Gibbard's Collapse Theorem for the Indicative Conditional: An Axiomatic Approach

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1 Background: Gibbard's (Informal) Argument

Gibbard [2] presents an argument to the effect that any conditional satisfying certain principles must be equivalent to the material (*viz.*, classical) conditional. Here is one rendition of Gibbard's (informal) argument.

Let $\supset$ be the classical material conditional, and let $\rightsquigarrow$ be the indicative conditional. Suppose that the indicative satisfies the *import-export law*. That is, suppose

(IE) $A \rightsquigarrow (B \rightsquigarrow C)$ is *logically equivalent* to $(A \,\&\, B) \rightsquigarrow C$.

If $\rightsquigarrow$ satisfies (IE), then (i) is equivalent to (ii).

(i) $(A \supset C) \rightsquigarrow (A \rightsquigarrow C)$.
(ii) $((A \supset C) \,\&\, A) \rightsquigarrow C$.

Substitutivity of logical equivalents then implies that (ii) [and $\therefore$ (i)] is equivalent to (iii).

(iii) $(A \,\&\, C) \rightsquigarrow C$.

So, if (iii) is a logical truth (as Gibbard supposes), then (i) and (ii) are too. Finally, suppose the indicative is at least as strong as the material conditional. That is, suppose $P \rightsquigarrow Q$ *entails* $P \supset Q$. Then, (i) entails (iv).

(iv) $(A \supset C) \supset (A \rightsquigarrow C)$.

Hence, (iv) is (also) a logical truth. Thus, $A \supset C$ entails $A \rightsquigarrow C$. Therefore, in general, $p \rightsquigarrow q$ entails $p \supset q$ and $p \supset q$ entails $p \rightsquigarrow q$. That is, in general, $\rightsquigarrow$ and $\supset$ are logically equivalent. *QED.*

In this note, I present a formal axiomatization of the (theoretic and meta-theoretic) assumptions, which I take to be *essential* to the Gibbardian collapse phenomenon. This will lead to a formal proof of what I will call *Gibbard's collapse theorem*. Our formal treatment will reveal that collapse to the *classical, material* conditional is *not* inevitable. In fact, when one looks more closely at the assumptions involved (*essentially*) in proving Gibbard's collapse theorem, one realizes that *both classical and intuitionistic* interpretations of the indicative conditional are compatible with Gibbard's collapse phenomenon. This non-classical aspect of Gibbardian collapse is hidden by traditional presentations, which tend to (implicitly) presuppose various classical (theoretic and meta-theoretic) principles that are inessential to the theorem.

2 Axiomatization of the Gibbardian Collapse Phenomenon

Let $\mathscr{L}$ be a sentential (object) language containing atoms 'A', 'B', ..., and two *logical* connectives '&' and '$\rightarrow$'. In addition to these two *logical* connectives, $\mathscr{L}$ will also contain another binary connective '$\rightsquigarrow$', which is intended to be interpreted as the English indicative. In the meta-language for $\mathscr{L}$, we will have two meta-linguistic relations: $\Vdash$ and $\vdash$. '$\Vdash$' will denote a binary relation between individual sentences in $\mathscr{L}$. Specifically, '$\Vdash$' will be interpreted as the *single premise deducibility* (or *entailment*) relation of $\mathscr{L}$. '$\vdash$' will denote a monadic property of sentences of $\mathscr{L}$. Specifically, '$\vdash$' will be interpreted as the property of *theoremhood* (or *logical truth*) in $\mathscr{L}$. We will not presuppose anything about the relationship between '$\Vdash$' and '$\vdash$'. Rather, we will state explicitly all assumptions about these meta-theoretic relations that will be required (essentially) for Gibbard's collapse theorem. More precisely, I will state eight (8) independent axioms for $\rightarrow$, $\rightsquigarrow$, &, $\Vdash$, and $\vdash$, which will be jointly sufficient for (and severally essential for the proof of) Gibbard's collapse theorem.

First, two preliminary remarks: (a) the "if... then" (which I'll sometimes abbreviate as "$\Rightarrow$") and "and" in the meta-meta-language of $\mathscr{L}$ will be assumed throughout to be *classical*, and (b) the eight axioms are *schematic* (*i.e.*, they are to be interpreted as allowing *any instances* that can be formed with sentences of $\mathscr{L}$). With those caveats in mind, here are the eight (8) axioms that will form the basis of my formalization of Gibbard's collapse theorem.

(1) $\vdash (p \,\&\, q) \rightarrow p$.
(2) $\vdash p \rightarrow (q \rightarrow r)$ if and only if $\vdash (p \,\&\, q) \rightarrow r$.
(3) $\vdash p \rightsquigarrow (q \rightsquigarrow r)$ if and only if $\vdash (p \,\&\, q) \rightsquigarrow r$.
(4) If $\vdash p \rightsquigarrow q$, then $\vdash p \rightarrow q$.
(5) $\vdash (p \,\&\, q) \rightsquigarrow q$.
(6) If $p \Vdash q$ and $p \Vdash r$, then $p \Vdash q \,\&\, r$.
(7) If $\vdash p \rightarrow q$, then $p \Vdash q$.
(8) If $p \Vdash q$ and $q \Vdash p$, then p and q are *inter-substitutable* (in the context of $\rightsquigarrow$ theorems).[1]

Before stating our collapse theorem, I will make a few remarks about the axioms (1)–(8). Axiom (1) is a (left) conjunction-elimination *axiom* for $\langle \rightarrow, \& \rangle$. This is valid in every conditional logic I can think of. Axioms (2) and (3) are *import-export* rules for $\langle \rightarrow, \& \rangle$-*theorems* of $\mathscr{L}$ and $\langle \rightsquigarrow, \& \rangle$-*theorems* of $\mathscr{L}$, respectively. They say that import-export is *validity preserving* for both conditionals in $\mathscr{L}$. This is one of the more controversial axioms in the list. It is valid in many logics (*e.g.*, both classical and intuitionistic logic), but it also fails in many logics (*e.g.*, various substructural logics). Axiom (4) says that the indicative

[1] More precisely, we only need a *special case* of inter-substitutivity (in $\rightsquigarrow$ theorems). Let $p \approx q =_{df}$ for all r, $\vdash p \rightsquigarrow r$ iff $\vdash q \rightsquigarrow r$. Then, all we need to assume in (8) is: If $p \Vdash q$ and $q \Vdash p$, then $p \approx q$. This is made clearer in the proofs found in the Appendix.

conditional is "at least as strong as" the logical conditional — but *only* in the sense that if an indicative conditional is a *theorem* of $\mathscr{L}$, then the corresponding logical conditional is *also* a *theorem* of $\mathscr{L}$. Axiom (5) is a (right) conjunction-elimination *axiom* for $\langle \rightsquigarrow, \& \rangle$. Like (1), this is a universally valid principle for conditional logics. Axiom (6) is a form of the *conjunction introduction rule*. This is valid for just about any (single-premise) entailment relation I can think of. Axiom (7) says (informally) that if a logical conditional is a logical truth (theorem), then its antecedent entails its consequent. This is one direction of the deduction theorem for the logical conditional.[2] It holds in many logical systems (including both classical and intuitionistic logic). Axiom (8) is the assumption of inter-substitutivity of logical equivalents (in indicatives). This axiom is valid in many (positive) logics, including both intuitionistic and classical logic. Finally, axioms (1)–(8) are *independent*. And, they *suffice* to ensure that the indicative conditional collapses to the logical conditional. To wit, the following theorem.[3]

Theorem. Axioms (1)–(8) are independent, and they jointly entail the following *collapse* of $\rightsquigarrow$ to $\rightarrow$

(9) $p \rightarrow q \Vdash p \rightsquigarrow q$ and $p \rightsquigarrow q \Vdash p \rightarrow q$.

Before closing this section, two crucial remarks about our formal Gibbardian collapse theorem are in order.

Remark 1. Axioms (1)–(8) do *not* entail collapse of $\rightsquigarrow$ to $\supset$. That is to say, collapse of the indicative to the *classical, material* conditional does *not* follow from (1)–(8), and is therefore *inessential* to the (core) Gibbaridan collapse phenomenon. More precisely, we can show that (1)–(8) are compatible with

(10) $\nvdash ((p \rightsquigarrow q) \rightsquigarrow p) \rightsquigarrow p$ and $\nvdash ((p \rightarrow q) \rightarrow p) \rightarrow p$.

That is, *Peirce's Law* is not guaranteed by (1)–(8) to be a theorem (for either $\rightsquigarrow$ or $\rightarrow$).

Remark 2. Axioms (1)–(8) *do* entail that the indicative conditional must collapse to a logical conditional that is *at least as strong as the intuitionistic conditional*. This follows from the fact that (1)–(8) entail the following three additional theorems:

(11) If $\vdash p$ and $\vdash p \rightsquigarrow q$, then $\vdash q$. [And, if $\vdash p$ and $\vdash p \rightarrow q$, then $\vdash q$.]
(12) $\vdash p \rightsquigarrow (q \rightsquigarrow p)$. [And, $\vdash p \rightarrow (q \rightarrow p)$.]
(13) $\vdash (p \rightsquigarrow (q \rightsquigarrow r)) \rightsquigarrow ((p \rightsquigarrow q) \rightsquigarrow (p \rightsquigarrow r))$. [And, $\vdash (p \rightarrow (q \rightarrow r)) \rightarrow ((p \rightarrow q) \rightarrow (p \rightarrow r))$.]

It is well-known that (11)–(13) suffice to derive all theorems of intuitionistic implication. Therefore, (1)–(8) entail that all theorems of intuitionistic

[2] The other direction of the deduction theorem for $\rightarrow$ also follows from axioms (1)–(8), but I will not give a proof of that here.

[3] In the Appendix, I provide a proof of this theorem (and the other technical results of the paper).

implication are theorems of both the indicative and the logical conditional. So, while the Gibbardian collapse phenomenon is compatible with a non-classical conditional, it does entail collapse to something that is no weaker (in terms of its theorems) than the intuitionistic conditional.

3 Concluding Remarks

We have given a rigorous formal rendition of the assumptions that we think are essential to a Gibbardian collapse theorem. This has revealed that the collapse phenomenon is not essentially classical in nature. But, it has also revealed that collapse to a conditional at least as strong as the intuitionistic conditional is essential to the phenomenon. This means that anyone who thinks that the indicative conditional does not have (at least) the logical strength of the intuitionistic conditional (*i.e.*, that the indicative lacks some theorems that the intuitionistic conditional has) is going to have to reject some of our axioms (1)–(8). The only axioms that seem plausibly deniable (to me — in the context of a sentential logic containing only conditionals and conjunctions) are axioms (2) and (3). These are the *import-export* laws, and they seem to be the most suspect of the bunch. I find it difficult to see how any of the other axioms could (plausibly) be denied (but I won't argue for that claim here). The two main purposes of this note have been (a) to reveal the non-classical nature of the (essence of the) Gibbardian collapse phenomenon, and (b) to make clear precisely what theoretic and meta-thoeretic assumptions underlie Gibbardian collapse.

4 Appendix: Proofs of Theorems

4.1 Proofs of the Independence of Our Axioms (1)–(8)

First, I must prove that the axioms (1)–(8) are *independent*. I will do so by providing eight countermodels.[4]

Independence of (1). We must show $\{(2),(3),(4),(5),(6),(7),(8)\} \not\Rightarrow (1)$. Here is a model on which (2)–(8) are **T**, but (1) is **F**.

&	0	1	2
0	0	1	2
1	0	1	2
2	2	2	2

$\rightarrow$	0	1	2
0	1	1	2
1	2	1	2
2	1	1	1

$\rightsquigarrow$	0	1	2
0	1	1	2
1	2	1	2
2	1	1	1

$\Vdash$	0	1	2
0	**T**	**T**	**F**
1	**F**	**T**	**F**
2	**T**	**T**	**T**

$\vdash$	0	1	2
	F	**T**	**F**

[4] All models and proofs in this Appendix were discovered and verified with the aid of the automated reasoning programs `paradox` [1], `vampire` [5], `prover9/mace4` [4], and `otter` [3]. All models are *smallest possible*, but there may be more elegant proofs of the theorems (I tried to find the simplest proofs I could, using various techniques for finding elegant proofs [7]).

Note: $\vdash (p \,\&\, q) \rightarrow p$ is **F** on this model, when $p \coloneqq 0$ and $q \coloneqq 1$. □

Independence of (2). We must show $\{(1),(3),(4),(5),(6),(7),(8)\} \not\Rightarrow (2)$. Here is a model on which $\{(1),(3),(4),(5),(6),(7),(8)\}$ are **T**, but (2) is **F**.

$\&$	0	1	2
0	0	1	0
1	1	1	1
2	0	1	2

$\rightarrow$	0	1	2
0	2	0	2
1	2	2	2
2	0	0	2

$\rightsquigarrow$	0	1	2
0	2	1	2
1	2	2	2
2	0	1	2

$\Vdash$	0	1	2
0	**T**	**F**	**T**
1	**T**	**T**	**T**
2	**F**	**F**	**T**

$\vdash$	0	1	2
	F	**F**	**T**

Note: $\vdash p \rightarrow (q \rightarrow r) \Rightarrow \vdash (p \,\&\, q) \rightarrow r$ is **F** on this model, when $p, q \coloneqq 0$ and $r \coloneqq 1$. □

Independence of (3). We must show $\{(1),(2),(4),(5),(6),(7),(8)\} \not\Rightarrow (3)$. Here is a model on which $\{(1),(2),(4),(5),(6),(7),(8)\}$ are **T**, but (3) is **F**.

$\&$	0	1	2
0	0	1	0
1	1	1	1
2	0	1	2

$\rightarrow$	0	1	2
0	2	1	2
1	2	2	2
2	0	1	2

$\rightsquigarrow$	0	1	2
0	2	0	2
1	2	2	2
2	0	0	2

$\Vdash$	0	1	2
0	**T**	**F**	**T**
1	**T**	**T**	**T**
2	**F**	**F**	**T**

$\vdash$	0	1	2
	F	**F**	**T**

Note: $\vdash p \rightsquigarrow (q \rightsquigarrow r) \Rightarrow \vdash (p \,\&\, q) \rightsquigarrow r$ is **F** on this model, when $p, q \coloneqq 0$ and $r \coloneqq 1$. □

Independence of (4). We must show $\{(1),(2),(3),(5),(6),(7),(8)\} \not\Rightarrow (4)$. Here is a model on which $\{(1),(2),(3),(5),(6),(7),(8)\}$ are **T**, but (4) is **F**.

$\&$	0	1
0	0	1
1	1	1

$\rightarrow$	0	1
0	0	1
1	0	0

$\rightsquigarrow$	0	1
0	0	0
1	0	0

$\Vdash$	0	1
0	**T**	**F**
1	**T**	**T**

$\vdash$	0	1
	T	**F**

Note: $\vdash p \rightsquigarrow q \Rightarrow \vdash p \rightarrow q$ is **F** on this model, when $p \coloneqq 0$ and $q \coloneqq 1$. □

Independence of (5). We must show $\{(1),(2),(3),(4),(6),(7),(8)\} \not\Rightarrow (5)$. Here is a model on which $\{(1),(2),(3),(4),(6),(7),(8)\}$ are **T**, but (5) is **F**.

$\&$	0	1
0	0	1
1	1	1

$\rightarrow$	0	1
0	0	1
1	0	0

$\rightsquigarrow$	0	1
0	1	1
1	1	1

$\Vdash$	0	1
0	**T**	**F**
1	**T**	**T**

$\vdash$	0	1
	T	**F**

Note: $\vdash (p \,\&\, q) \rightsquigarrow q$ is **F** on this model, when $p, q \coloneqq 0$. □

Independence of (6). We must show $\{(1),(2),(3),(4),(5),(7),(8)\} \not\Rightarrow (6)$. Here is a model on which $\{(1),(2),(3),(4),(5),(7),(8)\}$ are **T**, but (6) is **F**.

$\&$	0	1	2
0	1	1	1
1	1	1	1
2	0	1	2

$\rightarrow$	0	1	2
0	2	0	2
1	2	2	2
2	0	0	2

$\rightsquigarrow$	0	1	2
0	2	0	2
1	2	2	2
2	0	0	2

$\Vdash$	0	1	2
0	**T**	**F**	**T**
1	**T**	**T**	**T**
2	**F**	**F**	**T**

$\vdash$	0	1	2
	F	**F**	**T**

Note: $(p \Vdash q$ and $p \Vdash r) \Rightarrow p \Vdash q \,\&\, r$ is **F** on this model, when $p, q, r \coloneqq 0$. □

Independence of (7). We must show $\{(1),(2),(3),(4),(5),(6),(8)\} \not\Rightarrow (7)$. Here is a model on which $\{(1),(2),(3),(4),(5),(6),(8)\}$ are **T**, but (7) is **F**.

$\&$	0	1
0	0	0
1	0	0

$\rightarrow$	0	1
0	0	0
1	0	0

$\rightsquigarrow$	0	1
0	0	0
1	0	0

$\Vdash$	0	1
0	**F**	**F**
1	**F**	**F**

$\vdash$	0	1
	T	**F**

Note: $\vdash p \rightarrow q \Rightarrow p \Vdash q$ is **F** on this model, when $p, q \coloneqq 0$. □

Independence of (8). We must show $\{(1),(2),(3),(4),(5),(6),(7)\} \not\Rightarrow (8)$. Here is a model on which $\{(1),(2),(3),(4),(5),(6),(7)\}$ are **T**, but (8) is **F**.

$\&$	0	1
0	0	0
1	0	1

$\rightarrow$	0	1
0	1	1
1	0	1

$\rightsquigarrow$	0	1
0	1	1
1	0	1

$\Vdash$	0	1
0	**T**	**T**
1	**T**	**T**

$\vdash$	0	1
	T	**F**

Recall (see *fn.* 1) that the precise content of axiom (8) is the following:

(8) If $p \Vdash q$ and $q \Vdash p$, then $p \approx q$, where $p \approx q$ just in case, for all r, $\vdash p \rightsquigarrow r$ iff $\vdash q \rightsquigarrow r$.

Thus, a counterexample to (8) must involve a model containing a triple $\{p, q, r\}$ such that both $p \Vdash q$ and $q \Vdash p$ are **T**, but $\vdash p \rightsquigarrow r \Leftrightarrow \vdash q \rightsquigarrow r$ is **F**. This is just such a model, where $p, r \coloneqq 0$ and $q \coloneqq 1$. □

4.2 Proof of Our (Intuitionistic) Collapse Theorem

The following is a (unified) proof of the following two central theorems reported in the main text: (a) our Gibbardian collapse theorem (9), and (b) claim (13) for the indicative conditional. Claims (11) and (12) have easy proofs from (1)–(8), so I omit those here.[5] In this proof, I will present all axioms (and steps) in *clausal form*, and the only rule of inference I will use is *hyper-resolution*.[6]

[5] It can also be shown that *all* of our axioms (1)–(8) are *essential* to *any* proof of the collapse theorem. Moreover, it can be shown that axiom (2) is not needed to prove (13) for the indicative conditional, and Axiom (3) is not needed to prove of (13) for the logical conditional [but, in both cases, the remaining axioms *are* essential for proving (13)]. Obtaining (direct) axiomatic proofs of (13) was non-trivial. I thank Bob Veroff (and his *proof sketches* technique [6]) for his invaluable assistance in obtaining (direct) axiomatic proofs of (13) for both conditionals.

[6] The proof given here was discovered using McCune's theorem-prover `Otter` [3], and it was verified using his more recent `prover9` [4]. The requisite substitution instances are generally not too difficult to figure out for each hyper-resolution step. I omit those details, but they can be generated using McCune's `prooftrans` program [4]. Finally, I have posted two input files for exploring and verifying the results reported here. First, `http://fitelson.org/gibbard_fof.in` is a `tptp/fof` syntax input file for exploring the Gibardian collapse phenomenon (this file should work with most theorem-provers/model-finders that are available today). Second, `http://fitelson.org/gibbard_prover9.in` is a `prover9` input file, which allows for easy verification of the main proof of claims (9) and (13) reported below.

1. $\vdash (A \,\&\, B) \to A$	Axiom (1)
2. $\vdash (A \,\&\, B) \rightsquigarrow B$	Axiom (5)
3. $\nvdash A \to (B \to C) \;\vee\; \vdash (A \,\&\, B) \to C.$	Axiom (2).
4. $\vdash A \to (B \to C) \;\vee\; \nvdash (A \,\&\, B) \to C.$	Axiom (2).
5. $\nvdash A \rightsquigarrow (B \rightsquigarrow C) \;\vee\; \vdash (A \,\&\, B) \rightsquigarrow C.$	Axiom (3).
6. $\vdash A \rightsquigarrow (B \rightsquigarrow C) \;\vee\; \nvdash (A \,\&\, B) \rightsquigarrow C.$	Axiom (3).
7. $\nvdash A \rightsquigarrow B \;\vee\; \vdash A \to B.$	Axiom (4).
8. $A \nVdash B \;\vee\; A \nVdash C \;\vee\; A \Vdash B \,\&\, C.$	Axiom (6).
9. $\nvdash A \to B \;\vee\; A \Vdash B.$	Axiom (7).
10. $A \nVdash B \;\vee\; B \nVdash A \;\vee\; \nvdash A \rightsquigarrow C \;\vee\; \vdash B \rightsquigarrow C.$	Axiom (8).
11. $\vdash (((A \to B) \,\&\, C) \,\&\, A) \to B.$	3, 1.
12. $\vdash ((A \,\&\, (B \rightsquigarrow C)) \,\&\, B) \rightsquigarrow C.$	5, 2.
13. $\vdash (A \,\&\, B) \to B.$	7, 2.
14. $A \,\&\, B \Vdash A.$	9, 1.
15. $((A \to B) \,\&\, C) \,\&\, A \Vdash B.$	9, 11.
16. $\vdash ((A \,\&\, (B \rightsquigarrow C)) \,\&\, B) \to C.$	7, 12.
17. $A \,\&\, B \Vdash B.$	9, 13.
18. $(A \,\&\, (B \rightsquigarrow C)) \,\&\, B \Vdash C.$	9, 16.
19. $A \,\&\, B \Vdash A \,\&\, B.$	8, 14, 17.
20. $A \,\&\, B \Vdash B \,\&\, A.$	8, 17, 14.
21. $(A \,\&\, (B \rightsquigarrow C)) \,\&\, B \Vdash ((A \,\&\, (B \rightsquigarrow C)) \,\&\, B) \,\&\, C.$	8, 19, 18.
22. $A \,\&\, B \Vdash (A \,\&\, B) \,\&\, B.$	8, 19, 17.
23. $((A \to B) \,\&\, C) \,\&\, A \Vdash (((A \to B) \,\&\, C) \,\&\, A) \,\&\, B.$	8, 19, 15.
24. $\vdash (A \,\&\, B) \rightsquigarrow A.$	10, 20, 20, 2.
25. $\vdash (((A \to B) \,\&\, C) \,\&\, A) \rightsquigarrow B.$	10, 14, 23, 2.
26. $\vdash (((A \rightsquigarrow B) \,\&\, C) \,\&\, A) \rightsquigarrow B.$	5, 24.
27. $\vdash ((A \to B) \,\&\, A) \rightsquigarrow B.$	10, 14, 22, 25.
28. $\vdash ((A \rightsquigarrow B) \,\&\, A) \rightsquigarrow B.$	10, 14, 22, 26.
29. $\vdash ((((A \rightsquigarrow (B \rightsquigarrow C)) \,\&\, D) \,\&\, A) \,\&\, B) \rightsquigarrow C.$	5, 26.
30. $\vdash (A \to B) \rightsquigarrow (A \rightsquigarrow B).$	6, 27.
31. $\vdash ((A \rightsquigarrow B) \,\&\, A) \to B.$	7, 28.
32. $\vdash (((A \rightsquigarrow (B \rightsquigarrow C)) \,\&\, (A \rightsquigarrow B)) \,\&\, A) \rightsquigarrow C.$	10, 14, 21, 29.
33. $\vdash (A \to B) \to (A \rightsquigarrow B).$	7, 30.
34. $\vdash (A \rightsquigarrow B) \to (A \to B).$	4, 31.
35. $\vdash ((A \rightsquigarrow (B \rightsquigarrow C)) \,\&\, (A \rightsquigarrow B)) \rightsquigarrow (A \rightsquigarrow C).$	6, 32.
36. $A \to B \Vdash A \rightsquigarrow B$	9, 33.
37. $A \rightsquigarrow B \Vdash A \to B.$	9, 34.
38. $\vdash (A \rightsquigarrow (B \rightsquigarrow C)) \rightsquigarrow ((A \rightsquigarrow B) \rightsquigarrow (A \rightsquigarrow C)).$	6, 35. $\square$

4.3 Counterexample to Peirce's Law for $\rightsquigarrow$ and $\to$

Here is a model on which (1)–(8) are all **T**, but $\vdash ((p \rightsquigarrow q) \rightsquigarrow p) \rightsquigarrow p$ and $\vdash ((p \to q) \to p) \to p$ are **F**.

$\&$	0	1	2
0	0	1	0
1	1	1	1
2	0	1	2

$\to$	0	1	2
0	2	1	2
1	2	2	2
2	0	1	2

$\rightsquigarrow$	0	1	2
0	2	1	2
1	2	2	2
2	0	1	2

$\Vdash$	0	1	2
0	**T**	**F**	**T**
1	**T**	**T**	**T**
2	**F**	**F**	**T**

$\vdash$	0	1	2
	F	**F**	**T**

Note: $\vdash ((p \rightsquigarrow q) \rightsquigarrow p) \rightsquigarrow p$ and $\vdash ((p \to q) \to p) \to p$ are both **F** on this model, when $p \coloneqq 0$ and $q \coloneqq 1$. $\square$

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