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Smiley's Distinction Between Rules of Inference and Rules of Proof

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0 PREAMBLE

Since my days as an undergraduate and then a graduate student in England in the period 1968–1974 I have been an appreciative consumer of Timothy Smiley's work, though the first time I heard him referred to as Tim (or had a chance to meet him personally) was on the occasion of an Australasian Association for Logic conference organized by Graham Priest at the University of Western Australia in 1983. At that conference, I presented a version of the paper abstracted as Humberstone (1984), on the subject of the unique characterization of connectives, which is intimately connected to a theme of Smiley (1962), pursued also from a somewhat different angle in another classic paper of the same vintage, Belnap (1962). For a BPhil thesis at the University of York, I worked on a theme from Smiley (1963), trying to see how far his idea could be taken, of reducing the number of modal operators (broadly understood) to one—which we could loosely think of as expressing a kind of absolute necessity—in terms of which other modal notions could be processed as forms of relative necessity (absolute necessity given this or that statement, formally represented by a sentential constant). A descendant of this work appeared as Humberstone (1981), fixing a problem (pointed out by Kit Fine) in my early efforts with the aid of a suggestion from Dana Scott (my supervisor for yet another BPhil thesis, this time at Oxford); related and subsequent developments are surveyed in Humberstone (2004). Some ideas from Smiley (1996) on negation and rejection are taken up in Humberstone (2000*b*). Footnote 1 of Smiley (1962), concerning substitution and replacement, inspired much of Humberstone (in preparation).

The topic for what follows—a contrast between two kinds of rules, especially as rules appear in the axiomatic approach to logic—was (to my knowledge) first aired in Smiley (1963). While it has occasionally been alluded to in subsequent years, it has never received the sustained attention it deserves. Here I express my appreciation for Smiley's rewarding logical work by giving it at least some of that attention.

1 SMILEY ON RULES AND THE DEDUCTION THEOREM

For a proof system in the axiomatic or 'Hilbert' style for a given logic, in which certain formulas are laid down as axioms and certain rules are given for deriving theorems from the given axioms, an ancillary consequence relation is defined,[1] often called *deducibility* and associated with the Deduction Theorem. In the end, 'deducibility' may in the end turn out to be the wrong term (see Note 20 below). In any case, the direction of the relation is wrong here: we should say, more accurately, that the consequence relation in question holds between a set of formulas and an individual formula when the latter is deducible from the former. In the simplest case, illustrated here for the implicational fragment of intuitionistic propositional logic, one takes as axioms all formulas instantiating the following schemata.

(A1)	$A \longrightarrow (B \longrightarrow A)$
(A2)	$(A \longrightarrow (B \longrightarrow C)) \longrightarrow ((A \longrightarrow B) \longrightarrow (A \longrightarrow C))$

and as the sole rule, Modus Ponens: A, $A \longrightarrow B/B$. (Premises before the slash, conclusion after it.) One then defines the consequence relation alluded to above, $\vdash_{IL}$, say, thus: $\Gamma \vdash_{IL} B$ if and only if B can be obtained by a series of applications of the rule Modus Ponens starting from formulas which are either axioms or elements of Γ. Finally, then the Deduction Theorem for $\vdash_{IL}$ says that for any set of formulas Γ and any formulas A, B:

$$\textit{If } \Gamma, A \vdash_{IL} B \textit{ then } \Gamma \vdash_{IL} A \to B$$

This result (originally due to Herbrand and Tarski) is proved by induction on the number of applications of Modus Ponens used to obtain B from $\Gamma \cup \{A\}$ together with the axioms.[2] Strangely enough (A2), though originally used as an axiom by Frege well before the Deduction Theorem was explicitly contemplated (in Frege 1879), seems exactly tailor-made for enabling the inductive step of this proof to go through.

If we abbreviate '$\varnothing \vdash_{IL} B$' to '$\vdash_{IL} B$' (as suggested in Note 1), then the above definition of '$\Gamma \vdash_{IL} B$' has the expected effect that '$\vdash_{IL} B$' amounts to the claim that B is *provable* on the basis of the above axiomatization, and indeed can be taken as defining provability, since it asserts that B can be obtained by some number of applications of Modus Ponens from the axioms (there being nothing on the left of the '$\vdash$' to appeal to). On the other hand, if we first define provability in these terms, we might go on to characterize a consequence relation, $\vdash_{IL}^{*}$, say, in the following terms, in which 'theorem' is used in place of 'provable formula' for brevity: $\Gamma \vdash_{IL}^{*} B$ if and only if B can be obtained by a series of applications of the rule Modus Ponens starting from formulas which are either theorems or elements of Γ. This differs from the earlier definition only in that the word 'axioms' has been replaced by the word 'theorems'. One easily sees that for all formulas B and sets of formulas Γ:

$\Gamma \vdash_{IL} B$ and only if $\Gamma \vdash_{IL}^{*} B$,

so the above formulation of the Deduction Theorem (for this system) could equally well be formulated with '$\vdash_{IL}^{*}$' in place of '$\vdash_{IL}$'.

Now, as is well known, the presence of additional logical vocabulary (such as connectives $\wedge, \vee, \neg$) in the language, or of additional axioms—for instance giving the vocabulary concerned its intuitionistic properties, or giving that vocabulary ($\longrightarrow$ included) the properties associated with it in classical logic—does not present any obstacle to conducting the proof of the Deduction Theorem for the richer logic. We will soon be considering the addition of a modal operator $\Box$ to the language. But when it comes to adding further *rules* some care is needed, as Smiley (1963) observed. Alternatively, since for many purposes it is convenient to consider axiom schemata such as (A1) and (A2) above as 0-premise rules, one could put this point in terms of adding further proper rules, where a proper rule is understood as an *n*-premise rule with $n \geq 1$. (On this way of speaking, an axiom schema is a 0-premise sequential rule, with sequentiality as defined in the following section.) For what follows, however, it is more convenient to exclude 0-premise rules and the word 'proper' is occasionally used for emphasis only.

Where we have an axiomatization using (proper) rules in addition to

Modus Ponens the question arises as to what it would be for the Deduction Theorem to be satisfied. One way of responding would be to keep the definition exactly as above, and say the Deduction Theorem for the envisaged system requires that when B can be obtained by applications of Modus Ponens from the axioms and the formulas in some set $\Gamma \cup \{A\}$, then $A \longrightarrow B$ can be similarly obtained from the axioms and the formulas in Γ. A second reaction would be to focus not on the rule Modus Ponens itself in the original case but on the fact that this rule exhausted the set of proper rules used in the axiomatization, which gives rise to a second abstraction from the original case. Now we say that Deduction Theorem holds for the envisaged extended system provided that whenever a formula B can be obtained by applications of *any of the primitive rules of the system* from the axioms and the formulas in some set $\Gamma \cup \{A\}$, then $A \longrightarrow B$ can be obtained, by applying some of those rules, from the axioms and the formulas in Γ. Let us put this another way. Say that a consequence relation $\vdash$ satisfies the Deduction Theorem just in case for all sets of formulas Γ and formulas A, B, in the language of $\vdash$, we have:

(DT)	*If* $\Gamma, A \vdash B$ *then* $\Gamma \vdash A \longrightarrow B$

For an axiomatic system S to satisfy the Deduction Theorem is for a certain consequence relation $\vdash$ associated with S to satisfy (DT); however, we still use the phrase 'Deduction Theorem' even when dealing with a consequence relation presented otherwise, as long as the condition (DT) is satisfied.[3] The question is how to pass from an axiomatic system to the 'associated' consequence relation of interest. According to the first of the two reactions just described, even when S has primitive rules other than Modus Ponens, the relevant consequence relation fixes the consequences of a set as the results of applying only Modus Ponens to theorems of the logic and formulas in the set. According to the second reaction, the crucial consequence relation is characterized instead by replacing the privileged position of Modus Ponens and allowing derivability using any of the primitive rules of S.

Smiley (1963:115) discusses the modal logic S2, axiomatized using various schemata and, alongside Modus Ponens (there called R1), the further rule, R2: $\Box(A \longrightarrow B)/\Box(\Box A \longrightarrow \Box B)$. In defining the associated consequence relation, Smiley allows only the use of Modus Ponens and not also of this modal rule, in the definition of what it is for a formula to be deducible from a set of formulas. He makes the following remarks, in which the phrase ‘for material implication’ is a reminder of the fact that the Deduction Theorem makes reference to the connective ‘$\longrightarrow$’ (actually notated differently in Smiley 1963, as ‘$\supset$’), and the phrase ‘both systems’ is occasioned by the fact that Smiley is discussing not only S2 but another system, called by him OS2, the details of which do not matter for present purposes.

An almost immediate consequence of these definitions is that the deduction theorem for material implication holds in both systems. There are asser tions in the literature that the deduction theorem fails for S2, but they are the result either of treating R2 as a straightforward rule of inference (as in Moh 1950, p. 61) or else of a mistake.[4] (Smiley 1963:115)

At first sight, this may seem to be nothing but an endorsement of the first of the two reactions distinguished above: we give a privileged position to Modus Ponens in fixing the consequence relation required to satisfy (DT). This would not be very satisfying as a general account, placing, as it does, so much emphasis on a particular rule. Nor is it in fact Smiley’s position, as the words ‘treating R2 as a straightforward rule of inference’ in the above quotation betray. So we have to back up a little in Smiley (1963) and see what this phrase means.

On the page before that on which the above passage occurs, Smiley writes as follows:

In formulating OS2 and S2 in this way it is intended that the rule R2 is not to be used unrestrictedly, but only in the generation of further theorems from theorems. In this it resembles the rule of substitution for propositional variables, the rule ‘from A infer $\Box A$’ in S4, or indeed the rule of generalisation in the predicate calculus. These might all be called ‘rules of proof’ as opposed to proper ‘rules of inference’ like R1.[5] (Smiley 1963:114)

As Smiley proceeds to explain, the difference between the two kinds of rules emerges when the deducibility relation—the consequence relation that is pertinent to the Deduction Theorem, that is—associated with the axiomatic system is considered. In his own words: ‘B is deducible from $A_1, \ldots, A_n$ if there is a sequence of formulae ending in B, of which every member either is one of A_1, …, A_n or else is a theorem or else follows from preceding formulae by R1'. Incorporating the rules of proof/rules of inference distinction to give a general formulation of the idea, we have an axiomatic system as a triple $\langle Ax, R, R_{inf}\rangle$ with Ax a set of formulas (the axioms) of some language, and R and R_{inf} two sets of rules whose premises and conclusions are formulas of that language, and with $R_{inf} \subseteq R$. The subscript on 'R_{inf}' is of course intended to suggest '(rules of) inference'; you may call the rules in R 'rules of proof', or alternatively reserve this label for, as it were, rules of proof *proper* ('mere rules of proof', we might equally well say): those in R but not in R_{inf}, or more formally, those in $R \backslash R_{inf}$. The former course is adopted here. For $S = \langle Ax, R, R_{inf}\rangle$ the theorems of S are the formulas derivable from Ax by means of the rules in R, while the deducibility consequence relation associated with S, $\vdash_S$, is defined by: $\Gamma \vdash_S B$ just in case B can be derived from theorems of S together with formulas in by the application of rules in R_{inf}. This is like the earlier definition of '$\overset{*}{\vdash}_{\mathbb{IL}}$' in that it appeals to an already available definition of theoremhood or provability. But if a formulation along the lines given for $\vdash_{\mathbb{IL}}$ above is preferred, it can proceed in the following terms: $\Gamma \vdash_S B$ if and only if B can be obtained from formulas in $\Gamma \cup Ax$ by applying the rules in R subject to the condition that when premises figure in an application of a rule in $R \setminus R_{inf}$, none of those premises should be an element of Γ or have been derived by earlier applications of the rules to a formula in Γ. As Schurz (1994:387) puts it: the rules of proof proper should be 'applied only to those members of the proof which do not depend on premises in Γ'.[6]

While the conceptualization of axiomatic systems along the $\langle Ax, R, R_{inf}\rangle$ lines here distilled from Smiley's discussion is occasionally seen in the literature—for example, we find it in Gabbay (1981:9), as the distinction between 'consequence rules' (rules of inference) and 'provability rules' (rules of proof)—by far the more prevalent approach, evidenced especially in one stream of Polish logical work, is quite different.[7] In this work, such a system is conceived of as a pair $\langle Ax, R\rangle$ and the induced consequence relation matches the second way of characterizing the Deduction Theorem described above: use of any of the rules in R, together with the formulas in Ax, is permitted for deriving consequences according to this consequence relation. We will take up the contrast between the Smiley-inspired and the more widely prevalent conception of axiomatization in the next two sections.

2 DERIVABLE/ADMISSIBLE: ANOTHER DISTINCTION

Anderson and Belnap (1975), discussing the addition of a modal operator ($\Box$) for necessity to the then favoured system **E** of entailment (which is the intended reading of '$\longrightarrow$' for the quotation below), address the rule of necessitation (A / $\Box A$) in the following terms:

In the first place we are led by a strong tradition to believe that the necessity of any theorem (of a formal system designed to handle the notion of logical necessity at all) should also be a theorem; unless this requirement is met, the system simply has no theory of its own logical necessities. For this reason we would like to have it be true that, whenever A is provable, then necessity of A is also provable. This condition could be satisfied by incoherent brute force, as it is for example in systems like **M** (Feys–von Wright), where a rule of necessitation is taken as primitive. It could equally well be satisfied by taking $A \longrightarrow \Box A$ as an axiom. Both courses are equally odious, the latter because it destroys the notion of necessity, and the former because, if $A \longrightarrow \Box A$ is neither true nor a theorem, then we ought not to have—in a coherent formal account of the matter—a primitive rule to the effect that $\Box A$ does after all follow from A. This constraint would not bother us if we were *simply* trying to define the set of theorems of **E** recursively in such a way that a digital computer, or some equally intelligent being, could grind them out. But our ambitions are greater than this; we would like to have our theorems and our primitive rules dovetail in such a way that if **E** says or fails to say something, *we* don't contradict it or violate its spirit. (Note that neither $\longrightarrow$E nor &I does so). Nevertheless it should be *true*, as a lucky accident, so to speak, that whenever A is a theorem, $\Box A$ is likewise.[8] (Anderson and Belnap 1975:235f)

It doesn't strictly make sense to say, as here, that '$A \longrightarrow \Box A$ is neither true nor a theorem', even when a particular formula A is specified, but presumably the authors intend the (informal) claim that if a statement does not in general entail its necessitation—see the start of §3—then, to pick up the quotation verbatim 'we ought not to have—in a coherent formal account of the matter—a primitive rule to the effect that $\Box A$ does after all follow from A.' Well, perhaps we ought not to have a rule of *inference* to that effect, since we are supposing that the necessitation of a statement cannot in general be inferred from that statement, but how is this an objection to taking necessitation as a rule of *proof*? Could this passage from Anderson and Belnap be rewritten so as to take account of Smiley's distinction? Would the principle behind it, given that distinction, be something along the following lines: one should not use rules of proof one does not endorse as rules of inference? This would mean a ban on the use of all the examples in Smiley's list from the previous section: not just Necessitation, but also Uniform Substitution, and generalization-like rules in predicate logic. These last raise numerous complications of their own (issues about free variables and individual parameters) which are best avoided. Indeed in Smiley's own more recent thoughts on the matter there is considerable distaste shown for doing propositional logic with propositional variables and a rule of Uniform Substitution.[9] Here we ignore this change of heart and proceed with some reminders as to the semantic side of these rules (though Uniform Substitution will tend to take a back seat in the discussion). This will be attended to in §3, which presumes a basic familiarity with the (Kripke) semantics for normal modal logics. Before that, there is more to say about the passage from Anderson and Belnap quoted above.

The concluding sentence of the quotation says that despite not wanting Necessitation as a primitive rule, 'Nevertheless it should be *true*, as a lucky accident, so to speak, that whenever A is a theorem, $\Box A$ is likewise.' In other words, the rule should be *admissible* (i.e. the set of theorems should be closed under application of the rule).[10] All the attention to whether the rule is primitive is completely beside the point: the contrast that counts here is between rules that are derivable (primitive or derived) on the one hand, and rules which are merely admissible (admissible but not derivable) on the other.[11] Anderson and Belnap's antipathy to the rule of Necessitation should be expressed not as an objection to its being primitive in an axiomatization, but to its being derivable at all. (We return briefly in §3 to the question of whether this objection is well-founded.)

Let us review the notion of derivability as it applies to the simple conception of axiomatizations as pairs $\langle Ax, R\rangle$. We can think of a rule as the set of all its applications and, for an n-premise rule, take these applications to be $(n + 1)$-tuples of formulas of the language under consideration, the first n positions occupied by premise-formulas and the final position by the conclusion-formula.[12] Such a rule is *derivable* on the basis of $\langle Ax, R\rangle$ just in case for any application $\langle B_1, \ldots, B_n, C\rangle$ of the rule, we can obtain C from $\{B_1, \ldots, B_n\} \cup Ax$ by means of the rules in R. (This is just to say that $B_1, \ldots, B_n \vdash C$, for the consequence relation determined by $\langle Ax, R\rangle$ in the manner described in §1.) What becomes of this notion on the Smiley conception of axiomatic systems in the form $\langle Ax, R, R_{inf}\rangle$, in which a distinguished subset of the rules are deemed to be rules of inference and not just rules of proof? A derivable rule of proof should be, as in the simpler set-up just reviewed, any rule derivable on the basis of the reduct $\langle Ax, R\rangle$ of $\langle Ax, R, R_{inf}\rangle$. One might think that correspondingly, a derivable rule of inference should be a rule derivable on the basis of the reduct which discards rules of proof proper, i.e. on the basis of $\langle Ax, R_{inf}\rangle$. The idea is that we want to allow chaining together of *inference* steps without any admixture of *proof* steps which are not sanctioned as licensing inferences (by the given partition of R into R_{inf} and $R \setminus R_{inf}$). But the proposed definition does not implement the idea correctly, since it excludes the use of rules of proof proper in yielding theorems from which, together with the premises of an application of the would-be derived rule of inference, yield the conclusion with the aid of primitive rules of inference. In this case, the rule of proof is not itself applied to those premises and should not count against the derivability of the envisaged rule. Accordingly, a correct definition of the derivability of a rule of inference ρ should instead have it that the conclusion of any application of ρ can be obtained from its premises by successive applications of rules in R_{inf} together with theorems provable on the basis of $\langle Ax, R, R_{inf}\rangle$. We could equally well say: provable on the basis of $\langle Ax, R\rangle$ since $R_{inf} \subseteq R$ (and the selection of a particular subset of R as R_{inf} has no bearing on the set of theorems—only on the induced consequence relation).

Before passing to a consideration of the semantic side of these matters, let us review and streamline the terminology with which we have been working. Let us describe an axiom system or axiomatization $\langle Ax, R\rangle$ of the simple-minded kind as representing the *undifferentiated* (more explicitly: 'rule-undifferentiated') approach to the subject, and a system $\langle Ax, R, R_{inf}\rangle$

in the Smiley-inspired style as representing the *differentiated* approach. Anything that can be done on the former approach can be done on the latter, since we allow the possibility that $R = R_{inf}$; but the latter is more flexible in the manner already described as to how to associate a consequence relation with an axiomatic system. Although one usually speaks of a logic (or more generally a theory), understood as a set of formulas, as being axiomatized by an axiomatic basis, whether of the undifferentiated $\langle Ax, R\rangle$ type or of the differentiated type, it does no harm to speak of what we have been calling the associated consequence relation in either case as itself axiomatized by the basis in question.[13] Reviewing the contrast, we say on the undifferentiated approach that the consequence relation $\vdash_{\langle Ax, R\rangle}$ *axiomatized by* $\langle Ax, R\rangle$ is the least consequence relation (on the language in question) $\vdash'$ such that:

(1)	$\vdash' B$ for all $B \in Ax$, and
(2)	$\Gamma \vdash' B$ whenever for some $A_1, \ldots, A_n \in \Gamma, \langle A_1, \ldots, A_n, B\rangle \in \rho$, for some $\rho \in R$,

while on the differentiated approach we say that the consequence relation $\vdash_{\langle Ax, R, R_{inf}\rangle}$ *axiomatized by* $\langle Ax, R, R_{inf}\rangle$ is the least consequence relation (on the language in question) $\vdash' B$ such that:

(1)*	$\vdash' B$ for all B such that $\vdash_{\langle Ax, R\rangle} B$, and
(2)*	$\Gamma \vdash' B$ whenever for some $A_1, \ldots, A_n \in \Gamma, \langle A_1, \ldots, A_n, B\rangle \in \rho$, for some $\rho \in R_{inf}$

Note that (1)* here invokes the definition provided by (1) and (2) for the $\langle Ax, R\rangle$ reduct of the given $\langle Ax, R, R_{inf}\rangle$. It provides the theorems of the logic axiomatized; compare the characterization of $\dot{\vdash}_{IL}$ in §1. (Occasionally below, the set of such theorems will be referred to as the 'formula logic' concerned, for contrast with logics conceived of as themselves being consequence relations.) An *n*-premise rule ρ is *derivable* on the basis of $\langle Ax, R\rangle$ when for any $\langle A_1, \ldots, A_n, B\rangle \in \rho$, we have $A_1, \ldots, A_n \vdash_{\langle Ax, R\rangle} B$. For the differentiated approach we have two notions, depending on whether the derivability of ρ as a rule of proof or the derivability of ρ as a rule of inference is at issue. In the former case, the definition is as before, using the reduct $\langle Ax, R\rangle$ of the axiomatization

$\langle Ax, R, R_{inf}\rangle$ in question: we simply require that $A_1, \ldots, A_n \vdash_{\langle Ax,\, R\rangle} B$, whenever $\langle A_1, \ldots, A_n, B\rangle \in \rho$. For the latter case, we have ρ derivable as a rule of inference provided that $A_1, \ldots, A_n \vdash_{\langle Ax,\, R,\, R_{inf}\rangle} B$ whenever $\langle A_1, \ldots, A_n, B\rangle \in \rho$.

Digression. Let us pause to consider the question of admissibility. Since the admissibility of a rule is a matter of the set of provable formulas being closed under the rule, one naturally associates admissibility with the 'rule of proof' side of the picture. Could separate sense be made of something's being admissible as a rule of inference on the differentiated approach? Well, an alternative (though equivalent) definition of the admissibility of ρ on the basis of the undifferentiated axiomatization $\langle Ax, R\rangle$ is that $\langle Ax, R\rangle$ and $\langle Ax, R \cup \{\rho\}\rangle$ have the same theorems. On the differentiated approach one says the same thing, with R_{inf} as an idle parameter: $\langle Ax, R, R_{inf}\rangle$ and $\langle Ax, R \cup \{\rho\}, R_{inf}\rangle$ have the same theorems. This suggests a definition for the admissibility of ρ on the basis of $\langle Ax, R, R_{inf}\rangle$, namely that $\langle Ax, R, R_{inf}\rangle$ and $\langle Ax, R \cup \{\rho\}, R_{inf} \cup \{\rho\}\rangle$ axiomatize the same consequence relation.[14] We could make such a definition, but it does not lead to anything of interest, since the admissible rules, so defined, would coincide with the derivable rules. ***End of Digression.***

The rules of interest here, still keeping Uniform Substitution to one side, are all of them *sequential* rules in the sense of Łoś and Suszko (1958), which means that for each rule, ρ, say, there is a sequence of formulas—sometimes called the *skeleton* of ρ—$\langle A_1, \ldots, A_n, B\rangle$ such that the applications of are precisely those $\langle C_1, \ldots, C_n, D\rangle$ for which there is some substitution s with $\langle C_1, \ldots, C_n, D\rangle = \langle s(A_1), \ldots, s(A_n), s(B)\rangle$. For example, Modus Ponens and Necessitation have skeletons $\langle p, p \to q, q\rangle$ and $\langle p, \Box p\rangle$, respectively.[15] Note that for an axiomatization on either the differentiated or the undifferentiated approach, even when all the primitive rules (those in R or just those in R_{inf}) are sequential, the derivable rules are typically not sequential, since the union of any two n-premise derivable rules is a derivable rule. However, the derivable rules will still in this case be substitution-invariant in the sense that every substitution instance of an application of the rule is an application of the rule.[16]

Even when the rules R are all sequential, so that it is effectively decidable whether or not a putative application of one of them is indeed such an application, the additional requirement that the set R, as well as the set Ax of axioms, should also be recursive, is often imposed to do justice to the idea that it should be effectively decidable whether or not a putative proof is a proof. An especially simple way of satisfying this demand arises from various notions of finiteness for an axiomatization. Say that a differentiated axiomatization $\langle Ax, R, R_{inf} \rangle$ is *finite* if the sets Ax and R (and therefore also R_{inf}) are finite, and that $\langle Ax, R, R_{inf} \rangle$ is *schematically finite* if R is a finite set of sequential rules and Ax is the set of all substitution instances of a finite subset Ax_0 of Ax. The idea in the latter case is that we avoid the use of uniform substitution by describing the (in general) infinite set Ax by means of finitely many axiom-schemata, arising from Ax_0 by replacing distinct propositional variables occurring in the formulas thereof by distinct schematic letters. Since these definitions appeal only to Ax and R, we may take them over intact for undifferentiated axiomatizations $\langle Ax, R \rangle$. The notion of schematically finite axiomatizations will be employed in the following section.

3 SEMANTIC CONSIDERATIONS

Although the distinction between rules of proof and rules of inference was introduced by Smiley in discussion of what one might naturally consider to be a purely syntactical matter, the Deduction Theorem, there is a clear semantical motivation revealed by the choice of terminology. Informally, this can be illustrated in the case of Necessitation by considering an argument with premise, 'John F. Kennedy was shot in Dallas', say, and conclusion 'It is (logically or metaphysically) necessary that John F. Kennedy was shot in Dallas'. This contrasts with an argument having as premises a conditional and its antecedent, and as conclusion its consequent. In the latter case we think of the conclusion as something which can legitimately be *inferred from* the premises, and in the former case of the conclusion as not being something to be inferred from the premise. This difference makes it natural to describe Modus Ponens as being—and Necessitation as not being—a rule of inference. It is equally natural to want to continue with a gloss on the difference which invokes some notion of truth-preservation as characterizing transitions in the former case and not in the latter. Indeed a main focus of interest in connection with rules has always been over differences in which semantic features they preserve,[17] and the preservation behavior of the two rules mentioned in the Kripke semantics for normal modal logics is well known: Modus Ponens is *locally* truth-preserving in any model (preserves truth at any point—or 'world'—in the model) while Necessitation is only *globally* truth-preserving (preserving the property of being truth-at-all-points in the model, for any model). Thus we can refine the connection between inference and truth-preservation implicit in Smiley's classification of Modus Ponens and Necessitation (sometimes abbreviated to 'Nec' below) as positive and negative instances, respectively, of the 'rule of inference' category, by saying that it is local truth-preservation that matters for the intuitive idea of inferrability. It is well known[18] that since the local/global distinction applies not only to preservation of truth but also to preservation of validity, where a formula is valid at a point in a (Kripke) frame if it is true at that point in every model on the frame, and valid on a frame if it is valid at every point therein, what we end up with here is a fourfold distinction. Uniform Substitution preserves validity at a point, and *a fortiori* validity on a frame, but unlike Modus Ponens it does not preserve truth at a point, and unlike Necessitation it does not preserve truth throughout a model. So, lacking the local truth-preservation characteristic, it too gets classified in the second passage quoted from Smiley in §1 as a rule of proof which should not be counted as a rule of inference. In what follows attention will be specifically on the truth-based rather than the validity-based incarnation of the local/global contrast. (Note also that this summary has ignored further preservation characteristics arising for models with a distinguished point or set of points used in modal actuality logic, various non-normal modal logics, etc.)

Such verdicts are made available, but are not made inevitable, by the differentiated approach to axiomatization. After all, given R as our set of primitive rules, we can select as R_{inf}, consistently with the differentiated approach, any subset of R, even if on the intuitive grounds just rehearsed, the rules in question would not naturally be called rules (or principles) of inference. For example, we may want to axiomatize (in the sense explained at the end of §2) the 'model consequence' relation of global truth-preservation, which we will denote by superscripting a turnstile with the first three letters of 'global', and subscripting it with '**K**', the name of the smallest normal modal logic (considered as a set of formulas), to indicate that no restriction is imposed as to the models of interest. That is, we define:

$\Gamma \vdash^{glo}_{\mathbf{K}}$ iff for every Kripke model $\mathcal{M}$, if every formula in Γ is true at all points in $\mathcal{M}$, then so is *B*.

(Since this is a semantic characterization of the consequence relation in question, some may prefer to see '$\models$' in place of '$\vdash$'.) A mild variation on the canonical model method (see e.g. Kracht 1999: Proposition 3.1.3) shows that $\vdash^{glo}_{\mathbf{K}}$ has a schematically finite axiomatizaton in the shape of $\langle Ax, R, R_{inf}\rangle$ with *Ax* the set of instances of any finite set of schemata which together with Modus Ponens (or MP for short) yield all truth-functional tautologies along with all instances of the **K**-schema:

$\Box(A \longrightarrow B) \longrightarrow (\Box A \longrightarrow \Box B)$

and $R = R_{inf} = \{\text{MP, Nec}\}$. We could also supply a straightforwardly finite axiomatization of the same consequence relation by taking as *Ax* a finite set of axioms which together with Modus Ponens and US (Uniform Substitution) suffice for the truth-functional tautologies, and $R = \{\text{MP, Nec, US}\}$, $R_{inf} = \{\text{MP, Nec}\}$.

The result(s) just given can be formulated in terms of rule-soundness and rule-completeness: the derivable rules of inference of the two axiomatizations described are all and only those rules which preserve truth throughout an arbitrary model.[19] If we took the second axiomatization and pushed US into R_{inf} we should have a correct rule-soundness statement for arbitrary frames: all derivable inference rules preserve validity on an arbitrary frame. But the corresponding rule-completeness statement in this case would not be correct, in view of the existence of Kripke-incomplete normal modal logics.

The consequence relation $\vdash^{glo}_{\mathbf{K}}$ does not satisfy the Deduction Theorem, of course, since for example $p \longrightarrow \Box p$ is not true throughout every model, even though $\Box p$ is true throughout every model throughout which p is true, reflecting the distance we have traveled here from Smiley's original conception of the rule of inference/rule of proof distinction. A *rapprochement* is possible, however, if we repackage the above points in a more sophisticated version of the differentiated approach. Instead of adding what were rules of proof proper (i.e. rules in $R \setminus R_{inf}$) into the 'rule of inference' compartment R_{inf}, we could drop the idea of a single consequence relation associated with ('axiomatized by') a rule-differentiated basis $\langle Ax, R, R_{inf} \rangle$, and instead distinguish the induced *inferential* consequence relation—formerly just the consequence relation thereby axiomatized—from the induced *probatory* consequence relation, defined similarly but allowing arbitrary rules from *R*, rather than just those in R_{inf}, when working out the consequences of a set of formulas.[20]

Then we could say, concerning the first (schematically finite) axiomatization described above, that it has as its probatory consequence relation precisely the model consequence relation $\vdash^{glo}_{\mathbf{K}}$. (This would not be the case for the second axiomatization, since US does not preserve truth throughout a model. In the alternative terminology, this would make for a failure of rule-soundness with respect to the class of all models.)

Whichever version, naive or sophisticated, of the differentiated approach is taken—and from now on we revert to the naive version—we should certainly be wary of the claim of Anderson and Belnap quoted in §2. According to these authors, we should not have 'a primitive rule to the effect that $\Box A$ does after all follow from A', since even if (as the sophisticated version allows us to say) $\Box A$ cannot be inferred from A, there is a perfectly good sense—given by global truth-preservation—in which $\Box A$ does indeed follow from A. That is, the truth of $\Box A$ *throughout a model* follows from the truth of A *throughout the model*. The situation in this respect is quite different from the well-known converse of Necessitation, Denecessitation, i.e. the sequential rule with skeleton $\langle \Box p, p \rangle$, under which the set of theorems of **K** is also closed.[21] (Of course Anderson and Belnap were not discussing **K**, but the present point shows the weakness of their suggestion that Necessitation itself should, 'in a coherent formal account of the matter', have no higher status than that of an admissible rule.) If we threw this rule in with Necessitation, it would destroy the soundness half of the rule-soundness and rule-completeness result mentioned, since the truth throughout a model of $\Box A$ does not imply that A has this same property.

To conclude this section we illustrate the convenience of the differentiated approach in connection with the local analogue of $\vdash^{glo}_{\mathbf{K}}$, which we naturally call $\vdash^{loc}_{\mathbf{K}}$. That is, we define:

$\Gamma \vdash^{loc}_{\mathbf{K}} B$ iff for every Kripke model $\mathcal{M}$, and any point in $\mathcal{M}$, if every formula in Γ is true at that point in $\mathcal{M}$, then so is B.

Note that unlike its global counterpart, with which it agrees for $\Gamma = \varnothing$, this consequence relation does satisfy the Deduction Theorem,[22] and it relates Γ to B just in case the conjunction of finitely many formulas in provably implies B in the formula logic **K**. We have the following striking contrast:

(1) $\vdash_{K}^{loc}$ has no schematically finite undifferentiated axiomatization, while

(2) $\vdash_{K}^{loc}$ does have a schematically finite differentiated axiomatization.

Part (2) of this assertion is clear enough, since we have, for example, $\langle Ax, R, R_{inf}\rangle$ with *Ax* as above (*apropos* of which the K schema was mentioned) and *R* comprising Modus Ponens and Necessitation, and R_{inf} consisting of just Modus Ponens. For (1) we give just a sketch of the proof. If we have $\langle Ax, R\rangle$ as a schematically finite axiomatization of $\vdash_{K}^{loc}$, with $R = \{\rho_1, \ldots, \rho_m\}$ (each ρ_i sequential), then in view of the superscripted '*loc*', each ρ_i must be locally truth-preserving (in every model). Where ρ_i has skeleton $\langle C_1, \ldots, C_n, D\rangle$, we can then add all the formulas

$$s(C_1) \longrightarrow (s(C_2) \longrightarrow \ldots \longrightarrow (s(C_n) \longrightarrow s(D) \ldots))$$

to *Ax*, for every substitution *s*, calling the result Ax^+. This amounts to adding all instances of a schema with distinct schematic letters replacing distinct propositional variables in the implication with successive antecedents the C_j and consequent *D*. So $\langle Ax^+, \{MP\}\rangle$ would also be a schematically finite axiomatization of $\vdash_{K}^{loc}$. But that would mean that this new basis would provide a schematically finite axiomatization of the formula logic **K**, and a minor adaptation of a proof in Lemmon (1965:302, Theorem 2)[23] shows that there is no such axiomatization with Modus Ponens as the sole proper rule.[24]

4 CLOSING COMMENTS

The prevalence of the undifferentiated approach, with its attendant conflation of rules of proof with rules of inference, has had some pernicious effects, the most recent of which is perhaps the interpretation of the term *normal* as it applies to consequence relations in modal logic. Since traditionally the main consequence relation associated with a modal logic has been the local (i.e. locally truth-preserving) consequence relation (as is remarked in the opening sentence of Kracht 1999: ch. 3), it has been customary to regard a consequence relation $\vdash$ on the language of modal logic as normal if whenever $\Gamma \vdash A$, we have $\Box\Gamma \vdash \Box A$ (where $\Box\Gamma$ is $\{\Box C : C \in \Gamma\}$). The corresponding sequent-to-sequent rule is sometimes called *Scott's rule* and Scott (1974) specifically inveighed against confusing the special case of this rule in which Γ is empty—a *vertical* transition, in his terminology (think of a rule display with sequent-premise(s) above and sequent conclusion below a line)—with the condition that for all formulas *A*, we should have $A \vdash \Box A$: *a horizontal* transition. Yet just this last condition is imposed in work by Blok and Pigozzi and others since 1989 in abstract algebraic logic (or 'AAL') and has come to be regarded as defining the normality of a consequence relation.[25] I take this contrast between the acceptable vertical transition and the unacceptable (in the local case) horizontal transition to be Smiley's point again: Necessitation is fine as a rule of proof but not as a rule of inference. Like everyone else on the undifferentiated side of the fence, the AAL community sees a (formula-to-formula) rule in the axiomatization of a logic and only knows one thing to do with it: use it to obtain consequences of arbitrary sets of formulas. Giving this inappropriate definition of normality (for consequence relations) has conspired with the fact that the local consequence relations concerned are not (in general) algebraizable by the standards of Blok and Pigozzi (1989) to make them very much second class citizens of the AAL world, by comparison with their global counterparts.[26] There is no reason to tie normality to the global side of the local/global division in this way.

Scott's horizontal/vertical contrast suggests a notation which would decorate the skeleton of a sequential rule with information as to the status of the premises of its applications (as in Humberstone 2008:443): a superscripted *downward* arrow indicates a vertical transition from that premise and a superscripted *rightward* arrow, a horizontal transition. Thus Necessitation as a rule of proof would have the decorated skeleton $\langle p^{\Downarrow}, \Box p\rangle$ while for Necessitation as a rule of inference (if one were interested in such a thing—as in the naive implementation of the differentiated strategy in §3) the skeleton would be $\langle p^{\Rightarrow}, \Box p\rangle$. When we consider rules with more than one premise, however, the binary division into rules of proof and rules of inference loses its apparent exhaustiveness, since the premise positions may be differently tagged. For example, suppose that * is a binary connective and consider the four possible decorations of the rule-skeleton $\langle p, q, p * q\rangle$:

$$\langle p^{\Rightarrow}, q^{\Rightarrow}, p * q\rangle \quad \langle p^{\Rightarrow}, q^{\Downarrow}, p * q\rangle \quad \langle p^{\Downarrow}, q^{\Rightarrow}, p * q\rangle \quad \langle p^{\Downarrow}, q^{\Downarrow}, p * q\rangle.$$

The first and last of these are straightforwardly a rule of inference and a rule of proof respectively (essentially what we might think of as *and*-introduction and Adjunction, respectively, regarding * as representing conjunction),[27] but the second and third pair defy straightforward classification in either category, indicating sequent-to-sequent rules as given respectively below, with '$\vdash$' here used as though it were a sequent-separator, so as to avoid having to introduce further notation:

$$\frac{\vdash B}{A \vdash A * B} \qquad\qquad \frac{\vdash A}{B \vdash A * B}$$

The whole area of sequent-to-sequent rules—as in natural deduction and sequent calculus—lies well outside the boundaries of the present discussion. There is an interesting question as to how (or indeed whether) to apply the rule of inference/rule of proof distinction to them, now thinking of the premises for an application of a rule as being sequents rather than formulas.

Another question that would deserve attention, returning to formula-to-formula rules, concerns the closeness of the link between the Deduction Theorem and the intuitive idea of a rule of inference, suggested by the discussion in Smiley (1963) (and also—implicitly—by that of Schurz 1994). When $\langle A, B\rangle \in \rho \in R_{inf}$ we have $A \vdash B$ for the associated consequence relation (the associated inferential consequence relation, on the sophisticated version of the differentiated approach), but should we always have $\vdash A \longrightarrow B$? The supervaluational semantics of Thomason contains examples suggesting a negative answer, taking *B* as 'it is true that *A*' (1970:273) and 'it is inevitable that *A*' (275), though what they raise most acutely is the question of the status of what the inference to a certain conclusion is an inference from: does it represent a mere supposition, an envisaged piece of new information, or what?[28] In the end the phrase 'rule of inference' may itself turn into something of an umbrella term subsuming interestingly different cases, much as the term 'rule' itself did before Smiley articulated the distinction between rules of proof and rules of inference.

NOTES

1. The notion of a consequence relation is taken to be familiar here; see p. 15 of Shoesmith and Smiley (1978) for the defining conditions (called there Overlap, Dilution, and Cut for Sets). The usual notational liberties will be taken in connection with a consequence relation $\vdash$: for example 'Γ, A $\vdash$ B' and '$\vdash B$' abbreviate '$\Gamma \cup \{A\} \vdash B$' and '$\varnothing \vdash B$' respectively.
2. For the detailed proof, see Kleene (1952), §21. Kleene's exposition is especially clear on the precise inductive structure of the argument. Despite the sentence which follows, a version of (A2) with its two antecedents permuted would do equally well, and this is the version appearing in Kleene's discussion. (References to the original Herbrand and Tarski sources can be found in Kleene's 1952:98.) In recent years many generalizations of the Deduction Theorem have been considered—see Czelakowski (2001: ch. 2) for an extensive sampling—but in what follows we have in mind just the simple traditional version of the result.
3. The terminology is not wholly felicitous in the general case (as remarked in Humberstone 2006a: 46) but is nonetheless convenient.
4. Fn. 3 in Smiley (1963) at this point cites Barcan (1946) as confusing the claim that $\vdash A$ implies $\vdash B$ with the claim that $A \vdash B$.
5. For conformity with our notation, Smiley's '*L*' has been replaced by '$\Box$'. The last rule, A / $\Box A$, mentioned by Smiley in this passage is the rule of Necessitation. Note that in view of the terminological proposal made in the passage, this should not really be glossed (as Smiley does) as 'from A *infer* $\Box A$'. ('From A to $\Box A$' is better.) Nor should one formulate the rule as 'if $\vdash A$ then $\vdash \Box A$': this is rather the statement that the rule in question is admissible (and no rule should be confused with a statement about rules or indeed about anything else).
6. For 'proof' here, read 'deduction', this being the term (after which the Deduction Theorem is named) for a record of the derivation of the formula on the right of the '$\vdash$' from those on the left (with the help of axioms). Rather than alluding to rules of proof proper, Schurz speaks of rules different from Modus Ponens, since he is considering the (typical) case in which that is the only rule of inference (in Smiley's sense). Although deductions in the present sense are often described as sequences of formulas, they are better visualized as trees with the deduced formula at the root and axioms and elements of Γ at their leaves. Each non-leaf node is immediately dominated by nodes labeled with formulas which are premises for an application of a rule whose conclusion formula labels that node. A node (or the occurrence of a formula labeling it) is Γ-dependent if the subtree with it as root has some node labeled with an element of Γ. (Alternatively, see the definition of dependence in Kleene 1952:99.)
7. See e.g. Pogorzelski (1971), Prucnal (1972), Wojtylak (1983), and Pogorzelski and Wojtylak (2005). It should be added that the ordered pairs considered in this literature come as $\langle R, Ax\rangle$ rather than in the reverse order given in the following sentence of the main text, with the axioms first (which is more convenient for present purposes). Such pairs are referred to in various ways in the papers just cited, including 'systems of propositional calculus' and 'logical systems'. Even those who do not make explicit use of the ordered pair theme for identifying axiomatizations implicitly operate with the same conception (in §2 we will call it the undifferentiated approach) of how such an axiomatization is to induce a consequence relation, and this includes a great many authors, from Łoś and Suszko (1958) to Blok and Pigozzi

(1989) and beyond.

8. The final sentence in the quotation actually starts a new paragraph in the source text.

9. This seems to be at least part of the drift of Smiley (1982).

10. Curiously Anderson and Belnap do not employ the notion of admissibility in the discussion on p. 236, even though they have explained the notion on p. 54; instead we get this nonsense about it being a 'lucky accident' that $\Box A$ is provable whenever A is.

11. Many respectable logicians, especially writing before the mid-1960s, simply ignore this distinction. For example Kleene (1952) calls all admissible rules derivable (or derived), and when he wants to talk about derivable rules (in the present sense) uses the phrase 'directly derivable'. Kripke (1965) uses the terms 'admissible' and 'derivable' interchangeably (again, simply to mean *admissible*). Kleene (1952:92) says 'A metamathematical theorem of the simple form $\Delta \vdash E$ is a derived rule of the *direct* type', thereby conflating (contrary to the recommendation of Note 5 above) rules with statements. It is also worth noting that his indifference to the rule of proof/rule of inference distinction has led to cloudy formulations in this and other works—see Kielkopf (1972).

12. Since the order of the premises is immaterial, a cleaner treatment—though rather fussier than would here be desirable—might take an application of an n-premise rule to be not $\langle B_1, \ldots, B_n, C\rangle$ but rather $\langle [B_1, \ldots, B_n], C\rangle$ in which $[B_1,\ldots, B_n]$ is the multiset of the formulas concerned. Thus if $n = 2$ and B_1 is the same formula as B_2, this is a multiset in which the formula in question occurs twice.

13. This usage is not unheard of in the literature. The only danger is that of confusion with something more general, as when a proof system with initial sequents and sequent-to-sequent rules (as in the case of the sequent calculus approach) is referred to as an axiomatization of the sequent logic—and thus (except in the substuctural logics case) of the obviously associated consequence relation (and the initial sequents are similarly referred to as 'axioms'). In the present discussion, only formulas count as axioms, and sequent-to-sequent rules come up for discussion only in passing in the final section.

14. Note that we have to add ρ to the set of rules of proof as well as to the set of rules of inference, since we require the latter to be a subset of the former.

15. We suppose that all (sentential) languages under consideration have $p_1, p_2, \ldots, p_n, \ldots$ as propositional variables (sentence letters), and abbreviate the first two in this list to p, q. Note that as defined, there is really no unique skeleton for a given rule, since the propositional variables could be relettered—to say nothing of the arbitrariness of the order in which the premise formulas $A_1, \ldots, A_n$ in a skeleton $\langle A_1, \ldots, A_n, B\rangle$ appear (commented on in Note 12 above). Note that we speak of the *substitution instances* of a formula, but the *instances* of a schema. (Derivatively, we also speak of the substitution instances of an n-tuple of formulas, when the same substitution is applied to all of the formulas in the n-tuple.) Note that the sequential rules are those susceptible of schematic representation with the slash notation (in the 'A / $\Box A$' style).

16. Łoś and Suszko (1958) called substitution-invariant rules *structural*, but this makes for an unfortunate collision with the terminology of 'structural rules' as deployed, for example by Gentzen, in connection with sequent-to-sequent rules (namely for rules whose formulation does not involve any particular logical vocabulary). The same authors pointed out that uniform substitution itself is not (in the present terminology) a substitution-invariant rule.

17. Examples of the literature in this vein include Fagin et al. (1992), Brady (1994), Humberstone (1996). The last reference concerns sequent-to-sequent rules, however, rather than formula-to-formula rules, our main focus here. The first reference describes some interesting modal examples; see also the first new paragraph on p. 386 of Schurz (1994).
18. See Fitting (1983) or van Benthem (1985), for instance. Substitution-invariant or 'schematic' versions of the validity-preserving relations are described in Fagin et al. (1992).
19. This terminology, though not quite with the present understanding, can be found in Belnap and Thomason (1963).
20. In fact it is this consequence relation, rather than its inferential cousin, which is most commonly meant by talk of deducibility in practice, even though this is not the relation relevant to the Deduction Theorem and mentioned under the heading of 'deducibility' in §1. For example, the two occurrences of 'interdeducible (in the field of K)' on pp. 577 and 581 of Hughes (1980), appear in connection with pairs of formulas each of which can be derived from the other only with the aid of US, meaning that these derivations do not constitute suitable 'deductions' in the sense of the Deduction Theorem.
21. Schurz (1994:386, top paragraph) also gives this example, as do Font and Jansana (2001:438). Schurz's use of the example is an indication of how misleading his title is: 'Admissible versus Valid Rules'. By 'valid' is here meant locally truth-preserving (in the class of models for the logic in question): emphasis on the contrast between this and admissibility is exactly the mistake Anderson and Belnap make. But Schurz's discussion, its title notwithstanding, is alert to the significance of derivability as opposed to admissibility (and suggests that he would have been happy, had he been aware of it, to embrace Smiley's distinction between rules of proof and rules of inference).
22. This point does not depend on the 'K' subscript, but holds for the local and global consequence relations corresponding to any normal modal logic. An extended discussion of the relations between these two consequence relations for the case of (an expressively impoverished fragment of) S5 appears in Humberstone (2006*b*). For more general considerations on the current local/ global contrast, see the first section of ch. 3 of Kracht (1999).
23. Or see the Corollary to Theorem 7 of Kripke (1965:219). Lemmon (amongst others) calls what are here called schematically finite axiomatizations simply finite axiomatizations (intending, further, that the sole rule employed is Modus Ponens).
24. While the contrast between (1) and (2) may initially seem impressive, it must be conceded to depend in large part on the specific identification of axiom schemata with zero-premise sequential rules, and the way 'schematically finite' captures the idea of a finitude of schemata, so understood. On this reading, something like '$\Box^n(p \longrightarrow p)$' understood as summarizing the prefixing of any number of occurrences of '$\Box$' to the formula '$p \longrightarrow p$', would not count as an axiom schema, let alone the 'doubly schematic' version with 'A' in place of 'p'. Relaxing the requirement so as to admit these last as axiom schemata would allow the inferential consequence relation associated with the class of all models to have a schematically finite axiomatization with Modus Ponens as the sole rule, destroying the contrast between (1) and (2) in the text, because just as the use of schematic formula-letters exploits the fact that all applications of US can be made to precede any applications of MP, so this numerical schematicity exploits the fact that all applications of Nec can be made to precede any applications of MP. This device (essentially) was employed in Fitch (1973).
25. See the references given in Note 40 of Humberstone (2006*a*), where I have complained

about this before. (The text to which that note is appended contains a misprint: ‘Scott’s own recommendations on score’ should read ‘Scott’s own recommendations on this score’.)

26. Help is on the way—indeed, has already arrived—as far as this latter consideration is concerned: see Font and Jansana (2001).

27. Compare the ‘Four forms of Modus Ponens’ in Scott (1974), as refined by the discussion in §6 of Humberstone (2000*a*) as well as in Humberstone (2008). ‘Adjunction’ here is the Hilbert-style rule of that name in Anderson and Belnap (1975) and elsewhere.

28. For more on the supposition/update contrast just drawn, see Humberstone (2002: §3).

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