $(n + 1)$-ary relations on points.[5] This, we think, should lead us, at least at first, to expect that semantics for our binary conditional connective will appeal to a ternary relation. The expectation would be satisfied; in the semantics given by Routley and Meyer, the crucial ternary relation R is involved in the semantics as follows: for any

[3]We drop the M subscript when it's clear from context, writing $x \vDash A$.

[4]The way we set things up here, every sentence must either hold or not (and not both) at any point in any model, but we say nothing about negation, leaving open the possibility that A and its negation might both hold or both fail to hold at the same point in a model. In fact this is more than a mere possibility—it's crucial to a relevant treatment of negation. But this paper isn't about negation, so we let the point pass. (For a philosophical interpretation of the relevant 'star' semantics for negation, see [22].)

[5]This is the basis of Dunn's Generalized Galois Logics ('Gaggles'), a project begun in [7] – see [4] for a summary, references, and new work. See also [5].

sentences A and B at any point x in any model M:

$$x \vDash_M A \to B \text{ iff for all } y, z \text{ such that } Rxyz, \text{ if } y \vDash_M A, \text{ then } z \vDash_M B.$$

2.2 More is required

So far, so good. But, says Van Benthem [32], this is too easy.

> There appears to be an over-application of the Henkin method in intensional logic, generating facile possible world semantics. For instance, could it be that the Routley semantics lacks explanatory power, due to its lack of potential falsification?

So the semantics so far doesn't even ensure that the 'conditional' defined fails to be pure gibberish, let alone shed any light on conditionality.[6]

We agree, though we might quibble a bit about the details. But we concede the spirit of the point. (See [27] for an extreme example of formal semantics gone wild.) In order to provide a philosophically illuminating semantics of the relevant conditional, we need to say more about what these models are: what the points are, what the ternary relation R is, and why compound sentences—in particular conditionals—are evaluated in the way that they are. What's more, this explication had better make it clear how these models relate to conditionality; otherwise the semantics can be fairly accused of arbitrariness, or of ad hocness, or of simply copying the phenomenon to be explained. In short, the semantics are 'merely formal' and philosophically unilluminating—at least if we want to understand the *meaning* of a conditional. So more is required.

3 Three parsings of conditionality

In order to provide this 'more', we now look at three very standard understandings of conditionality: those that arise in the context of modal (absence-of-counterexample), intuitionist, and conditional logics. We will show how each familiar way is connected to the Routley-Meyer ternary relation.

Natural understandings of ternary relations, $Rxyz$, are often not egalitarian. For many ternary relations, two of the three arguments naturally cluster together, the other being out on its own. So it is in the interpretations of the ternary relation we will consider. In the first interpretation we consider, the second and third arguments go together; in the second interpretation, it is the first and second that go together, and in the third, it is the first and third. Let us see how.

[6] We hasten to add that there is no hint of a sneer in Van Benthem. The paper is a very open-minded and fair one. And indeed, see Section 4 below, where Van Benthem seems to have later had a kind of conversion experience regarding ternary accessibility with his 'arrow logic'.

3.1 Modal (absence-of-counterexample) conditionals

One way to think of a conditional 'If A then B' is as asserting an *absence of counterexamples*. A *counterexample*, in this context, is a *relevant 'point'* at which A (the antecedent) is true and B (consequent) 'untrue' or 'false' (in some sense). What makes a point relevant turns on the details of the conditional in question. For some (extensional) conditionals, like the (classical) material conditional, the class C of relevant points—the class of candidate counterexamples—is a singleton comprising only the point of evaluation (i.e., the 'place' at which the conditional is asserted): $x \models A \supset B$ iff there's no counterexample in $C = \{x\}$ iff x is not such that $x \models A$ and $x \not\models B$.[7]

Many conditionals consider a much broader class of candidate counterexamples than the point-of-evaluation singleton. We can modal up the hook ($\supset$) to expand our class of candidate counterexamples: the familiar (classical modal) strict conditional $A \mathrel{-\!\!3} B$, defined as $\Box(A \supset B)$, expands the class of relevant points beyond the point of evaluation: in particular, our class C of strict-hook relevant possibilities (i.e., candidate counterexamples for the strict hook) is now the class of *$\Box$-relevant possibilities*, the class containing all points y in the 'sight' of our $\Box$. In short, $A \mathrel{-\!\!3} B$ is true at a point x iff there's no relevant counterexample, no x-accessible y in $C = \{y : R_\Box xy\}$, that is:

$$x \models A \mathrel{-\!\!3} B \text{ iff there's no } y \text{ such that } R_\Box xy \text{ and } y \models A \text{ and } y \not\models B.$$[8]

These examples are familiar, and so too is the given pattern. The question at hand is whether this absence-of-counterexample pattern is at work (or may be seen as such) in the Routley–Meyer ternary semantics. On the surface, the answer seems to be negative, since we seem not to be searching for counterexamples in the target sense—single 'points' (however broadly construed) at which the antecedent is true but consequent untrue. As above, the clause for our relevant conditional $A \to B$ goes like this:

$$x \models A \to B \text{ iff there's no } y, z \text{ such that } Rxyz, y \models A \text{ and } z \not\models B.$$

But *counterexamples*, on the target conception, are supposed to be single 'points' or 'places' at which the antecedent is true and consequent untrue. Of course, if we *require* of all models that $Rxyz$ only if $y = z$, then the given truth-at-a-point condition is perfectly in line with the absence-of-counterexample pattern: the 'ternary' R is simply picking out the relevant points—the class of candidate counterexamples. The trouble is that such a requirement makes for irrelevant logic (in the technical sense of 'irrelevant'): we need for y and z to come apart

[7] These truth-at-a-point conditions are inappropriate for characterizing the material conditional in target relevant logics, since $x \models \neg A$ is not the same as $x \not\models A$, but we use them here because they're familiar and illustrate the pattern we're highlighting in the absence-of-relevant-counterexample conception of conditionals.

[8] Again, the characterization is not strictly appropriate for a *relevant* account of $\Box(\neg A \vee B)$, but our concern here is more with the familiar pattern. (Besides, we are ignoring, as much as we can, issues of negation in this paper.)

if our conditional is to behave 'relevantly' (i.e., if we're to give a semantics for the target relevant logics).

The absence-of-counterexample pattern is nonetheless at work in the Routley–Meyer ternary semantics. What's going on (or what may be seen as such) is that our conditional calls for a broader perspective on our universe of candidate counterexamples; it calls us to recognize 'pair points' in addition to our 'old' points.[9] In particular, for any $x, y \in W$, let us say that $\langle x, y\rangle$ is a *half point* iff $x = y$, and that $\langle x, y\rangle$ is a *duo point* if $x \neq y$. Using 'pair point' to cover both half and duo points, let us define *truth at a pair point* and *untruth/falsity at a pair point* as follows, where $\models$ is the standard truth-at-a-point relation in the Routley–Meyer semantics.

- Truth at a pair point: $\langle x, y\rangle \models_1 A$ iff $x \models A$.
- Untruth/Falsity at a pair point: $\langle x, y\rangle \models_0 A$ iff $y \not\models A$.

Worth noting is that these derivative truth- and untruth/falsity-at-a-pair relations collapse into our 'old' relation $\models$ for all half points:

$$\langle x, x\rangle \models_1 A \text{ iff } x \models A$$

and

$$\langle x, x\rangle \models_0 A \text{ iff } x \not\models A$$

But half points are not enough for relevance: we need duo points to break irrelevancies such as $A \rightarrow . B \rightarrow B$ and the like.

And now the absence-of-counterexample pattern comes into focus. The pattern is present in the 'ternary' semantics; it's just that the semantics asks us to broaden our conception of candidates—broaden the 'points' that serve as candidate counterexamples. In particular, the 'ternary' semantics asks us to extend our notion of counterexample to pair points: a counterexample to $A \rightarrow B$ is a relevant point w at which A is true and B 'untrue' or 'false' on the given relations: $w \models_1 A$ and $w \models_0 B$. In many cases (e.g., $p \rightarrow q$, etc.), half points serve as counterexamples; but there are other cases (e.g., $p \rightarrow . q \rightarrow q$, etc.) where the only counterexample w is a duo point $w = \langle y, z\rangle$ for some $y \neq z$.

Contrary to initial appearances, the 'ternary' semantics fully exhibits the absence-of-counterexample pattern. What the semantics calls for is recognition of 'pair points' in addition to our 'old' points. The 'ternary' truth-at-a-point conditions can now be seen as talking about the derivative relations $\models_1$ and $\models_0$ at pair points: $x \models_1 A \rightarrow B$ iff there's no x-accessible counterexample, that is,

- $x \models_1 A \rightarrow B$ iff there's no $w \in W^2$ such that Rxw and $w \models_1 A$ but $w \models_0 B$.

On this way of looking at things, the familiar absence-of-counterexamples conception of conditionality is respected in a fairly familiar fashion whereby the candidate counterexamples are picked out by an 'access' relation. The 'ternary' relation is really just a familiar binary access relation; the difference is that

[9] This idea is briefly noted in [3, Ch. 2].

we now look beyond our familiar points, acknowledging another sort—duo points. This pattern of expanding our perspective is familiar. Just as the 'nature of necessity' calls us to see more points beyond this one (i.e., beyond the point of evaluation), so too with the nature of relevant conditionals: they call us to see points in addition to our regular half points—to see duo points.[10]

3.2 Conditionals as operators

In the last section, it was natural to think of the relation as a binary relation of the form $Rx\langle y,z\rangle$; the $\langle y,z\rangle$ gives the loci of evaluations of the antecedent and the consequent, and the x gives us a perspective on this, so to speak. In the understanding of the ternary relation we will consider in this section, the natural grouping is $R\langle x,y\rangle z$, the x and y cooperating to generate z.

One may think of a conditional, $A \to B$, as something that provides a route from A to B, an operator that, when applied to A, gives B. Thus, for example, in intuitionist logic, $A \to B$ is standardly conceptualized as a construction which takes you from a proof of A to a proof of B. The verificationist spin is unnecessary. The operation can be thought of, realistically, simply as one which takes you from the proposition expressed by A to the proposition expressed by B. Thus, we may think of a conditional as a function which, when applied to the antedent, gives you the consequent. (What the function does to other arguments we can leave as a matter of indifference.)[11] On its own, of course, this is not enough: there are many functions which when applied to the proposition expressed by A give the proposition expressed by B, for instance the function which only takes A to B, that is, the singleton of the ordered pair $\langle A,B\rangle$, namely, $\{\langle A,B\rangle\}$. What it is for the proposition expressed by the conditional $A \to B$ to *hold* in a situation, world, state of information or the like is for the function it embodies to correspond to an *inference* which that situation makes available or supports.

In the sequent calculi for relevant logic, there are two ways of grouping formulas together, extensional—often denoted by a comma—and intensional—often denoted by a semi-colon.[12] A,B can be thought of as something like the extensional conjunction of A and B. $A;B$ can be thought of as something like the functional application of A to B.[13]

An analogous notion of 'application' makes sense for worlds, where these serve to determine both what the facts are and which inferences are available. Where x and y are worlds, we may think of the compound $x;y$ as the set of propositions, body of information or partial setup resulting from y by applying

[10]This still leaves open the question of what, exactly, pair points are. One way of looking at them is as pairs of situations coordinated by an information connection [16, 21].

[11]See, further, [20].

[12]These two ways originated in the work of Dunn and Mints around 1970, where they showed how to provide a cut-free Gentzen system for the positive fragment of the relevant logic **R**. [23, Ch. 2] contains (p. 125) a history of how this led to Belnap's Display Logic and other developments. See also [29].

[13]But only something like: if we had $(A \to B);A = B$, then we would have both $(A \to B);A \vdash B$ and $B \vdash (A \to B);A$. The first is relevantly valid; the second is not even classically valid.

to it all the inferences provided by x. Classically, this amounts to the operation of closure under logical consequence applied to $x \cup y$, but in the more general setting of relevant logics it behaves more like functional application of x to y, without the special combinatory properties of set union.[14] If there is an implication operator $\rightarrow$ on propositions expressing a function from antecedent to consequent corresponding to inference, therefore, $x; y$ should be, or at any rate warrant the assertion of, those propositions b such that for some a, $a \rightarrow b$ is warranted by x and a by y.

The obvious thing is then to define $Rxyz$ as meaning that z is, or has the information content of, $x; y$. This won't quite do, because there is no reason why, in this case, z should be one of the worlds. Crucially, we may have $a \vee b$ (the proposition expressed by $A \vee B$) in z, even though neither of a nor b is—as one would require for the standard truth conditions for disjunction. One can try to get around this problem by non-standard, or otherwise contrived, treatments of disjunction. However, it is simpler to define $Rxyz$ for worlds x, y and z, as meaning that *at least* everything warranted by $x; y$ holds at z, or in symbols, given a reading of worlds as sets, $x; y \subseteq z$. That is, the ternary relation we want is that z is at least as strong as x and y put together—where 'put together' is capable of several readings including the one we prefer in terms of inference. The Routley-Meyer truth condition for the implication connective falls out naturally.[15]

3.3 Conditional logics

In the last two sections we have examined in turn what happens when we group together the last two terms $Rx\langle y, z\rangle$ and then what happens when we group together the first two terms $R\langle x, y\rangle z$. In the present section we consider grouping the first and last terms together, in an ugly notation: $Rx\rangle y\langle z$.

One way to look at the conditional in so called conditional logics is as describing some kind of relativized necessity—in effect, a link along the lines of $A \rightarrow B$ iff $\Box_A B$ iff B is necessary-in-an-A-ish-way. This is the point of view taken in [19, Ch. 5], where they are given semantics as follows. Where R_A is a binary relation on points determined by A (e.g., in the familiar framework of Lewis [14], $R_A xy$ could, given some assumptions, be defined as *y is one of the A-satisfying worlds closest to x*):

$$x \vDash A \Rightarrow B \text{ iff } z \vDash B \text{ for all } z \text{ such that } R_A xz$$

Without further restriction, this imposes certain logical relations on the consequents of conditionals; for example, if $A \rightarrow B$ and $A \rightarrow C$ both hold at a point,

[14]See [17] for an account of relevant implication in terms of such an application function. In that paper, in the spirit of seeking 'applied' semantics, Routley-Meyer models were considered as a treatment of multi-agent reasoning, where one agent provides the inferences and another the premises.

[15]Naturally, but not trivially: the fact that $x; y$ is, so to speak, the intersection of all the worlds that extend it is a key lemma in the usual completeness proof for relevant logics.

then $A \rightarrow B \wedge C$ must hold there too, and vice versa. Also, if C holds at every point where B does (which it will if B entails C), then if $A \rightarrow B$ holds at a point, so too must $A \rightarrow C$.[16] But no logical relations are yet imposed on the antecedents of conditionals by this formulation, since there is no requirement that $\Box_A$ and $\Box_B$ have anything to do with each other, no matter what logical relations hold between A and B. Thus, we can even have logically equivalent sentences A and B such that $A \rightarrow C$ holds at a point but $B \rightarrow C$ fails there!

·· *Parenthetical historical note.* The equivalence of conditionals with the same consequent and equivalent antecedents has a complicated history in the literature. It is valid on Lewis's semantics, since his R_A is defined in terms of worlds *satisfying A*, but the axiomatization in the first edition of [14] does not yield it, as was pointed out by Erick C. W. Krabbe [12]; a patch is added in the second edition. In Stalnaker and Thomason's related systems [30], the equivalence is forced in the application by the rule that $A \rightarrow C$ may be inferred from $A \rightarrow B$ and $B \rightarrow A$; there was some (unpublished) discussion of the intuitive validity of this principle in the 1970s. Van Fraassen [34] gave a semantics for a conditional logic without this, essentially by indexing relations with formulas. *End note* ··

Although this gives us a great deal of flexibility, it gives too much. If we want to focus on conditionality, a more restrictive model is appropriate.[17] A first step might be to require intersubstitutability of logical equivalents. This is easy to do; we simply let our relations be indexed not by sentences, but instead by sets of points. (These sets of points can be considered what Alan Anderson dubbed a 'UCLA proposition', i.e., the set of points at which the proposition is true.) Where $|A|_M$ is the set of points in the model M at which A holds,[18] we can then restate the above clause like so:

$$x \vDash A \Rightarrow B \text{ iff } z \vDash B \text{ for all } z \text{ such that } R_{|A|}xz.$$

Since logically equivalent sentences hold at all the same points, this will have the desired effect.

There is more logical structure to be had, though. Antecedents of conditionals, for example, are a classic example of a downward-entailing environment: if A entails B, then if $B \rightarrow C$ is true at a point, $A \rightarrow C$ should be true there too. We can ensure this straightforwardly as follows: we require of our models that, for any sets X and Y of points, if $X \subseteq Y$, then $R_X \subseteq R_Y$.

So far so good. But this still allows for $A \rightarrow C$ and $B \rightarrow C$ to both hold at a point without $A \vee B \rightarrow C$ holding there. This is out of line with our understanding of conditional sentences. To rule out cases like this, we can impose a further

[16] That is, $\Box_A$ is a normal modal necessity, for any sentence A.

[17] Some of the restrictions we make here are contentious, but we think alleged counterexamples to the inferences in question trade on shifts in context, and so fail as demonstrations of invalidity. (For an overview of the alleged counterexamples, and substantially this response, see [15].) If that doesn't persuade you, you can read us as describing a certain sort of conditional, the sort that holds just when the antecedent is *sufficient* for the consequent.

[18] We drop the subscript M when no confusion will arise, which is nearly always.

restriction on our models, where W_is are sets of points: $R_{\bigcup\{W_i\}} \subseteq \bigcup\{R_{W_i}\}$.[19]

In fact, we can combine these last two restrictions into one: where W_is are sets of points, we require that $R_{\bigcup\{W_i\}} = \bigcup\{R_{W_i}\}$. One direction of this equality is simply our second requirement; the other direction is equivalent to our first requirement.[20]

At this point, we've restricted our models so that certain inferences come out valid. Here, we show that these restrictions bring with them our familiar ternary-relational semantics. To remind, we have:

- $x \vDash A \Rightarrow B$ iff for all z such that $R_{|A|}xz, z \vDash B$
- $R_{\bigcup\{W_i\}} = \bigcup\{R_{W_i}\}$

Given, by a model, relations R_Y for each set Y of points, we can define, for that model, a ternary relation R on points as follows: $Rxyz$ iff for all sets Y of points such that $y \in Y$, R_Yxz. Thus, this way of getting to a ternary relation takes the middle term, y, to be the 'special' one; it thus fills out the three possible perspectives on the relation. Now, take x to be any point in a model meeting our restrictions, then:

(*) $z \vDash B$ for all z such that $R_{|A|}xz$ iff $z \vDash B$ for all y, z such that $Rxyz$ and $y \vDash A$.

Proof LTR: Assume there is a point x in some model such that $Rxyz$, $y \vDash A$, but $z \nvDash B$. Since $y \vDash A$, $y \in |A|$. And since $Rxyz$, it must be that $R_{|A|}xz$.

Proof RTL: Assume there is a point x in some model such that $R_{|A|}xz$ and $z \nvDash B$. Suppose there is no $y \in |A|$ such that $Rxyz$. Then, for every $y_i \in |A|$, there must be some set Y_i such that $y_i \in Y_i$ and $\langle x, z\rangle \notin R_{Y_i}$. Since this is so for each Y_i, it must be that $\langle x, z\rangle \notin \bigcup\{R_{Y_i}\}$, and so $\langle x, z\rangle \notin R_{\bigcup\{Y_i\}}$. But we can see $|A| \subseteq \bigcup\{Y_i\}$, so $\langle x, z\rangle \notin R_{|A|}$. Contradiction. Thus, there must be a $y \in |A|$ such that $Rxyz$.

The fact (*) shows us that, when our models are properly restricted for conditionals, the conditional-logic clause invoking binary relations indexed by sets of points can be replaced by the familiar Routley-Meyer clause invoking a ternary relation definable from the indexed binary relations.

·· *Parenthetical note.* We remind, for readers thinking along with us via a comparison with Lewis framework [14], that Lewis's semantics is often formulated

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[19] This is slightly stronger than what we'd need, so long as there's no infinitary disjunction in the language. But it's clear, we think, that if there is infinitary disjunction in the language, it should work in this way too; that is, that if $A_i \rightarrow C$ holds at a point for every A_i, then $\bigvee\{A_i\} \rightarrow C$ holds there too. That requires the condition we've given, so we'll stick with it.

* *Comparative note wrt [14].* For the reader familiar with Lewis's semantics [14], which we've noted in passing above, we note that that semantics does not guarantee that $A \rightarrow C$ and $B \rightarrow C$ will always hold at x if $(A \vee B) \rightarrow C$ does; counterexamples arise when one of A and B is more 'far-fetched' than the other. This was criticized as counterintuitive by Nute [18]. The ternary relation semantics for relevance logics, and its generalizations in Dunn's Gaggle Theory [7], do validate 'distribution' equivalences like this. *End note* *

[20] *Proof*: Assume our first requirement, and that $y \in \bigcup\{R_{W_i}\}$. Then $y \in R_{W_k}$, for some k. Since $W_k \subseteq \bigcup\{W_i\}$, $y \in R_{\bigcup\{W_i\}}$, by our first requirement. Now, assume $\bigcup\{R_{W_i}\} \subseteq R_{\bigcup\{W_i\}}$, and that $X \subseteq Y$. This means $X \cup Y = Y$, so $R_X \cup R_Y = R_{X \cup Y} = R_Y$, and thus $R_X \subseteq R_Y$.

in terms of a ternary relation between worlds: y is more similar (or *is counterfactually closer to*) x than z is. It is not, however, a special case of the Routley-Meyer semantics: the quantificational structure of the clause giving the truth conditions for Lewis's arrow in terms of his ternary relation is more complicated than that of the Routley-Meyer clause, which is what allows such pathologies as that in the *comparative note wrt [14]* in footnote 19. *End note* ··

4 Bringing the threads together

What we have now done is show, as promised, that whatever it takes to be a conditional—at least in modal, intuitionist, and conditional, logics—the ternary R has got it. But is there a perspective which unifies these three perspectives? Here is one suggestion.[21]

Consider the relation R to be a relation of relative relative possibility; that is, take it that $Rxyz$ just when z is possible relative to y, relative to x.[22]

·· *Parenthetical historical note.* This was suggested to Routley and Meyer by Dunn in response to his reading an early draft of their first paper on the semantics of entailment. They cite him in their published version [26, p. 206]:[23]

> Consider a natural English rendering of Kripke's binary R. xRy 'says' that 'world' y is possible relative to world x. An interesting ternary generalization is to read $xRyz$ to say that 'worlds' y and z are compossible (better, maybe, compatible) relative to x. (The reading is suggested by Dunn.)

Peter Woodruff soon after suggested to Dunn that the ternary accessibility relation be viewed as an indexed set of binary accessibility relations. See [9]. To spell this out a little, we could *define* a Routley-Meyer frame simply as a Kripke frame for a (normal) multi-modal logic together with a function assigning to each world x one of the binary accessibility relations R_x and thus one of the necessity operators $\Box_x$ with $y \models \Box_x B$ iff $\forall z(R_x yz \Rightarrow z \models B)$. Where G is the real (base) world of a (reduced) frame, the appropriate box for G is just truth.

[21]On the other hand, we could keep the three perspectives separate to see whether they motivate different logics. For example, consider the functional interpretation of $\rightarrow$. If we allow ourselves some 'type-lifting' we can take the proposition A to be a function from $A \rightarrow B$ to B. Thus, on this interpretation, we might want to say that any point that satisfies A also satisfies $(A \rightarrow B) \rightarrow B$. We can enforce that by placing the condition on frames that if $R < x, y > z$ then $R < y, z > z$. Of course, type-lifting is not forced on us by the functional interpretation, but it seems *natural* in that context, whereas on the other two interpretations it seems convoluted at best.

[22]It is important to note that this cannot be understood as 'relative product': $xSy \,\&\, ySz$. In the case of the relevant logic **R** it is an easy exercise to show from Routley and Meyer's conditions on R that it would then be a congruence connecting every set of points on the frame, and so effectively there would be just one point and we would be back to classical logic. Incidentally, it is also known that the ternary relation cannot be defined as $xSz \,\&\, yTz$. See [6]. It would be nice to have a more general result about R not being first-order definable using just binary relations.

[23]We change the original notation to conform to our notation in this paper.

That is, $\Box_G B$ is (or is equivalent to) B. Other worlds, however, have their own perspectives on where B is required to hold if $\Box_x B$ is to hold at y.

Then the relevant implication connective is such that

$$x \models A \rightarrow B \text{ iff } \forall y (y \models A \Rightarrow y \models \Box_x B).$$

Even more neatly, $A \rightarrow B$ holds at x iff $A \rightarrow \Box_x B$ is true (holds at G), where $A \rightarrow B$ holds at G iff B holds everywhere that A holds.

A little more is required for the technical care and feeding of the monotonicity postulate on Routley-Meyer frames, but the point that this construal makes modal sense of relevant conditionals is clear enough.

The necessity operator with the single world x indexing a single binary relation can be generalized to an operator $[X]$ with a set of worlds X indexing many binary relations, as in Dunn's interpretation of Pratt's dynamic logic in [9]. It has the truth condition $y \models [X]B$ iff $\forall x \in X$, $\forall z (R_x yz \Rightarrow z \models B)$. Equivalently, we can reduce the many binary relations to a single one, defining $R^X yz$ to mean $\exists x \in X (R_x yz)$, and then $y \models [X]B$ iff $\forall z (R^X yz \Rightarrow z \models B)$ We may view X as indexing a binary relation, and so we get a special case of the operator $\Box_A$ of section 3.3 when the sentence A is replaced, as ultimately suggested there, by a set of points X.

Dunn has more recently exploited Woodruff's construction as a duality between data (static) and computation (dynamic). The basic idea is that propositions can be viewed as either sets of states or as the set of actions these states index. See [9] for a general explanation, and [10] and [8] for concrete applications. Barwise [2] had a formally similar idea, which was of two 'sites' being connected by a 'channel.' He did not rule out the case where channels might also be sites. See also [21]. Mention should also be made of van Benthem's 'arrow logic' [33]. *End note* ··

To get a grip on what relative relative possibility is, it will help to return to the modal case: simple relative possibility. When is a point z possible relative to y? When everything required (necessary) at y holds at z. Note that nothing in this picture guarantees that what's required at a point holds at that point. For example, A might be required at y without holding at y; then y would not be possible relative to itself. We write $\Box B$ to record B's being required somewhere, and there is only this one constraint on relative possibility.

What happens, then, if we want to consider different constraints beyond just $\Box B$'s requiring B? What if we wanted to consider the phenomenon of constraint in general, of A's requiring B? (This, it should be clear, is where conditionality arrives on the scene.) Well, as before, z is possible relative to y iff everything required at y holds at z; that is, iff for every constraint whose antecedent holds at y, its consequent holds at z. Again, nothing here guarantees that what's required at a point holds at that point. If there's a constraint whose antecedent holds at y and whose consequent doesn't, then y isn't possible relative to itself. But now, since we want to explore conditionality in general, we want different constraints to hold at different points. This requires that whether z is possible relative to y can itself only be answered relative to a point x with its constraints.

Thus, we see built in to the very notion of conditionality the idea of relative relative possibility—the ternary relation we've been calling R. It's no surprise, then, that R keeps on turning up when we examine conditionality.

All three of our foregoing readings of relation R can be seen in this light. Think of our first gloss, in terms of counterexamples. For the pair point $\langle y, z\rangle$ to count as a counterexample, according to x, is simply for z to be relatively possible from y, according to x: if everything required at y, according to x, holds at z, then if A holds at y but B doesn't hold at z, A can't require B according to x.

When it comes to our second gloss, in terms of functions, we can now offer a story as to what *kind* of function is in play. The conditional $A \to B$ is not just any function from A to B; it's a function which, by outputting B given A, tells us that B is *required* by A. If $Rxyz$ represents relative relative possibility, then x and y fix the two parameters possibility is relative to. z is possible (relative to x and y) iff everything required (at y, according to x) holds at z. (This offers us another explanation for why $Rxyz$ is better defined as $x \circ y \subseteq z$ than as $x \circ y = z$. To be possible relative to x and y, z only needs to meet the requirements imposed by x at y; there's nothing stopping it from going beyond these requirements.)

Third, this way of thinking clarifies the conditional-logic picture as well. We can read $R_Y xz$ as telling us that everything that x requires to follow from the proposition Y holds at z. Supposing (as in §3.3) that x's requirements are suitably organized, this boils down to $Rxyz$ again: everything x requires to follow from *any* proposition that holds at y holds at z. This is, once more, relative relative possibility.

The idea of relative relative possibility thus gives us a single picture of the ternary relation R. As we've seen, we can elaborate this picture in any of three ways, depending on which argument place we wish to focus on. But however you slice it, relative relative possibility is intimately connected with the idea of conditionality: what it is for one thing to require another. This is captured by the ternary relational semantics for $\to$. It is anything but an empty formalism; it is at the very core of the nature of conditionality.

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