Paradoxes of Consistency & (Revising) The Logic of Belief

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Lottery & Preface Paradoxes
Indifference Paradoxes
Framework
Self-Reference
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1This talk draws heavily on joint work with Kenny Easwaran (USC).

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Before we dive in — some notation and basic assumptions.

- \( B(p) \equiv S \) believes that \( p \). \( D(p) \equiv S \) disbelieves that \( p \).

We will be using simple, idealized models of epistemic agents (viz., epistemically rational agents). As with all models, only some aspects of epistemic agents are modeled.

I’ll assume the following about \( B, D \). [The first 4 are integral, but the last one is just for simplicity. More on these later.]

- **Logical Omniscience.** If \( p \equiv q \), then \( B(p) \equiv B(q) \).
- **Incompatibility.** \( B(p) \Rightarrow \neg D(p) \).
- **Opinionation.** \( B(p) \vee D(p) \).
- **Accuracy conditions.** \( B(p) \) \([D(p)] \) is accurate iff \( p \) is \( T[F] \).

- (NC) \( D(p) \equiv B(\neg p) \).

Our \( S \)’s make judgments regarding all \( p \)’s in some finite Boolean algebra \( B \) (generated by some \( L_p \)). I will use \( \mathcal{B} \) to denote the set of all of \( S \)’s judgments (\( B \)’s and \( D \)’s) over \( B \).

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We will consider several (4) kinds of epistemic paradoxes, which seem to show that (classical) deductive consistency is not a rational (formal, coherence) requirement for belief.

- These paradoxes will be in (what I take to be) increasing order of difficulty. Only the final one involves self-reference. The first three involve (large-ish) minimal inconsistencies.

- After introducing the first three paradoxes, I will discuss two common types of responses to them (or responses to paradoxes like them) that have appeared in the literature.

- I will explain why I don’t find those responses completely satisfying. And, this will lead to a new proposal for revising the classical “logic of belief” (viz., classical coherence).

- I will present a new framework [8] for grounding formal coherence requirements. It is a natural and non-ad hoc generalization (to belief) of Joyce’s [13, 12] framework for grounding epistemic coherence requirements for credence.

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Classical deductive consistency is implicated in some well-known paradoxes. The two most infamous paradoxes (of the sort I have in mind) are the Lottery and the Preface.

(i) **Lottery Paradox** ([15],[6]). For each ticket \( i \), it is highly probable that \( i \) is a loser (\( L_i \)). So, it would seem reasonable to be such that \( B(L_i) \), for each \( i \). However, this inevitably renders our set \( \mathcal{B} \) inconsistent, since we know that \( (\exists i)(\neg L_i) \).

(ii) **Preface Paradox** ([17],[4]). Let \( B \subset \mathcal{B} \) be the set containing all of your reasonable (1st-order) beliefs. This \( B \) is an incredibly rich and complex set of judgments. You’re fallible (i.e., your 1st-order evidence is sometimes misleading). So, it seems reasonable to believe that some \( B \)’s in \( B \) are false. However, adding this (2nd-order) belief to \( B \) renders \( \mathcal{B} \) inconsistent.

- There has been a ton of discussion of these paradoxes in the literature. Roughly, there are two sorts of responses:

  (1) Dogmatically maintain consistency as a (global) CR for \( \mathcal{B} \).

    - Pollock ([18]): all inconsistent \( \mathcal{B} \)’s engender “collective defeat,” which mandates suspension of belief (wrt some \( p \)’s).
- Pollock-style responses are implausible. As Christensen [4] nicely explains, it just seems wrong to claim that suspension is reasonable/supported, for any salient p. [More generally, (1) has implausible consequences about evidential support.]

(2) Radically abandon all (formal) coherence requirements for \( B \).
- Christensen [4] suggests we do everything with credences, which (a) do have a formal coherence requirement (namely, probabilism), and (b) have no trouble dealing with the Lottery or the Preface. We should learn to live without B/D.
- Kolodny [14] thinks we cannot live without B/D, but we should learn to live without (formal, rational) coherence requirements for them. Instead, they have only evidential requirements, which are satisfied in the Lottery & Preface.

- I would prefer a different response than either (1) or (2). But, while (2) goes too far, there is a kernel of truth to it.
  - Christensen’s focus on credences is useful. We can learn a lot from how their coherence requirements are grounded.
  - Kolodny’s focus on evidence/justification is also crucial.

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- Here is another epistemic paradox, but this one is couched (first) in terms of comparative confidence. I will use ‘\( p \sim q \)’ to express the claim that ‘\( S \) is equally confident in \( p \) and \( q \)’.
- (III.1) \( S \) observes a bank robbery. \( S \) gets a good look at the robber \( (r_0) \), who has a full head of hair. The police create \( n \) perfect duplicates of the robber \( (r_i) \). They remove \( i \) hairs from the head of \( r_i \), and make a line-up: \( r_0, r_1, r_2, r_3, \ldots, r_N \). They show \( S \) only pairs: \( (r_0, r_1), (r_1, r_2), (r_2, r_3), \ldots, (r_{N-1}, r_N) \). Let \( p_i \equiv r_i \) be the robber (\( p_i \equiv r_i = r_0 \)). On the basis of the visual evidence \( S \) obtains from seeing the pairs, \( S \) comes to be such that \( p_1 \sim p_{i+1} \), for each \( i \). But, at the end of the day, the police show \( S \) one last pair: \( (r_0, r_N) \), where \( r_N \) has zero hairs on his head. Of course, \( S \) is such that: \( p_0 \sim p_N \).

- This one is not so easy for the probabilist to handle. All forms of probabilism will require that \( \sim \) is transitive [10].
- Interestingly, there is a full belief version of (III) that the probabilist can handle (Christensen-style). We will also be able to handle the B-version of (III) in our framework.

Instead of using \( p \sim q \), we will use \( B(p \equiv q) \), where this is interpreted as: \( S \) believes that \( p \) and \( q \) are equally likely.

(III.2) \( S \) observes a bank robbery. \( S \) gets a good look at the robber \( (r_0) \), who has a full head of hair. The police create \( n \) perfect duplicates of the robber \( (r_i) \). They remove \( i \) hairs from the head of \( r_i \), and make a line-up: \( r_0, r_1, r_2, r_3, \ldots, r_N \). They show \( S \) only pairs: \( (r_0, r_1), (r_1, r_2), (r_2, r_3), \ldots, (r_{N-1}, r_N) \). Let \( p_i \equiv r_i \) be the robber (\( p_i \equiv r_i = r_0 \)). On the basis of the visual evidence \( S \) obtains from seeing the pairs, \( S \) comes to be such that \( B(p_i \equiv p_{i+1}) \), for each \( i \). But, at the end of the day, the police show \( S \) one last pair: \( (r_0, r_N) \), where \( r_N \) has zero hairs on his head. Of course, \( S \) is s.t.: \( D(p_0 \equiv p_N) \).

- It’s interesting that this one becomes easier to handle (both for probabilists and for us [10]), if it is couched in terms of beliefs about likelihoods, rather than comparative confidence.
- In any case, (III.2) is similar to the Lottery and the Preface, because it involves a large-ish, minimal inconsistent \( B \).
  [Here, I presuppose \( S \) knows the relation \( \approx \) is transitive.]

We would suggest that these paradoxes indicate that consistency is too strong to be a CR for full belief.

- Typically, such paradoxes involve a conflict between a consistency requirement and an evidential requirement, which requires believing what is justified/supported.

\( \Diamond \box{} \) Ideally, we want coherence requirements for full belief that are entailed by both alethic and evidential considerations.

- Our framework yields just such CR’s, in “3 easy steps”.
  - Step 1: Define the vindicated (viz., perfectly accurate) judgment set, at \( w \). [“Judgments of the omniscient \( S \) at \( w \).”]
    \( \hat{B}_w \) contains \( B(p) \) iff \( B(p) \) is true (false) at \( w \).
  - Step 2: Define a notion of “distance between \( B \) and \( \hat{B}_w \)”. That is, a measure of distance from vindication \( d(B, \hat{B}_w) \).
    \( d(B, \hat{B}_w) \) \( \equiv \) the number of inaccurate judgments in \( B \) at \( w \).
  - Step 3: Adopt a fundamental epistemic principle, which uses \( d(B, \hat{B}_w) \) to ground a formal coherence requirement for \( B \).
Given our choices at Steps 1 and 2, there is a choice we can make at Step 3 that will yield consistency as a CR for $\mathfrak{B}$.

**Possible Vindication (PV).** There exists some possible world $w$ at which all of the judgments in $\mathfrak{B}$ are accurate. Or, to put this more formally in terms of $d$: $(\exists w)[d(\mathfrak{B}, \mathfrak{B}_w) = 0]$.

Possible vindication is one way we could go here. But, our framework is much more general than the classical one. It allows for many other choices of fundamental principle.

Like Joyce [13, 12] — who makes the analogous move with credences, to ground probabilism — we retreat from (PV) to the weaker: avoidance of (weak) dominance in $d(\mathfrak{B}, \mathfrak{B}_w)$.

**Weak Accuracy-Dominance Avoidance (WADA).**

There does not exist an alternative belief set $\mathfrak{B}'$ such that:

1. $(\forall w)[d(\mathfrak{B}', \mathfrak{B}_w) \leq d(\mathfrak{B}, \mathfrak{B}_w)]$, and
2. $(\exists w)[d(\mathfrak{B}', \mathfrak{B}_w) < d(\mathfrak{B}, \mathfrak{B}_w)]$.

Completing Step 3 in this way leads to a new CR for $\mathfrak{B}$.

(WADA) leads to a coherence requirement that is strictly weaker than consistency — even if (NC) $D(P) \equiv B(\neg P)$. [We think (WADA) + (NC) is equivalent to (Rs). See Extras (15-17).]

Moreover, in general, minimal inconsistent $\mathfrak{B}$’s (of cardinality greater than 2) will not violate (WADA)/(Rs).

So, if we adopt (WADA)/(Rs), rather than (PV)/consistency, “paradoxes” (I)-(III) do not imply violations of our CR for $\mathfrak{B}$.

But, if $S$ does violate (WADA)/(Rs), this reveals two defects:

1. There must exist some judgment in $\mathfrak{B}$ that is inaccurate. [This alethic defect follows from: (PV) $\Rightarrow$ (WADA)/(Rs).]
2. And, there must exist some judgment in $\mathfrak{B}$ that is not supported by “the evidence” — whatever “the evidence” is! [This evidential defect follows from: (EB) $\Rightarrow$ (WADA)/(Rs).]

Our coherence requirements are points of agreement between alethic vs evidential perspectives (i.e., points of contact between truth/accuracy vs justification/evidence).
So far, our “paradoxes” are far removed from semantic paradoxes. [But, (III) does flirt with the sorites paradox.]


There are analogous examples in our full belief framework.

In order to be able to say anything interesting about \((P)\), we’ll need to relax our simplifying assumption (NC).

Moreover, if \(S\) is opinionated, then (next slide) the best she can do (from an accuracy-dominance-avoidance perspective) is to be such that either \(\{B(P), B(\neg P)\}\) or \(\{D(P), D(\neg P)\}\).

So, we will relax both \((\text{NC})\) and Opinionation. We will now allow \(S\) to suspend judgment regarding \(P\) \([S(P)]\). But, it’s not clear whether adding suspension will help us here.

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- There are judgment sets that are non-d-dominated, but inconsistent.
- This is true, even if we assume opinionation & (NC) \(D(p) = B(\neg p)\).
- The simplest possible examples of this kind involve \(S\)’s with at least two atomic sentences (e.g., \(X, Y\)).
- The table shows the 8 sets that are non-dominated, but inconsistent.

Of course, as we look at larger and larger languages/algebras, we start seeing more interesting examples.

But, even these toy examples exhibit a “preface-like” structure.

The logically stronger propositions tend to be disbelieved (and the logically weaker ones believed).

- There are nine (9) possible pairs of attitudes (viz., \(\mathcal{B}\)’s) a non-opinionated agent can have toward \(\{P, \neg P\}\). They are:
  1. \(\mathcal{B}_1 = \{B(P), B(\neg P)\}\) \(P\) is false. 1 right, 1 wrong, 0 (?).
  2. \(\mathcal{B}_2 = \{B(P), D(\neg P)\}\) \(P\) is false. 0 right, 2 wrong, 0 (?).
  3. \(\mathcal{B}_3 = \{B(P), S(\neg P)\}\) \(P\) is false. 0 right, 1 wrong, 1 (?).
  4. \(\mathcal{B}_4 = \{D(P), B(\neg P)\}\) \(P\) is true. 0 right, 2 wrong, 0 (?).
  5. \(\mathcal{B}_5 = \{D(P), D(\neg P)\}\) \(P\) is true. 1 right, 1 wrong, 0 (?).
  6. \(\mathcal{B}_6 = \{D(P), S(\neg P)\}\) \(P\) is true. 0 right, 1 wrong, 1 (?).
  7. \(\mathcal{B}_7 = \{S(P), B(\neg P)\}\) \(P\) is true. 0 right, 1 wrong, 1 (?).
  8. \(\mathcal{B}_8 = \{S(P), D(\neg P)\}\) \(P\) is true. 1 right, 0 wrong, 1 (?).
  9. \(\mathcal{B}_9 = \{S(P), S(\neg P)\}\) \(P\) is true. 0 right, 0 wrong, 2 (?).

- The “(?)” indicates my uncertainty about how to evaluate suspensions here. [Question: should we have \(S(P) \equiv S(\neg P)\)?]

- Anyhow, among the opinionated \(\mathcal{B}\)’s [1, 2, 4, 5], only non-dominated \(\mathcal{B}\)’s are [1, 5], which violate (PV) and \((\mathcal{R}_3)\)!

- This wreaks havoc with our framework, as it now stands.

\[(\text{TB}) S\] ought believe \(p\) just in case \(p\) is true.

\[(\text{PV}) (\exists w)[d(\exists\mathcal{B}, \mathcal{B}_w) = 0]\]. That is, \(\mathcal{B}\) is deductively consistent.

\[(\text{SADA}) \not\exists \mathcal{B}'\] such that: \((\forall w)[d(\exists\mathcal{B}', \mathcal{B}_w) < d(\exists\mathcal{B}, \mathcal{B}_w)]\).

\[-(\exists\mathcal{B} \square \text{MI}) \exists \beta \subseteq \mathcal{B} \text{ s.t. } (\forall w)[\frac{1}{2} > \text{ of the members of } \beta \text{ are inaccurate at } w] \]

\[(\mathcal{R}_3) \exists \text{ a probability function } Pr(\cdot) \text{ such that, } \forall p \in \mathcal{B}: \quad B(p) \text{ iff } Pr(p) > \frac{1}{2}, \text{ and } D(p) \text{ iff } Pr(p) < \frac{1}{2}.\]

\[(\text{EB}) S\] ought believe \(p\) just in case \(p\) is supported by \(S\)’s evidence.

Note: this assumes only \((\exists \text{Pr})(\exists E)[\text{Pr}(p \mid E) > \frac{1}{2} \text{ iff } B(p)]\).

\[(\text{WADA}) \not\exists \mathcal{B}' \text{ s.t. } (\forall w)[d(\exists\mathcal{B}', \mathcal{B}_w) < d(\exists\mathcal{B}, \mathcal{B}_w)] \cup (\exists w)[d(\exists\mathcal{B}', \mathcal{B}_w) < d(\exists\mathcal{B}, \mathcal{B}_w)]\).

\[(\mathcal{R}_w) \exists \text{ a probability function } Pr(\cdot) \text{ such that, } \forall p \in \mathcal{B}: \quad B(p) \text{ iff } Pr(p) > \frac{1}{2}, \text{ and } D(p) \text{ iff } Pr(p) < \frac{1}{2}.\]

\[-(\exists\mathcal{B} \square \text{HI}) \not\exists \beta \subseteq \mathcal{B} \text{ s.t. } (\forall w)[\frac{1}{2} \geq \text{ of the members of } \beta \text{ are inaccurate at } w] \]

\[(\text{NC}) S\] disbelieves \(p\) iff \(S\) believes \(\neg p\) \([i.e., D(p) \equiv B(\neg p)]\).
• Here is what the logical relations look like, among all of the 10 norms for (opinionated) $\mathcal{B}$. [Double (single) arrows represent known (conjectured) entailments. And, if there is no path, then we believe (or conjecture) that there is no entailment.]

\[
\begin{align*}
& (TB) \\
\downarrow & (PV) \\
\neg (\exists \beta \square HI) & \iff (R_S) & \iff (WADA) + (NC) \\
\downarrow & (WADA) \\
\downarrow & (R_W) \\
\neg (\exists \beta \square MI) & \iff (SADA)
\end{align*}
\]

- We (along with Rachael Briggs and Fabrizio Cariani) [1] are investigating various applications of this new approach.
- One interesting application is to judgment aggregation. E.g.,
  - Majority rule aggregations of the judgments of a bunch of agents — each of whom satisfy (PV) — need not satisfy (PV).
- Q: does majority rule preserve our notion of coherence, viz., is (WADA) preserved by MR? A: yes (on simple, atomic + truth-functional agendas), but not on all possible agendas.
  - There are (not merely atomic + truth-functional) agendas $A$ and sets of judges $J$ ($|A| \geq 5$, $|J| \geq 5$) that (severally) satisfy (WADA), while their majority profile violates (WADA).
  - But, if a set of judges is (severally) consistent [i.e., satisfy (PV)], then their majority profile must satisfy (WADA).

**Recipe.** Wherever $\mathcal{B}$-consistency runs into paradox, substitute coherence (in our sense), and see what happens.

• Another possible application of our new notion of coherence is to (at least partially) fill-out David Ripley and Greg Restall’s proposal about entailment. They suggest:

\[\Gamma \vdash \Delta \equiv \text{if an agent } S \text{ asserts each member of } \Gamma \text{ and denies each member of } \Delta, \text{ then } S \text{ is (thereby) “out of bounds”.} \]

• I do not want to get into a discussion about norms for assertion/denial. But, I can offer the following alternative that seems similar in spirit to the Ripley/Restall idea:

\[\Gamma \vdash \Delta \equiv \text{if } S \text{ believes each member of } \Gamma \text{ and disbelieves each member of } \Delta, \text{ then } S \text{ is (thereby) incoherent (in our sense).} \]

• I’m not sure how this definition would combine with Ripley’s ideas about “classical logic” and intro-elim vs structural rules involving $\models$, etc. We should talk about it.

• I doubt this is going to get Ripley everything he wants. But, if classical consistency is not a requirement of epistemic rationality, then this may be the closest he can get.

• One disanalogy with Joyce’s argument for $b$-probabilism, is that his argument is very flexible about weights it can assign to different $p$’s in $\mathcal{B}$. This seems like an advantage.

• After all, aren’t some propositions “more important” (“more important” to have accurate beliefs about) than others?

Sure. But, precisely what are we missing? 3 possibilities:

1. Something of pragmatic value. E.g., beliefs about my wife’s health vs. beliefs about the # of pebbles on pebble beach.
2. Something of instrumental epistemic value. E.g., being right about $p$ leads to a greater # of accurate beliefs/disbeliefs.
3. Something (X) of intrinsic epistemic value — where X can’t supervise on considerations of accuracy (+ evidence).

• Only (3) could be a problem for us. But, what could X be?

• Perhaps X is “informativeness”? I think this is orthogonal to pure accuracy, and requires separate treatment. [Here, we’d be getting into “synchronic closure” requirements, etc.]
Kenny is writing a paper [7] that explains how to relax the assumption of opinionation in our framework.

Our present approach is equivalent to assigning (in)accurate judgments an “accuracy score” of \((-w) + r\) (where \(w > r > 0\)), and calculating the “overall accuracy score” for \(B\) (at \(w\)) as the sum of “accuracy scores” over all \(p \in B\) (at \(w\)).

Kenny’s idea: allow \(S\) to suspend on \(p\) \([S(p)]\), and then “score” suspensions with a neutral “accuracy score” of zero.

On this neutral accuracy scheme for suspensions, we get a nice generalization of our representation Theorem (II).

(III) **Theorem.** An agent \(S\) will avoid (strict) dominance in “total accuracy score” if their \(B\) can be represented as follows:

There exists a probability function \(\Pr(\cdot)\) such that, \(\forall p \in B:\)

\[
B(p) \text{ iff } \Pr(p) > \frac{w}{1 + w}, \\
D(p) \text{ iff } \Pr(p) < 1 - \frac{w}{1 + w}, \\
S(p) \text{ iff } \Pr(p) \in \left[1 - \frac{w}{1 + w}, \frac{w}{1 + w}\right].
\]