# How to Model the Epistemic Probabilities of Conditionals

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- Lewis's Two Conceptions of "Rational Requirement"
  - 1976: "Probabilities of Conditionals & Conditional Probabilities"
    - Very strong conception, which assumes that all rational requirements hold (fully) *resiliently* — leads to trivialities.
  - 1980: "A Subjectivist's Guide to Objective Chance"
    - A weaker conception, which does *not* require (full) *resiliency* of rational requirements. It identifies "inadmissible" *exceptions* to resiliency, and thereby *avoids* triviality.
- The 1976 paper led many to *reject* The Equation.
- The 1980 paper led many to *accept* the Principal Principle.

☞ I will explain how to apply Lewis's 1980 conception to The Equation. The result is a *triviality-free* way of understanding The Equation (as a rational requirement for credence).





Consider the following constraint [1, 10] on a (prior) probability function $\Pr(\cdot)$, over a language containing two factual atoms $P$ and $Q$, and a third atom "$P \to Q$" (where "$P \to Q$" is interpreted *extra-systematically* as the indicative "if $P$, then $Q$").

**The Equation.** $\Pr(P \to Q) = \Pr(Q \mid P)$.

☞ Lewis [5] assumes that *if* **The Equation** is *rationally required*, *then* it must hold (fully) *resiliently* — *i.e.*, that the following *strengthening* of **The Equation** must be a constraint on $\Pr(\cdot)$.

**The Resilient Equation.**[1] For all $x$ (Boolean-definable in terms of $P, Q$) such that $\Pr(P \,\&\, x) > 0$,

$$\Pr(P \to Q \mid x) = \Pr(Q \mid P \,\&\, x).$$

Various triviality results have been derived from **The Resilient Equation.** The strongest possible such triviality result [3] is this.

**Triviality.** If $\Pr(P \,\&\, Q) > 0$ and $\Pr(P \,\&\, \neg Q) > 0$, then

$$\Pr(P \,\&\, (Q \equiv (P \to Q))) = 1.$$





In 1976, Lewis assumed that any rational requirement on initial (*viz.*, prior) credence $\Pr(\cdot)$ must be *fully resilient*. What he *says* in connection with this *doesn't entail* the desired (full) resiliency:

> the ... class of all those probability functions that represent possible systems of beliefs ... is closed under conditionalizing. Rational change of belief never can take anyone to a subjective probability function outside the class; and ... the change of belief that results from coming to know an item of new evidence should take place by conditionalizing on what was learned.

Be that as it may, Lewis's supposition of full resiliency leads to a *very* strong conception of "rational requirement."

Of course, *some* constraints *do* satisfy even this very strong conception. For instance, *probabilism itself* must satisfy it.

For *it is a theorem of the probability calculus* that if an initial credence function is a probability function $\Pr(\cdot)$, then so is $\Pr(\cdot \mid x)$, provided only that $\Pr(\cdot \mid x)$ well-defined. [The above quotation from Lewis (1976) articulates something close to this truism.]

My reconstruction: Lewis (1976) assumed (for *reductio*) that **The Equation** is a rational requirement on $\Pr(\cdot)$. Then, he used the full resiliency requirement to complete his *reductio*.

(1) **The Equation** is a rational requirement for $\Pr(\cdot)$.

(2) $\therefore$ **The Equation** must hold in a (fully) *resilient* way.

(3) $\therefore$ **The Resilient Equation** is a rational requirement for $\Pr(\cdot)$.

(4) But, **Triviality** is *not* a rational requirement for $\Pr(\cdot)$.

(5) Contradiction. [Since (3) entails ¬(4).]

(6) $\therefore$ **The Equation** is *not* a rational requirement for $\Pr(\cdot)$. □

The implicit assumption Lewis (1976) makes — that (1) *implies* (2) — is questionable. Indeed, in 1980, Lewis *rejects* it [7].

☞ Lewis (1980) is *not* moved by an analogous "*reductio* of the Principal Principle" as a rational constraint on prior/initial credence functions. Indeed, he goes to great lengths to *avoid* assuming that the Principal Principle holds (fully) *resiliently*.





Lewis [6] argues that the Principal Principle (PP) is a rational requirement on initial/prior credence functions $\Pr(\cdot)$.

(PP) $\Pr(p \mid \mathrm{Ch}(p) = c) = c$.

Lewis knows that if we require (PP) to hold (fully) *resiliently*, then we get something *probabilistically incoherent*. To wit:

($\mathrm{PP}_x$) $\Pr(p \mid x \mathbin{\&} \mathrm{Ch}(p) = c) = c$.

**Resilient (PP)** asserts that ($\mathrm{PP}_x$) holds *for all $x$ such that* ($\mathrm{PP}_x$) *is well-defined*. *That* principle [$(\forall x)\ \mathrm{PP}_x$] is *incoherent*.

Suppose $\mathrm{Ch}(p) = c < 1$ and $\Pr(p \mathbin{\&} \mathrm{Ch}(p) = c) > 0$, and let $x := p$. Then $(\forall x)\mathrm{PP}_x$ entails $\Pr(p \mid p \mathbin{\&} \mathrm{Ch}(p) = c) < 1$. But, *probability calculus* entails $\Pr(p \mid p \mathbin{\&} \mathrm{Ch}(p) = c) = 1$. □

For this reason, Lewis argues that we must *restrict the domain of the quantifier* $(\forall x)$ in $(\forall x)\mathrm{PP}_x$ to ***admissible*** *evidence*, which *rules-out* conditionalizing on various $x$'s, *e.g.*, $x := p$.





Here are some key differences between **The Resilient Equation** *vs* **Resilient (PP).** These differences are rather surprising in light of the conclusions drawn/invited by Lewis (1976) *vs* (1980).

- **The Resilient Equation** is *not incoherent* (merely "trivial").
- **Resilient (PP)** *is incoherent* (*i.e.*, *not* merely "trivial").
- The notion of "admissibility" that determines the requisite quantifier restriction on $(\forall x)\mathrm{PP}_x$ — *needed to avoid incoherence of* **Resilient (PP)** — is a very subtle and complex metaphysical notion, which cannot be neatly formalized.
- But, there *is* a simple, purely formal way to characterize the requisite quantifier restriction on **The Resilient Equation**, so as to avoid (Lewisian) triviality.
- This follows from the main result I proved in my [3], which shows that *only two instances* of **The Resilient Equation** are required for **Triviality**: $x := \neg Q$, and $x := P \supset Q$.





☞ This leads to a *triviality-free* way to model **The Equation** — as a rational requirement on initial/prior credence functions $\Pr(\cdot)$.

(TE) All rational initial credence functions $\Pr(\cdot)$ should satisfy the following *restricted* version of **The Resilient Equation.**

For all factual propositions $p$ and $q$, and for all propositions $x$ that are Boolean-definable in terms of $p, q$:

$$\Pr(p \to q \mid x) = \Pr(q \mid p \mathbin{\&} x)$$

*provided* that $x$ satisfies the following three constraints:

(i) $\Pr(p \mathbin{\&} x) > 0$,

(ii) $\Pr(\neg q \mid x) < 1$,

(iii) $\Pr(p \supset q \mid x) < 1$.

Constraint (i) ensures *conditional probabilities are well-defined.*

Constraints (ii) and (iii) ensure that *no Lewisian trivialities can be deduced* from (TE). As I show in [3], *all* Lewisian trivialities involve $x$'s which violate *at least one of* {(ii), (iii)}.

Hájek [4] proves an *independent* (combinatorial) triviality result for **The Equation.** Our approach also sidesteps Hájek's triviality.

Hájek shows that if we model the indicative conditional → as a *relation* between pairs of (factual) propositions, then **The Equation** must fail *for combinatorial reasons* (there won't be sufficiently many unconditional probabilities around to be paired off with conditional probabilities — as required by **The Equation**).

This *combinatorial* argument of Hájek's is independent of Lewis's argument, and so it requires a separate treatment.

Happily, our (TE) *already avoids* Hájek's combinatorial argument (here I am deeply indebted to recent discussions with A.H.).

For each pair of (factual) propositions $\langle p, q\rangle$ in the algebra, we *introduce a new atomic sentence* "$p \to q$" which is (*extra-systematically*) interpreted as the indicative conditional.

☞ Because we do *not* model → as a *relation* between propositions, we *can* have sufficiently many unconditional probabilities.





Richard Bradley [2] endorses the following rational requirement (assuming the same conventions I used in my discussion of Lewis above).

**Preservation.** $\Pr(\cdot)$ satisfies the following constraint:

If $\Pr(P) > 0$ and $\Pr(Q) = 0$, then $\Pr(P \to Q) = 0$.
If $\Pr(P) > 0$ and $\Pr(Q) = 1$, then $\Pr(P \to Q) = 1$.

**Preservation** is considerably weaker than **The Equation** (*e.g.*, it imposes *no* constraint on regular agents). However, if we require **Preservation** to hold *resiliently*, we get **Resilient Preservation.**

**Resilient Preservation.** $\Pr(\cdot)$ satisfies the following, for all $x$ (Boolean-definable in terms of $P, Q$) such that $\Pr(x) > 0$.

If $\Pr(P \mid x) > 0$ and $\Pr(Q \mid x) = 0$, then $\Pr(P \to Q \mid x) = 0$.
If $\Pr(P \mid x) > 0$ and $\Pr(Q \mid x) = 1$, then $\Pr(P \to Q \mid x) = 1$.

**Resilient Preservation** yields its own Lewisian trivialities. While these are weaker than Lewis's, they are strong enough to suggest that **Resilient Preservation** is not a rational requirement.





The following table illustrates the differences between **The Equation**, **The Resilient Equation**, and **Resilient Preservation.**

| $P$ | $Q$ | $P \to Q$ | $\Pr(\cdot)$ | $\Pr(\cdot)$ + **The Equation** | $\Pr(\cdot)$ + **R. Preservation** | $\Pr(\cdot)$ + **R. Equation** |
|---|---|---|---|---|---|---|
| T | T | T | $a$ | $a$ | $a$ | $a$ |
| T | T | F | $b$ | $b$ | 0 | 0 |
| T | F | T | $c$ | $c$ | 0 | 0 |
| T | F | F | $d$ | $d$ | $d$ | $1-a$ |
| F | T | T | $e$ | $e$ | $e$ | 0 |
| F | T | F | $f$ | $f$ | 0 | 0 |
| F | F | T | $g$ | $\frac{a+b}{a+b+c+d} - a - c - e$ | 0 | 0 |
| F | F | F | $1-\sum$ | $1-\sum$ | $1-\sum$ | 0 |

**The Equation** reduces the number of $\Pr(\cdot)$'s degrees of freedom by 1 (from 7 to 6), **Resilient Preservation** reduces it by 4, and **The Resilient Equation** reduces it by 6. Moreover, **The Resilient Equation** is *strictly stronger* than **Resilient Preservation.**





Here is a *triviality-free* way to model **Preservation** — as a rational requirement on initial/prior credence functions $\Pr(\cdot)$.

(P) All rational initial credence functions $\Pr(\cdot)$ should satisfy the following *restricted* version of **Resilient Preservation.**

For all factual propositions $p$ and $q$, and for all propositions $x$ that are Boolean-definable in terms of $p, q$:

If $\Pr(p \mid x) > 0$ and $\Pr(q \mid x) = 0$, then $\Pr(p \to q \mid x) = 0$.
If $\Pr(p \mid x) > 0$ and $\Pr(q \mid x) = 1$, then $\Pr(p \to q \mid x) = 1$.

*provided* that $x$ satisfies the following three constraints:

(i) $\Pr(x) > 0$,

(ii) $\Pr(\neg q \mid x) < 1$,

(iii) $\Pr(q \mid x) < 1$.

Constraint (i) ensures *conditional probabilities are well-defined.*

Constraints (ii) and (iii) ensure that *no Lewisian trivialities can be deduced* from (P). It can be shown that *all* Bradley-style Lewisian trivialities involve $x$'s which violate *at least one of* {(ii), (iii)}.

Stern, Hartmann & Fitelson [11] discuss a new impossibility result for the indicative conditional, which involves:

**MPI.** $\Pr(\cdot)$ satisfies the following two constraints (for all factual propositions $p$ and $q$):

(MP) $\Pr(q \mid p \,\&\, (p \to q)) = 1$.

(IND) $\Pr(p \to q \mid p) = \Pr(p \to q)$.

(MP) has been defended/accepted by *almost everyone* (note that it is restricted to *factual* $p$'s and $q$'s). And (IND) has been defended by many (esp. in the presence of **The Equation**) [8].

**Resilient MPI.** $\Pr(\cdot)$ satisfies the following, for all factual $p, q$, and all $x$ (Boolean-definable in terms of $p, q$) s.t. $\Pr(p \,\&\, x) > 0$.

(MP) $\Pr(q \mid p \,\&\, (p \to q)) = 1$.

($\text{IND}_x$) $\Pr(p \to q \mid p \,\&\, x) = \Pr(p \to q \mid x)$.

As it happens, **Resilient MPI** is *probabilistically incoherent* [11].





Here is one (impossibility-free) way to model **MPI** — as a rational requirement on initial/prior credence functions $\Pr(\cdot)$.

(M) All rational initial credence functions $\Pr(\cdot)$ should satisfy the following *restricted* version of **Resilient MPI**.

For all factual propositions $p$ and $q$, and for all propositions $x$ that are Boolean-definable in terms of $p, q$:

(MP) $\Pr(q \mid (p \,\&\, (p \to q)) \,\&\, x) = 1$.
(IND) $\Pr(p \to q \mid p \,\&\, x) = \Pr(p \to q \mid x)$.

*provided* that $x$ satisfies the following two constraints:

(i) $\Pr((p \,\&\, (p \to q)) \,\&\, x) > 0$,

(ii) either $\Pr(q \mid x) < 1$ or $\Pr(\neg q \mid x) < 1$.

Constraint (i) ensures *conditional probabilities are well-defined.*

Constraints (ii) and (iii) ensure that *no incoherence can be deduced* from (M). This is because the incoherence of **Resilient MPI** stems entirely from violations of (ii) [11].





Here is a compelling counterexample to **The Resilient Equation** (thanks to Paolo Santorio). A fair die (Die) was tossed.

($P$) Die landed on either 1, 3, 5, or 6.

($Q$) Die landed on 6.

($X$) Die landed even.[2]

| $P$ | $Q$ | $\Pr(\cdot)$ |
|---|---|---|
| **T** | **T** | $1/6$ |
| **T** | **F** | $1/2$ |
| **F** | **T** | 0 |
| **F** | **F** | $1/3$ |

**The Equation** $\Rightarrow \Pr(P \to Q) = \Pr(Q \mid P) = 1/4$. $\therefore \Pr((P \to Q) \,\&\, X) \le 1/4$.

$$\therefore \Pr(P \to Q \mid X) = \frac{\Pr((P \to Q) \,\&\, X)}{\Pr(X)} \le \frac{1/4}{1/2} = 1/2 < 1 = \Pr(Q \mid P \,\&\, X).$$

$\therefore$ This is an *intuitive counterexample to* **The Resilient Equation.** But, because $\Pr(P \supset Q \mid X) = 1$, this is no counterexample to (TE).

[2] $X$ is Boolean-definable in terms of $P, Q$, since $X$ is equivalent to $P \supset Q$.






[1] E. Adams, *The Logic of Conditionals*, 1965.

[2] R. Bradley, *A Preservation Condition for Conditionals*, 2000.

[3] B. Fitelson, *The Strongest Possible Lewisian Triviality Result*, 2015.

[4] A. Hájek, *Probabilities of Conditionals — Revisited*, 1989.

[5] D. Lewis, *Probabilities of Conditionals and Conditional Probabilities*, 1976.

[6] ———, *A Subjectivist's Guide to Objective Chance*, 1980.

[7] I. Nissan-Rozen, *Jeffrey Conditionalization, the Principal Principle, the Desire as Belief Thesis, and Adams's Thesis*, 2013.

[8] D. Rothschild, *Do Indicative Conditionals Express Propositions?*, 2010.

[9] B. Skyrms, *Resiliency, Propensities, and Causal Necessity*, 1977.

[10] R. Stalnaker, *A Theory of Conditionals*, 1968.

[11] R. Stern, S. Hartmann, and B. Fitelson, *A New Impossibility Result for the Indicative Conditional*, manuscript.