(General Purpose) Automated Reasoning in Modal Logics

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Overview of Presentation

- Brief Background on AR (first-order syntax, Otter notation, clauses)
- Propositional Modal Logics
  - Axiomatic Approaches
    - AR in Hilbert-style systems
    - Proofs & Models
  - Semantical Approaches
    - AR involving Kripke translations
    - Proofs & Models
  - Application of Axiomatic Methods to Interpretability Logics
- Other, "special purpose" approaches
- Challenge problems and Open Questions
- References

Brief Background on AR I

- Basic Notation (Otter syntax in parens):

Predicates	A, B, C (`A`, `B`, `C`)	Constants	a, b, c (`a`, `b`, `c`)
Variables	x, y, z (`x`, `y`, `z`)	Functions	f, g, h (`f`, `g`, `h`)
Quantifiers	$\forall$, $\exists$ (na)	Connectives	$\wedge$, $\rightarrow$, $\vee$, $\neg$, $=$ (na, na, `|`, `-`, `=`)

- Formulas *vs* Clauses (quantifier elimination and CNF)

Formula	Clause (Otter — Q-free, and CNF)
$(\forall x)(Px \rightarrow Gx)$	`-P(x)	G(x).`
$(\exists x)(Px \wedge Gx)$	`P(a). G(a).` (two clauses, new "a")
$(\forall x)(\exists y)(Rxy \vee x \neq y)$	`R(x,f(x))	-(x = f(x)).` (new "f")
$(\forall x)(\forall y)(\exists z)(Rxyz \wedge Rzyx)$	`R(x,y,f(x,y)). R(f(x,y),x,y).` (new "f")

- See chapters 1 and 10 of Kalman's recent book [11], and McCune's Otter user manual [13] for details on Otter's clause notation and syntax.

Brief Background on AR II

- Otter implements many rules of inference and strategies (see [11]). For our purposes (for now), it will suffice to discuss just one of these.
  - *Hyperresolution* [11, chapter 2] is a generalization of disjunctive syllogism in classical logic. Here are some examples:

```
-P | M.        -P(x) | M(x).     -L(x,f(b)) | L(x,f(a)).
P.             P(s).             L(y,f(y)).
------------   ---------------   -------------------------
∴ M.           ∴ M(s).           ∴ L(b,f(a)).
```

```
-P(x) | P(L(x)).        -P(i(x,y)) | -P(x) | P(y).
P(i(x,x)).              P(i(i(i(x,y),i(y,z)),i(x,z))).
-------------------     P(i(i(i(x,y),x),y)).
∴ P(L(i(x,x))).         ---------------------------------
                        ∴ P(i(x,x)).
```

  - In $\mathcal{N}_1 | \ldots | \mathcal{N}_n$, $\mathcal{S}$, $\therefore \mathcal{R}$, $\mathcal{N}_1 | \ldots | \mathcal{N}_n$ is the *nucleus*, $\mathcal{S}$ (may be a set) is the *satellite*, and $\mathcal{R}$ (may be non-literal) is the *hyperresolvent*.

Proving Theorems in Hilbert-Style Sentential Logics I

- As our last example shows, hyperresolution is the perfect rule for reasoning about sentential logical calculi (in Hilbert-Style).
- For instance, classical sentential logic can be axiomatized using only hyperresolution, and the following four clauses (see [17], and [11, ch. 8]):
  MP. `-P(i(x,y)) | -P(x) | P(y).`
  $Ł_1$. `P(i(i(x,y),i(i(y,z),i(x,z)))).`
  $Ł_2$. `P(i(x,i(n(x),y))).`
  $Ł_3$. `P(i(i(n(x),x),x)).`
- In recent years, we (at Argonne) have used OTTER to prove lots of new results in a wide variety of sentential logics (see [6], [5], [8], [4], [7]).
- Even simple logical calculi can involve *very* difficult proofs (see [20] for a nice survey of challenging problems, and powerful strategies for attacking them). We can prove all the theorems in [15, Appendix I] using OTTER.

Proving Theorems in Hilbert-Style Sentential Logics II

- Here's a simple but non-trivial OTTER proof of `P(i(n(n(x)),x))` in Ł:

```
1 [MP] -P(i(x,y)) | -P(x) | P(y).
2 [Ł1] P(i(i(x,y),i(i(y,z),i(x,z)))).
3 [Ł2] P(i(x,i(n(x),y))).
4 [Ł3] P(i(i(n(x),x),x)).
5 [2,2,1] P(i(i(i(i(x,y),i(z,y)),u),i(i(z,x),u))).
6 [3,2,1] P(i(i(i(n(x),y),z),i(x,z))).
7 [4,2,1] P(i(i(x,y),i(i(n(x),x),y))).
8 [5,5,1] P(i(i(x,i(y,z)),i(i(u,y),i(x,i(u,z))))).
9 [6,5,1] P(i(i(x,n(y)),i(y,i(x,z)))).
10 [7,6,1] P(i(x,i(i(n(n(x)),n(x)),y))).
11 [7,5,1] P(i(i(x,y),i(i(n(i(y,z)),i(y,z)),i(x,z)))).
12 [7,4,1] P(i(i(n(i(n(x),x)),i(n(x),x)),x)).
13 [9,6,1] P(i(x,i(y,i(n(x),z)))).
```



```
14 [10,8,1] P(i(i(x,i(n(n(y)),n(y))),i(y,i(x,z)))).
15 [11,8,1] P(i(i(x,i(n(i(y,z)),i(y,z))),i(i(u,y),i(x,i(u,z))))).
16 [13,2,1] P(i(i(i(x,i(n(y),z)),u),i(y,u))).
26 [13,12,1] P(i(x,i(n(i(i(n(i(n(y),y)),i(n(y),y)),y)),z))).
17 [16,14,1] P(i(n(x),i(x,i(y,z)))).
18 [26,15,1] P(i(i(x,i(n(i(n(y),y)),i(n(y),y))),i(z,i(x,y)))).
19 [17,11,1] P(i(i(n(i(i(x,i(y,z)),u)),i(i(x,i(y,z)),u)),i(n(x),u))).
20 [18,18,1] P(i(x,i(i(n(y),i(n(i(n(y),y)),i(n(y),y))),y))).
21 [20,19,1] P(i(n(n(x)),x)).
```

- This is a shorter proof than the one Łukasiewicz reports in [17]. To give you a feel for a *hard* problem in this area, try to prove that the following *single* axiom [14] is sufficient (with MP) to derive $Ł_1$–$Ł_3$.

  `P(i(i(i(i(i(x,y),i(n(z),n(u))),z),v),i(i(v,x),i(u,x)))).`

  The shortest known proof of this theorem is 41 steps long, and was found (from scratch) by Larry Wos using OTTER [6]. Wos's OTTER proof is simpler (in various ways) than the proof reported by Meredith in [14].

Proofs in Hilbert-Style Sentential Modal Logics I

- Sentential modal logics are just simple extensions of classical sentential logic. The new connectives "$\Box$" (we'll use "L" in OTTER) and "$\Diamond$" (we'll use "M" in OTTER) are added to the stock of classical connectives.
- All "normal" modal logics add the following rule of inference and the following axiom to classical sentential logic (OTTER notation):
  RN. `-P(x) | P(L(x)).`
  K. `P(i(L(i(x,y)),i(L(x),L(y)))).`
- OTTER performs best with minimal sets of connectives. So, I will use only {i, n, L} [`M(x) = n(L(n(x)))`] to characterize sentential modal logics.
- Other systems of interest add some or all of the following axioms:

D. `P(i(L(x),n(L(n(x))))).`
T. `P(i(L(x),x)).`
G. `P(i(L(n(L(n(x)))),n(L(n(L(x)))))).`
4. `P(i(L(x),L(L(x)))).`
5. `P(i(n(L(n(x))),L(n(L(n(x)))))).`
B. `P(i(x,L(n(L(n(x)))))).`

Proofs in Hilbert-Style Sentential Modal Logics II

1.	$\vdash p \to (q \to (p \,\&\, q))$	PL Tautology
2.	$\vdash \Box(p \to (q \to (p \,\&\, q)))$	1, RN
3.	$\vdash \Box(p \to q) \to (\Box p \to \Box q)))$	K axiom
4.	$\vdash \Box(p \to (q \to (p \,\&\, q))) \to (\Box p \to \Box(q \to (p \,\&\, q)))$	3, Subst.
5.	$\vdash \Box p \to \Box(p \to (p \,\&\, q))$	2, 4, MP
6.	$\vdash \Box(q \to (p \,\&\, q)) \to (\Box p \to \Box(p \,\&\, q))$	3, Subst.
7.	$\vdash \Box p \to (\Box q \to \Box(p \,\&\, q))$	5, 6, "PL"
8.	$\vdash (\Box p \,\&\, \Box q) \to \Box(p \,\&\, q)$	7, "PL"

- This **K** "proof" [2, page 35] is missing many steps, and it contains the connective "&", which we are not using in our Otter representation. **Exercise** (hard!): represent "$\vdash (\Box p \,\&\, \Box q) \to \Box(p \,\&\, q)$" in our Otter notation, and try to find a *proof* of it in our system **K** above (44 steps?).
- **Note**: almost all of the complexity is in the *non*-modal *PL reasoning*.

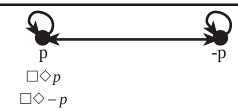
Finding Matrix Models for Sentential Modal Logics: S5 & G

i	0	1	2	3
0	3	3	3	3
1	2	3	2	3
2	1	1	3	3
*3	0	1	2	3

x	0	1	2	*3
L(x)	0	0	0	3

x	0	1	2	*3
-x	3	2	1	0

2-element S5 Kripke model in which G fails:



- McKinsey's axiom G cannot be expressed [2] as a (first order) constraint on the accessibility relation *R* in Kripke frames (∴ using Kripke translations to find such models automatically will not work, see below).
- I found the logical matrices above using John Slaney's special purpose matrix finder for implicational logics MaGIC [16], and verified them with Bill McCune's general first order model finder Mace [12] (a companion to Otter, which takes Otter input). I found the Kripke model by hand.

Automated Reasoning with Kripke Translations

- Many (but not all: *e.g.*, G and Löb) interesting modal formulae correspond to first-order conditions on relations *R* in Kripke frames:

D. $\Box p \to \Diamond p$	*R* is serial. $[(\forall x)(\exists y)Rxy]$
T. $\Box p \to p$	*R* is reflexive. $[(\forall x)Rxx]$
B. $p \to \Box\Diamond p$	*R* is symmetric. $[(\forall x)(\forall y)(Rxy \to Ryx)]$
4. $\Box p \to \Box\Box p$	*R* is transitive. $[(\forall x)(\forall y)(\forall z)((Rxy \,\&\, Ryz) \to Rxz)]$
5. $\Diamond p \to \Box\Diamond p$	*R* is euclidean. $[(\forall x)(\forall y)(\forall z)((Rxy \,\&\, Rxz) \to Ryz)]$

- These correspondences can allow us to (automatically) find proofs and countermodels more easily than with "pure" axiomatic techniques.
- **Exercises**: (1) Prove that all serial, symmetric, euclidean *R*s are reflexive and transitive. (2) Prove that some serial, euclidean *R*s are not transitive. Then, prove the *syntactic analogues* of (1) and (2), *i.e.*, (1′) prove **KDB5** ⊢ T, and **KDB5** ⊢ 4; and, (2′) show **KD5** ⊬ 4 — using logical *matrices*. (1′) and (2′) are much harder.

Application of Axiomatic Methods to Interpretability Logics I

- Interpretability logics (see [18] and [10]) are propositional modal logics with an additional, binary modal operator "▷" ("`I`" in `Otter`).
- The basic system **IL** is **K4** + the following axioms (Otter notation):

  `Löb. P(i(L(i(L(x),x)),L(x))).`

  J_1. `P(i(L(i(x,y)),I(x,y))).`

  J_2. `P(i(n(i(I(x,y),n(I(y,z)))),I(x,z))).`

  J_3. `P(i(n(i(I(x,y),n(I(z,y)))),I(i(n(x),z),y))).`

  J_4. `P(i(I(x,y),I(n(L(n(x))),n(L(n(y)))))).`

  J_5. `P(I(n(L(n(x))),x)).`

- Other formulas of interest in this context include:

  `P. P(i(I(x,y),L(I(x,y)))).`

  `M. P(i(I(x,y),I(n(i(x,n(L(z)))),n(i(y,n(L(z))))))).`

  `W. P(i(I(x,y),I(x,n(i(y,n(L(n(x)))))))).`

  P_0. `P(i(I(x,n(L(n(y)))),L(I(x,y)))).`

  M_0. `P(i(I(x,y),I(n(i(n(L(n(x))),n(L(z)))),n(i(y,n(L(z))))))).`

Application of Axiomatic Methods to Interpretability Logics II

- The Kripke semantics for interpretability logics is much less tractable (from a first order perspective) than it was for "normal" modal logics.
- So, we are pressured to use *axiomatic* methods of (automated) proof and model finding. Here, OTTER, MAGIC, and MACE can be very useful.
- The following can be shown pretty easily, using OTTER and MAGIC.

(i) **IL** ⊢ `I(i(n(x),n(L(n(x)))),x)` (ii) **IL** ⊬ `P`

(iii) **IL** ⊬ `M` (iv) **IL** ⊬ `W`

- Difficult problems (not yet solved with automated reasoning): (vii) Axiom 4 is dependent in **IL** (known), (viii) Some pair of {W, P_0, M_0} implies the third, in **IL** (OPEN), (ix) **IL** ⊬ P_0 (known), (x) **IL** ⊬ M_0 (known).

(C) `i(I(x,n(L(n(y)))),L(I(x,n(L(n(y)))))).`

(xi) **ILP** ⊢ C, and (xii) **ILM** ⊢ C (known). See [18] for more problems.

Some Other Approaches & Some More Challenge Problems

- Automated theorem proving (and model finding) for modal logics have been studied extensively in the last few decades ([9], [1], [19]).
- Typically, the focus has been on *special-purpose* provers and finders. Such systems essentially "hard code" the structures of particular logics.
- While this may lead to faster programs, it sacrifices *generality*. We'd like to see more work done on making *general purpose* techniques effective.

- Two more (known) ⊬ problems. Find logical matrices which establish that (a) **KTB** ⊬ 5 or (b) **KTB** ⊬ 4 (there are 3-element kripke models).
- More (known) ⊢ problems in **IL**+. Show any of the following: (c) **ILM** ⊢ W, (d) **ILP** ⊢ W, (e) **ILM** ⊢ P_0, (f) **ILP** ⊢ P_0, (g) **ILM** ⊢ M_0, (h) **ILP** ⊢ M_0.
- One more reference. See [3] for a general survey of propositional logics.
- See `http://philosophy.wisc.edu/fitelson/modal.htm` for files, *etc*.

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