

A DECISION PROCEDURE FOR PROBABILITY CALCULUS WITH APPLICATIONS^a

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^aThe paper behind this talk was recently published in the *Review of Symbolic Logic* [14].

Overview of Today's Talk

- Motivation: The Problem(s)
 - Bayesian confirmation theory (overview)
 - The Problem of Measure-Sensitivity
 - Lots of questions about the validity of Bayesian arguments
 - Leading to lots of problems in the probability calculus
 - Solutions needed!
- The Solution: PrSAT – A decision procedure for Probability Calculus
 - Probability calculus: axiomatic and algebraic approaches
 - Algebra, quantifier elimination, and the probability calculus
 - Implementation of PrSAT in *Mathematica*
 - Demonstration of PrSAT on some examples
 - Future Work: optimizing and scaling-up PrSAT

Some Bayesian Background I

- Orthodox Bayesianism (*i.e.*, Bayesian *epistemology*) assumes that the degrees of belief (or credence) of rational agents are (*Kolmogorov* [25]) *probabilities*.
- $\Pr(H | K)$ is the degree of belief that a rational agent with background knowledge K (S_K) assigns to H . This is S_K 's *prior* probability of H .
- $\Pr(H | E \& K)$ is the degree of belief S_K assigns to H , *on the supposition that E* (*i.e.*, the d.o.b. that S_K would assign to H upon learning E). This is S_K 's *posterior* probability of H (on E). I'll drop the " K "s now, for simplicity.
- A simple toy example (just to help fix our ideas): Let H be the hypothesis that a card (drawn at random from a standard 52-card deck) is a spade, and let E be the (evidential) proposition that the card is the ace of spades.
- Given standard assumptions about random card draws, $\Pr(H) = 1/4$ and $\Pr(H | E) = 1$. So, learning E raises the probability of (indeed, *verifies*) H .

Some Bayesian Background II

- In (contemporary) Bayesian confirmation theory, evidence E *confirms* (or *supports*) a hypothesis H if learning E *raises the probability of H* .
- If learning E *lowers* the probability of H , then E *disconfirms* (or *counter-supports*) H , and if learning E *does not change* the probability of H , then E is confirmationally *neutral* regarding H . This is a *Pr-relevance* theory.
- Within (*Kolmogorov!* [10], [13]) probability theory, there are many *logically equivalent* ways of saying that E confirms H . Here are a few:
 - E confirms H if $\Pr(H | E) > \Pr(H)$.
 - E confirms H if $\Pr(E | H) > \Pr(E | \sim H)$.
 - E confirms H if $\Pr(H | E) > \Pr(H | \sim E)$.
- By taking differences, (log-)ratios, *etc.*, of the left and right sides of these (or other equivalent) inequalities, a *plethora* of candidate *relevance measures of degree of confirmation* can be formed. Relevance measures are such that:

$$(\mathcal{R}) \ c(H, E) \leq 0 \text{ iff } \Pr(H | E) \leq \Pr(H).$$

Four Popular and Representative Relevance Measures

- *Dozens* of Bayesian relevance measures have been proposed in the philosophical literature (see [26] for a survey). Here are four popular ones.^a
 - *Difference*: $d(H, E) =_{df} \Pr(H | E) - \Pr(H)$
 - *Log-Ratio*: $r(H, E) =_{df} \log \left[\frac{\Pr(H | E)}{\Pr(H)} \right]$
 - *Log-Likelihood-Ratio*: $l(H, E) =_{df} \log \left[\frac{\Pr(E | H)}{\Pr(E | \sim H)} \right]$
 - “Normalized Difference”: $s(H, E) =_{df} \Pr(H | E) - \Pr(H | \sim E) = \frac{1}{\Pr(\sim E)} \cdot d(H, E)$
- Logs are taken to ensure easy satisfaction of relevance criterion (\mathcal{R}). They are merely a useful convention (they’re inessential, but they simplify things).
- The first part of our story concerns the *disagreement* exhibited by these measures, and its ramifications for Bayesian confirmation theory ...

^aUsers of d include [9], [8], and [22]. Users of r include [19], [27], and [20]. Users of l include [17], [32], and [12]. Users of s include [23] and [5]. See [10], [11], and [12] for further references.

Disagreement Between Alternative Relevance Measures

- What kind of disagreement between relevance measures is important?
- Mere *numerical* differences between measures are not important, since they need not affect *ordinal* judgments of what is more/less well confirmed than what (by what). Think about C vs F temperatures (numbers vs comparisons).
- *Ordinal* differences are crucial. Such *comparative* differences affect the cogency of many arguments surrounding Bayesian confirmation theory.
- For instance, it is part of Bayesian Lore that the observation of a black raven (E_1) confirms the hypothesis (H) that all ravens are black *more strongly than* the observation of a red herring (E_2) does (given “actual corpus” K).
- Given the standard background assumptions (K) in Bayesian accounts of Hempel’s ravens paradox, this conclusion [$c(H, E_1) > c(H, E_2)$] follows for *some* measures of confirmation c , but it *fails to follow* for some choices of c .
- Such arguments are said to be *sensitive to choice of measure* [11].

Tabular Summary Some Measure-Sensitive Arguments

Argument	Valid wrt relevance measure:			
	$d?$	$r?$	$l?$	$s?$
Horwich [19] <i>et al.</i> on Hempel’s Ravens Paradox	YES	YES	YES	No
Eells [9] on Goodman’s “Grue” Paradox	YES	No	No	YES
Sober [33] on Goodman’s “Grue” Paradox	YES	No	YES	YES
Rosenkrantz [30] on Irrelevant Conjunction	YES	No	No	YES
Earman [8] on Irrelevant Conjunction	YES	No	YES	YES
Horwich [19] <i>et al.</i> on the Variety of Evidence	YES	YES	YES	No
Christensen [5] on the Old Evidence Problem	No	No	YES	YES
Popper-Miller’s [29], [16] <i>Critique</i> of Bayesianism	YES	No	No	YES

Axiomatic Probability Calculus

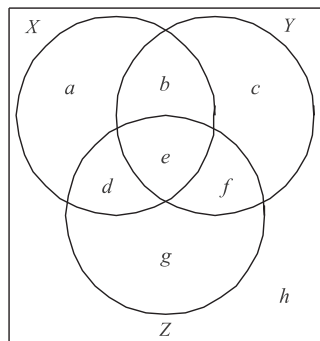
- All well-known problems in Bayesian confirmation theory can be represented in small, finite Kolmogorov probability models $\mathcal{M} = \langle \mathcal{B}, \Pr \rangle$.
- Such \mathcal{M} ’s are just small, finite Boolean algebras \mathcal{B} (of propositions), with a function $\Pr: \mathcal{B} \mapsto [0, 1]$ satisfying the following three axioms [25]:
 1. For all $X \in \mathcal{B}$, $\Pr(X) \geq 0$.
 2. $\Pr(\top) = 1$, where \top is any tautological proposition in \mathcal{B} .
 3. For all $X, Y \in \mathcal{B}$, if X and Y are mutually exclusive, then:

$$\Pr(X \vee Y) = \Pr(X) + \Pr(Y)$$
- Conditional probabilities $\Pr(\cdot | \cdot)$ are then *defined* in terms of $\Pr(\cdot)$ as:

Definition. $\Pr(X | Y) = \frac{\Pr(X \& Y)}{\Pr(Y)}$
- Thus, \Pr is a finitely additive measure over some \mathcal{B} of propositions. This allows \Pr ’s to be interpreted (roughly) as “areas” in Venn Diagrams ...

Algebraic Probability Calculus I

- Almost all problems in Bayesian confirmation theory are expressible in Kolmogorov probability models with just *three* atomic propositions.
- Such \mathcal{M} 's can be interpreted using Venn Diagrams ([28], [1]) in a simple way. For the remainder of the talk, I will focus on the 3-proposition case.



$$h = 1 - (a + b + c + d + e + f + g)$$

$$\Pr(X) = a + b + d + e$$

$$\Pr(Y) = b + c + e + f$$

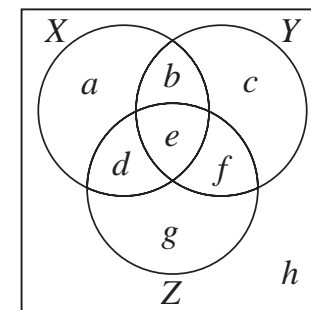
$$\Pr(Z) = d + e + f + g$$

$$\Pr(X|Y) = \frac{\Pr(X \& Y)}{\Pr(Y)} = \frac{b + e}{b + c + e + f}$$

$$\Pr(Y|\sim Z) = \frac{\Pr(Y \& \sim Z)}{\Pr(\sim Z)} = \frac{b + c}{1 - (d + e + f + g)}$$

- Using this technique, we can translate any equation, inequation, or inequality in (3-event) probability calculus into a simple algebraic formula in terms of the 7 variables $a, b, c, d, e, f,$ and g .
- Moreover, a probability function over such a 3-event space is simply an assignment of real numbers on $[0, 1]$ to a, \dots, h such that $a + \dots + h = 1$.
- I prefer *stochastic truth tables* to Venn Diagrams for representing probability models (easier to generalize to $n > 3$). Example:

X	Y	Z	States	$\Pr(s_i)$
T	T	T	s_1	$\Pr(s_1) = e$
T	T	F	s_2	$\Pr(s_2) = b$
T	F	T	s_3	$\Pr(s_3) = d$
T	F	F	s_4	$\Pr(s_4) = a$
F	T	T	s_5	$\Pr(s_5) = f$
F	T	F	s_6	$\Pr(s_6) = c$
F	F	T	s_7	$\Pr(s_7) = g$
F	F	F	s_8	$\Pr(s_8) = h$



Algebraic Probability Calculus II

- The class of questions I've been discussing are of the following form:
 - Do there exist real numbers a, \dots, g satisfying all members of a set of n simple algebraic formulae $\mathcal{S} = \{P_1(a, \dots, g), \dots, P_n(a, \dots, g)\}$?
 - In other words, we're interested in claims of the following form:

$$(\exists a \in \mathbb{R}) \cdots (\exists g \in \mathbb{R}) [P_1(a, \dots, g) \& \cdots \& P_n(a, \dots, g)]$$
 where the P_i are simple algebraic statements in terms of a, \dots, g .
 - The first eight statements in \mathcal{S} will state that a, \dots, h are on $[0, 1]$, and that they sum to 1 — *i.e.*, that the a, \dots, h are *basic probabilities*.
- Example. Are there $a, \dots, h \in (0, 1)$ such that $a + \dots + h = 1$, and
 1. $\Pr(X \& Y) = \Pr(X) \cdot \Pr(Y)$ [$b + e = (a + b + d + e) \cdot (b + c + e + f)$],
 2. $\Pr(X \& Z) = \Pr(X) \cdot \Pr(Z)$ [$d + e = (a + b + d + e) \cdot (d + e + f + g)$],
 3. $\Pr(Y \& Z) = \Pr(Y) \cdot \Pr(Z)$ [$e + f = (b + c + e + f) \cdot (d + e + f + g)$],
 4. $\Pr(X \& (Y \& Z)) \neq \Pr(X) \cdot \Pr(Y \& Z)$ [$e \neq (a + b + d + e) \cdot (e + f)$]?]

Algebraic Probability Calculus III

- Surprising (less trivial) Example: Do all relevance measures c satisfy (\dagger)?

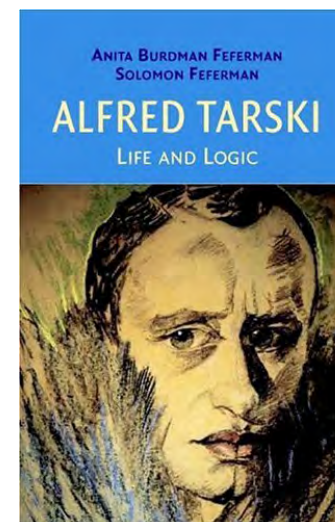
$$(\dagger) \quad \Pr(H|E_1) \geq \Pr(H|E_2) \implies c(H, E_1) \geq c(H, E_2).$$
- Bayesians have *assumed* that (\dagger) holds for all four of our c 's (and many others). But, using PrSAT (see below), we have shown this to be false.
- How can we express (\dagger) algebraically (let $X = H, Y = E_1,$ and $Z = E_2$)?
 - First, translate the left hand side of (\dagger):

$$(1) \quad \frac{b + e}{b + c + e + f} \geq \frac{d + e}{d + e + f + g}$$
 - Second, pick a measure c , and translate the right side (here, $c = s$):

$$(2) \quad \frac{b + e}{b + c + e + f} - \frac{a + d}{1 - b - c - e - f} \geq \frac{d + e}{d + e + f + g} - \frac{a + b}{1 - d - e - f - g}$$
- Question: Do there exist $a, \dots, h \in (0, 1)$ which sum to one and which make (1) true and (2) false? This is the algebraic equivalent of asking whether (\dagger) is false (for $c = s$) in some 3-event model \mathcal{M} . Answer: YES!

Algebraic Probability Calculus IV

- Problems like ours, on the satisfiability of sets of statements in probability calculus are expressible in the *theory of real-closed fields* (TRCF).
- A *real-closed field* (RCF) is a structure $\langle F, 0, 1, +, -, *, ^{-1}, < \rangle$ such that:
 1. $\langle F, 0, 1, +, -, *, ^{-1} \rangle$ is a field.
 2. (a) $x \not< x$
 (b) $x < y \ \& \ y < z \Rightarrow x < z$
 (c) $x < y \Rightarrow x + z < y + z$
 (d) $x, y > 0 \Rightarrow x * y > 0$ $[x > 0 \text{ iff } 0 < x]$
 (e) $x > 0 \vee x = 0 \vee 0 > x$
 3. Every positive element of F has a square root in F and every odd degree polynomial in $F[x]$ has a root in F . [The set \mathbb{R} forms an RCF.]
- TRCF is overkill for us. We only need the fragment of TRCF that involves *existentially quantified, simple algebraic claims over \mathbb{R}* .



A Decision Procedure for the Probability Calculus I

- Tarski [35] described a decision procedure for the theory of real-closed fields. Thus, in principle, Tarski's method gives us a way to determine whether an arbitrary argument in probability calculus is valid.
- Tarski's idea is called *elimination of quantifiers*. He showed that a formula $(\exists x)P(x, a)$, where $P(x, a)$ is quantifier-free, is equivalent to a quantifier free formula $Q(a)$. [a can stand for several variables a_1, \dots, a_n]
- Simple Example: $P(x, a)$ might have the form $f(x, a) = 0 \ \& \ g(x, a) = 0$, where f and g are polynomials, so we are asking for the condition(s) on a under which the polynomials f and g have a common root.
- It is a classical result of algebra that there is a polynomial $Q(a)$, called the *resultant* of f and g , which vanishes exactly when f and g have a common zero. Tarski's method is a (vast) generalization of this result.
- Tarski not only showed Q exists, he showed how to *compute* Q from P .

- Applying this procedure again and again, we can strip off one quantifier after another (from the inside out), eliminating all the quantifiers in a formula with nested quantifiers (note: we can write \forall as $\sim \exists \sim$).
- The crux of the matter, then, is the elimination of a single (\exists) quantifier.
- This is a piece of algebra whose historical roots go back to Sturm's theorem, which counts the number of roots of a polynomial in an interval in terms of the alternations of signs in the coefficients.
- I won't get into the details of Tarski's method. But, I will say something about its complexity. Unfortunately, it is *very* complex. Its complexity cannot be bounded by any tower of exponentials – see [7] for discussion.
- Even for our simple class of problems (strings of 7 \exists 's binding 7 variables), Tarski's method is (in general) not really feasible.
- The intuitive reason why Tarski's method is so inefficient is that it eliminates “one quantifier at a time,” and the formula “expands doubly-exponentially” each time a quantifier is eliminated.

A Decision Procedure for the Probability Calculus II

- George Collins invented an improved quantifier-elimination method known as *cylindric algebraic decomposition* (CAD) [6].
- The method decomposes a set defined in Tarski's language into a disjoint union of finitely many cells [a CAD], such that the polynomials involved in the definition of the original set do not change sign on any cell.
- Geometrically, \exists -quantification corresponds to projection onto a subspace with fewer variables. The projections of a set defined by a CAD are also defined by a CAD, as is the complement of a set defined by a CAD.
- Collins's CAD algorithm is "only" double exponential in the # of variables (polynomial in #/degree of polynomials, bit length of coefficients, and # of atomic formulas). This is the *lower bound* on the complexity of quantifier elimination in the (*general*) TRCF [7].
- Unlike Tarski's method, Collins's CAD method in some sense "eliminates all the quantifiers at once." This is why it is so much more efficient.

- Hong [18] improved further on Collins's CAD [*partial CAD* qepcad].
- This program has subsequently been further improved upon by many others, and is now publicly available on the Web [3] (linux/unix/PC).
[<http://www.cs.usna.edu/~wcbrown/research/qebycad/Tutorial/Tutorial.html> is a nice tutorial on quantifier elimination algorithms, including CAD.]
- Some of qepcad's functionality was implemented (by Strzebonski [34]) in *Mathematica* 4.1 (Experimental), and has since become part of the main Kernel in versions 5+. It's pretty good on our class of problems.
- *Mathematica* now contains a suite of CAD functions, including `CylindricalDecomposition` (which computes CADs), `Resolve` (which eliminates quantifiers using CAD), and `FindInstance` (which finds assignments to variables that satisfy a set of formulae in TRCF).
- In light of these developments (& Moore's Law), Michael Beeson [2] remarks "there is some hope of solving interesting, even open, problems that are too hard for humans, before the exponential behavior of the algorithm takes its toll."

A Decision Procedure for the Probability Calculus III

- This last function `FindInstance` is particularly useful for our purposes, since it is designed, specifically, for settling pure \exists -questions, like ours.
- `FindInstance` takes as its argument a finite set \mathcal{S} of equations, inequations, inequalities (in TRCF) over a finite set of real variables \mathcal{V} .
- `FindInstance` outputs an assignment of real numbers (if one exists) to the variables in \mathcal{V} , which satisfies all of the members of \mathcal{S} . If \mathcal{S} is unsatisfiable, then `FindInstance` outputs "{ }".
- Using the translation procedure above (and the `FindInstance` function), I developed a *Mathematica* function `PrSAT`, which takes as input a set \mathcal{S} of equations, inequations, or inequalities in probability calculus.
- If \mathcal{S} is satisfiable (in TRCF), then `PrSAT` returns a probability model satisfying all the members of \mathcal{S} . If not, `PrSAT` returns "{ }".

A Decision Procedure for the Probability Calculus IV

- As you might imagine, `PrSAT` is quite useful for a practicing Bayesian confirmation theorist! [I used an early prototype in my dissertation [12].]
- `PrSAT` is particularly useful for finding models of complicated sets of probabilistic equations, inequations, and inequalities. While `PrSAT` does not generate *proofs*, it does *verify* theoremhood (which is also useful!).
- With J. Alexander's and B. Blum's help, `PrSAT` is now a *Mathematica package* [15], with additional functionality (*e.g.*, a random search algorithm — see below).
- **Demo.** The first example is quite well-known ([31], p. 85). Problem: show that X can be independent of each of Y and Z , but *dependent* on $Y \& Z$.
- The second example is the "surprising" one I mentioned above. The model I found using `PrSAT` came as a shock to Bayesians [24, *fn.* 11].
- `PrSAT` can now easily decide all questions mentioned here. I use it to teach, and to find new results (some mentioned in my HPS talk this morning).

A Decision Procedure for the Probability Calculus V

- PrSAT is highly effective for problems involving ≤ 3 propositions (I've not seen any problems of this size that I couldn't solve w/PrSAT).
- PrSAT becomes inefficient for larger spaces. But, Ben Blum has developed a random search add-on to PrSAT, which is included in the latest version. This algorithm finds models (generally) in *linear* time.
- Galen Huntington just finished a thesis [21] on decision procedures for the \exists -fragment of TRCF. There are single-exponential algorithms for \exists TRCF, and Galen is (as far as we know) the first to fully implement one.
- At some point, I need to port my own *Mathematica* code (which is interpreted and slow), and merge it with Galen's (Haskell) code.
- My interest in such methods is largely *instrumental*. It grew out of a desire to be able to reconstruct, understand, and improve upon arguments involving the probability calculus that appear in the PoS literature.
- I think this is a promising area for collaborative research in several fields.

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