Formal Epistemology Workshop
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Tutorial 2

Hyperreals & Their Applications

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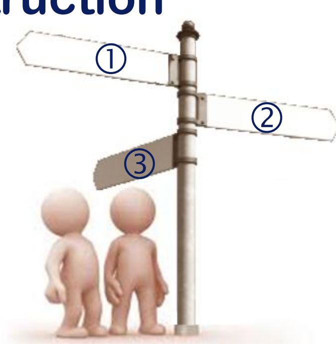
Overview

Three ways to introduce hyperreals:

① Existence proof (Model Theory)
② Axiom systems
③ Ultrapower construction



Overview

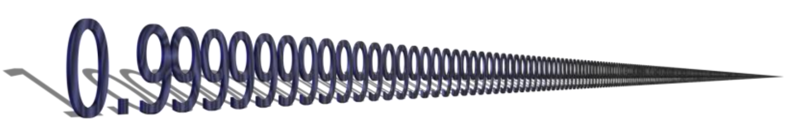
Many ways to motivate hyperreals:

① History of calculus
② Infinitesimal intuitions
③ Paradoxes of infinity
④ Formal epistemology
⑤ Philosophy of science
⑥ & Much more

Applications

We will get back to our last topic of the first tutorial:

Hyperreals and 0.999…



But first, we examine the underlying topic:
Infinitesimals and null-sequences

Infinitesimals and null-sequences

In classical analysis:

The sequence

$$\langle 0.1, 0.01, 0.001, \ldots, 10^{-n}, \ldots \rangle$$

converges to 0; it's a null-sequence.

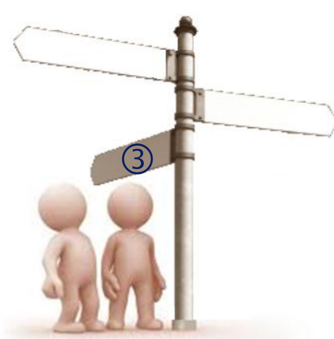
Can NSA teach us something about the common intuition that this sequence is a non-zero infinitesimal?

Yes, that is exactly what NSA does.

Introduction ③

The ultrapower construction of $^*\mathbb{R}$

First, we review the constructions of $\mathbb{R}$ of $^*\mathbb{Q}$



Construction of $\mathbb{R}$

We will review the construction of $\mathbb{R}$ via Cauchy sequences on $\mathbb{Q}$.

There are multiple other constructions for $\mathbb{R}$ starting from $\mathbb{Q}$, including Dedekind cuts or Weierstrass' construction.

Construction of $\mathbb{R}$

$\mathbb{Q}^{\mathbb{N}}$ = set of ω-sequences of rationals
$= \{ \langle q_1, q_2, \ldots, q_n, \ldots \rangle \mid \forall n \in \mathbb{N}\ (q_n \in \mathbb{Q}) \}$

☹ As such, $\mathbb{Q}^{\mathbb{N}}$ does not form a field
E.g., **$\langle 1,0,0,0,\ldots \rangle \times \langle 0,1,1,1,\ldots \rangle = \langle 0,0,0,0,\ldots \rangle$**

❶ Look at a particular subset of $\mathbb{Q}^{\mathbb{N}}$:

C = set of Cauchy sequences in $\mathbb{Q}^{\mathbb{N}}$
$= \{ \langle q_1, q_2, \ldots, q_n, \ldots \rangle \mid \forall n \in \mathbb{N}\ (q_n \in \mathbb{Q}) \wedge \forall \varepsilon > 0 \in \mathbb{Q}, \exists N \in \mathbb{N}, \forall n,m > N: |q_m - q_n| < \varepsilon \}$

Construction of $\mathbb{R}$

❷ Define equivalence relation ~ on C:

$\langle q_n \rangle \sim \langle s_n \rangle \Leftrightarrow$

$\forall \varepsilon > 0 \in \mathbb{Q}, \exists N \in \mathbb{N}, \forall n > N: |q_n - s_n| < \varepsilon$

❸ Define equivalence classes on C:

$[\langle q_n \rangle]_\sim = \{ \langle s_n \rangle \mid \langle q_n \rangle \sim \langle s_n \rangle \}$

❹ Now, we can define $\mathbb{R}$:

$\mathbb{R} = \{ [\langle q_n \rangle]_\sim \mid \langle q_n \rangle \in C \}$ "set of equivalence classes"

$= C/\sim$ "quotient ring"

❺ Embed $\mathbb{Q}$ in $\mathbb{R}$: $\forall q \in \mathbb{Q}, q = [\langle q,q,q,\ldots \rangle]_\sim$

Construction of $\mathbb{R}$

Question:

Could we have considered a different kind of equivalence relation, defined on all of $\mathbb{Q}^{\mathbb{N}}$?

Answer:

Yes, we will see an example of this: the ultrapower construction of $^*\mathbb{Q}$.

Construction of $^*\mathbb{Q}$

❶ Start from *all* sequences of rational numbers, $\mathbb{Q}^{\mathbb{N}}$

❷ Define equivalence relation on $\mathbb{Q}^{\mathbb{N}}$:
First, fix a free ultrafilter, $\mathcal{U}$, on $\mathbb{N}$
Then, define equivalence under $\mathcal{U}$

$$\langle q_n \rangle \sim_{\mathcal{U}} \langle s_n \rangle \Leftrightarrow \{\, n \mid q_n = s_n \,\} \in \mathcal{U}$$

We will come back to the definition of a free ultrafilter soon; it defines 'large' index sets.

Construction of $^*\mathbb{Q}$

❸ Define equivalence classes on $\mathbb{Q}^{\mathbb{N}}$

$$[\langle q_n \rangle]_{\sim\mathcal{U}} = \{\langle s_n \rangle \mid \langle q_n \rangle \sim_{\mathcal{U}} \langle s_n \rangle \}$$

❹ Now, we can define $^*\mathbb{Q}$:

$$^*\mathbb{Q} = \{\, [\langle q_n \rangle]_{\sim\mathcal{U}} \mid \langle q_n \rangle \in \mathbb{Q}^{\mathbb{N}} \,\}$$

"set of equivalence classes"

$$= \mathbb{Q}^{\mathbb{N}}/\sim_{\mathcal{U}}$$

"quotient ring"

❺ Embed $\mathbb{Q}$ in $^*\mathbb{Q}$:

$$\forall q \in \mathbb{Q},\ q = [\langle q,q,q,\ldots \rangle]_{\sim\mathcal{U}}$$

Construction of $^*\mathbb{R}$

At this point, it is easy to construct $^*\mathbb{R}$:

Just follow the recipe for $^*\mathbb{Q}$,
but start from all sequences of real numbers $\mathbb{R}^{\mathbb{N}}$ (instead of $\mathbb{Q}^{\mathbb{N}}$).

Filters

filter, $\mathcal{U}$, on $\mathbb{N}$

- $\mathcal{U} \subset \mathcal{P}(\mathbb{N})$
- $\emptyset \notin \mathcal{U} \wedge \mathbb{N} \in \mathcal{U}$ “proper, non-empty”
- $A, B \in \mathcal{U} \Rightarrow A \cap B \in \mathcal{U}$ “closure under finite meets”
- $A \in \mathcal{U} \Rightarrow \forall B \supset A,\ B \in \mathcal{U}$ “upper set”

Filters

ultrafilter, $\mathcal{U}$, on $\mathbb{N}$

- $\mathcal{U}\subset\mathcal{P}(\mathbb{N})$
- $\emptyset\notin\mathcal{U} \wedge \mathbb{N}\in\mathcal{U}$
- $A,B\in\mathcal{U} \Rightarrow A\cap B\in\mathcal{U}$
- $A\in\mathcal{U} \Rightarrow \forall B\supset A,\ B\in\mathcal{U}$
- $\forall A\subseteq\mathbb{N}\ (A\notin\mathcal{U} \Rightarrow A^C\ (=\mathbb{N}\backslash A)\in\mathcal{U})$

Filters

principal ultrafilter, $\mathcal{U}$, on $\mathbb{N}$
(or fixed)

- $\mathcal{U}\subset\mathcal{P}(\mathbb{N})$
- $\emptyset\notin\mathcal{U} \wedge \mathbb{N}\in\mathcal{U}$
- $A,B\in\mathcal{U} \Rightarrow A\cap B\in\mathcal{U}$
- $A\in\mathcal{U} \Rightarrow \forall B\supset A,\ B\in\mathcal{U}$
- $\forall A\subseteq\mathbb{N}\ (A\notin\mathcal{U} \Rightarrow A^C\ (=\mathbb{N}\backslash A)\in\mathcal{U})$
- $\exists n\in\mathbb{N},\ \forall A\in\mathcal{U}: n\in A$

Filters

Non-principal ultrafilter, $\mathcal{U}$, on $\mathbb{N}$
(or free)

- $\mathcal{U} \subset \mathcal{P}(\mathbb{N})$
- $\emptyset \notin \mathcal{U} \wedge \mathbb{N} \in \mathcal{U}$
- $A,B \in \mathcal{U} \Rightarrow A \cap B \in \mathcal{U}$
- $A \in \mathcal{U} \Rightarrow \forall B \supset A,\ B \in \mathcal{U}$
- $\forall A \subseteq \mathbb{N}\ (A \notin \mathcal{U} \Rightarrow A^C\ (= \mathbb{N} \setminus A) \in \mathcal{U})$
- $\nexists n \in \mathbb{N},\ \forall A \in \mathcal{U}: n \in A$

Filters

Non-principal ultrafilter, $\mathcal{U}$, on $\mathbb{N}$
(or free)

- $\mathcal{U} \subset \mathcal{P}(\mathbb{N})$
- $\emptyset \notin \mathcal{U} \wedge \mathbb{N} \in \mathcal{U}$
- $A,B \in \mathcal{U} \Rightarrow A \cap B \in \mathcal{U}$
- $A \in \mathcal{U} \Rightarrow \forall B \supset A,\ B \in \mathcal{U}$

  Equivalence relation on $\mathbb{R}^{\mathbb{N}}$
- $\forall A \subseteq \mathbb{N}\ (A \notin \mathcal{U} \Rightarrow A^C\ (= \mathbb{N} \setminus A) \in \mathcal{U})$

  Trichotomy on $^*\mathbb{R}$
- Intersection of all sets in $\mathcal{U} = \emptyset$

  $\mathbb{R} \neq {}^*\mathbb{R}$

Filters

Remark: The existence of a free ultrafilter requires Zorn's Lemma, which is equivalent to the Axiom of Choice (Tarski, 1930).

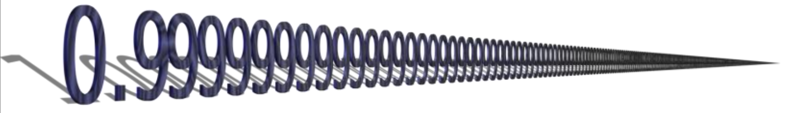
The first model of NSA only used a Fréchet filter (filter of all cofinite sets), which is free but not ultra. This gives a weaker theory, which is still interesting for constructivists.

Source: Matthew Bond, http://bondmatt.wordpress.com

Applications

Back to:

Hyperreals and 0.999…



Hyperreals and 0.999…

In classical analysis:

"0.999…" is exactly equal to (or just a different notation for) "1.000…"

Can NSA teach us something about the common intuition that 0.999… is infinitesimally smaller than unity?

intuition inside

Hyperreals and 0.999…

In $\mathbb{R}$:

$\langle 0.9, 0.99, 0.999, \ldots \rangle \sim \langle 1, 1, 1, \ldots \rangle$

Hence, $0.999\ldots = 1.000\ldots$

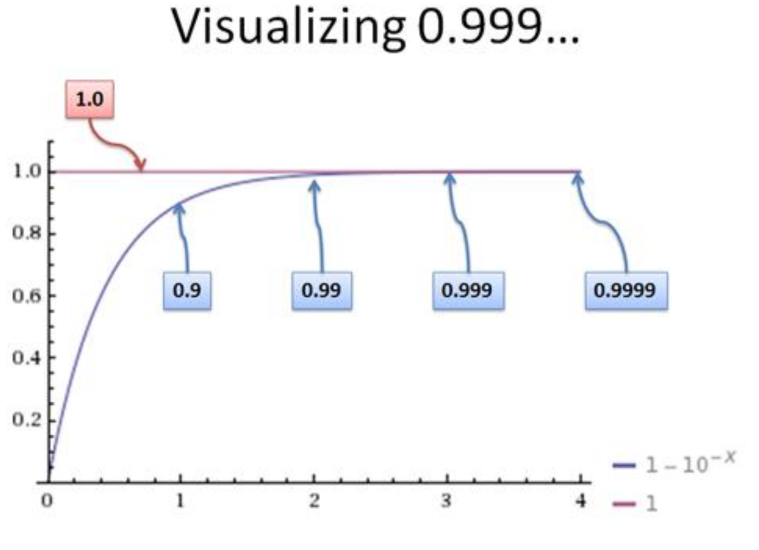
In $^*\mathbb{Q}$ (or $^*\mathbb{R}$):

$\langle 0.9, 0.99, 0.999, \ldots \rangle \nsim_{\mathcal{U}} \langle 1, 1, 1, \ldots \rangle$

Hence, $[\langle 0.9, 0.99, 0.999, \ldots \rangle]_{\sim\mathcal{U}} \neq 1$

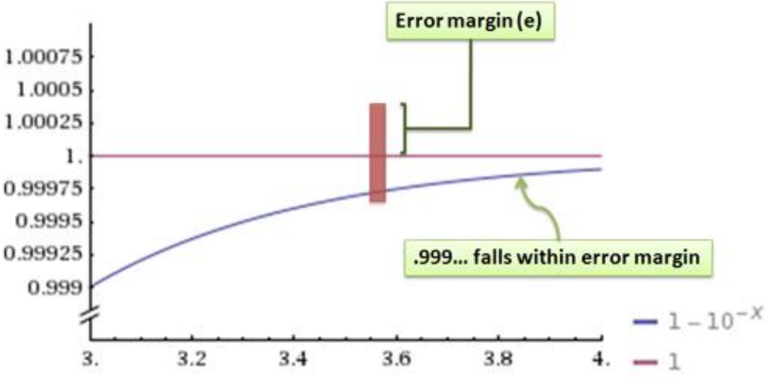
Be careful! The number $[\langle 0.9, 0.99, \ldots \rangle]_{\sim\mathcal{U}}$ is not equal to the real number 0.999… It is a hyperreal number with standard part 1; the number itself is smaller than 1 by an infinitesimal: $[\langle 0.1, 0.01, \ldots, 10^{-n}, \ldots \rangle]_{\sim\mathcal{U}}$.

Hyperreals and 0.999…



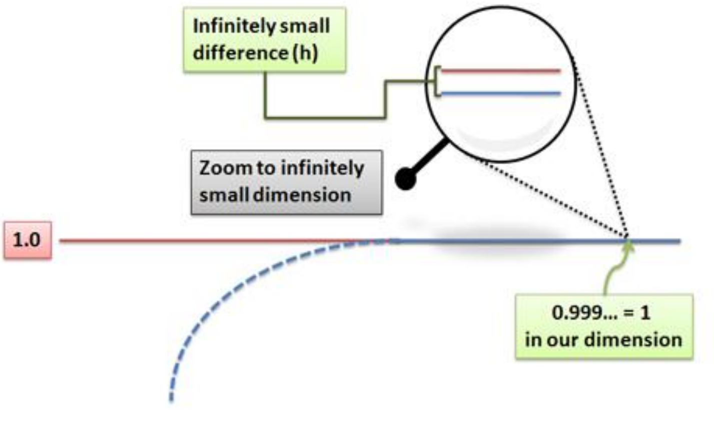
Hyperreals and 0.999…

Beating The Error Margin



Hyperreals and 0.999…

The Infinitesimal Difference



Applications

Paradoxes of infinity

Cardinality versus numerosity

Counting

Two principles for comparing sets:

Euclidean part-whole principle
If A is a proper subset of B,
then A is strictly smaller than B.

Humean one-to-one correspondence
If there is a 1-1 correspondence between A and B, then A and B are equal in size.

☺ For finite sets, these principles lead to equivalent ways of counting

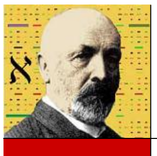
Counting

Two principles for comparing sets:

Euclidean part-whole principle
If A is a proper subset of B,
then A is strictly smaller than B.

Humean one-to-one correspondence
If there is a 1-1 correspondence between A and B, then A and B are equal in size.

☹ For infinite sets, these principles are incompatible



Cardinality

Cantor preserved one principle:

~~**Euclidean part-whole principle**~~
~~If A is a proper subset of B,~~
~~then A is strictly smaller than B.~~

Humean one-to-one correspondence
If there is a 1-1 correspondence between A and B, then A and B are equal in size.

This is the basis for ‘counting’ infinite sets, according to Cantor’s cardinality theory

Cardinal numbers

A recent paper by Mancosu asks: “Was Cantor’s theory of infinite number inevitable?”

The road least taken

Numerosity

Can we preserve the other principle?

Euclidean part-whole principle
If A is a proper subset of B,
then A is strictly smaller than B.

~~**Humean one-to-one correspondence**~~
~~If there is a 1-1 correspondence between A and B, then A and B are equal in size.~~

The answer is "Yes": this is the basic idea of Benci's numerosity theory.

Numerosity

Axioms Benci, Forti, Di Nasso (2006)

A numerosity function is a function *num* : $\mathscr{P}(\mathscr{Ord}) \to \mathcal{A}$ (ordered semi-ring) taking nonnegative values, which satisfies:

[Half Humean] If *num*(A) = *num*(B),
then A is in 1-1 correspondence with B

[Unit] $\forall$ o $\in \mathscr{Ord}$, *num*({o}) = 1

[Sum] If $A \cap B = \emptyset$,
then *num*($A \cup B$) = *num*(A) + *num*(B)

[Product] If τ is a θ-tile, then $\forall$ A $\subseteq \tau$,
$\forall$ B $\subseteq \delta < \theta^\omega$, *num*($A \otimes_\tau B$) = *num*(A).*num*(B)

Consequence: part-whole principle holds

Numerosity

Assume $\mathbb{N} = \{1,2,3,\ldots\}$ and *num*($\mathbb{N}$) = α

Basic examples

Then: *num*($\mathbb{N}\backslash\{1\}$) = α-1

num($\mathbb{Z}$) = $2\alpha+1$

num($\mathbb{N}\times\mathbb{N}$) = α^2

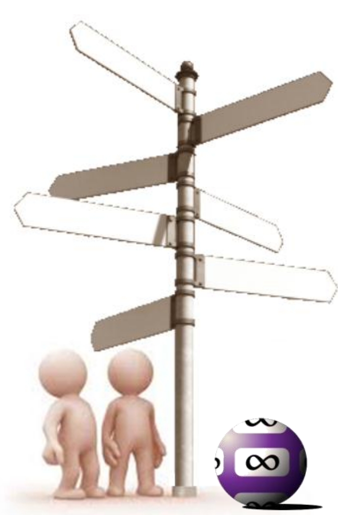
num($\{1,\ldots,n\}$) = n

Numerosity can be regarded as the ideal value of a real-valued ω-sequence (some type of non-Archimedean limit).
This idea is developed further in Alpha-theory, an axiomatic approach to NSA.

Applications

Probability

Can infinitesimals help us to build regular probability functions?



Infinitesimals and probability

My initial motivation for Regularity

- Probability theory starts from determining possible outcomes (*i.e.*, fixing a sample space).
- It seems odd that the distinction between possible and impossible can get lost once we start assigning probability values.
- Yet, it does in standard probability theory (with countable additivity).

Infinitesimals and probability

My initial motivation for NSA:

Can we get regular probability functions using infinitesimals?

As it turned out, this idea was not new: Skyrms (1980), Lewis (1980).

Infinitesimals and probability

Problems of interest:

- drawing a random number from $\mathbb{N}$ (de Finetti's infinite lottery) and conditionalizing on an even number, or on a finite subset;
- throwing darts at $[0,1]_{\mathbb{R}}$ and conditionalizing on $[0,1]_{\mathbb{Q}}$;
- a fully specific outcome (*e.g.*, 'all heads') of an ω-sequence of tosses with a fair coin.

Infinitesimals and probability

Four categories of probability theories:

Domain / Range	Standard	Non-standard
Standard ($\mathbb{R}$)	Kolmogorov	Loeb
Non-standard ($^*\mathbb{Q}$ or $^*\mathbb{R}$)		Nelson

Legend: Standard
Non-standard: internal
Non-standard: external

Infinitesimals and probability

Observe:

None of the existing approaches can describe a fair lottery on $\mathbb{N}$ or $\mathbb{Q}$ (or any countably infinite sample space).

Moreover:

None of the existing approaches can describe a fair lottery on a <u>standard</u> infinite sample space (of any cardinality) in a regular way.

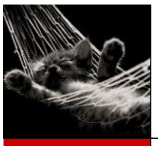
Infinitesimals and probability

Four categories of probability theories:

Domain / Range	Standard	Non-standard
Standard ($\mathbb{R}$)	Kolmogorov	Loeb
Non-standard ($^*\mathbb{Q}$ or $^*\mathbb{R}$)	NAP	Nelson

"Non-Archimedean Probability"

Together with Vieri Benci and Leon Horsten



Non-Archimedean Probability (NAP)

NAP0 Domain & Range

Probability is a function P, from $\mathscr{P}(\Omega)$ to $[0,1]_{\mathfrak{R}}$ with $\mathfrak{R}$ a superreal field

NAP1 Positivity

$\forall\, A \in \mathscr{P}(\Omega),\ \ P(A) \geq 0$

NAP2 Normalization & Regularity

$\forall\, A \in \mathscr{P}(\Omega),\ \ P(A) = 1 \Leftrightarrow A = \Omega$

NAP3 Finite Additivity (FA)

$\forall\, A,B \in \mathscr{P}(\Omega),\ A \cap B = \emptyset \Rightarrow P(A \cup B) = P(A) + P(B)$

NAP4 Non-Archimedean Continuity

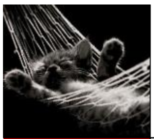
$\exists$ algebra homomorphism J: $\mathfrak{F}(\mathscr{P}_{fin}(\Omega),\mathbb{R}) \to \mathfrak{R}$
such that $\forall\, A \in \mathscr{P}(\Omega)$, $P(A) = J(\, p(\, A \mid \cdot\,)\,)$

$\forall\, \lambda \in \mathscr{P}_{fin}(\Omega) \setminus \varnothing,\ \ p(A \mid \lambda) = P(A \cap \lambda) / P(\lambda) \in \mathbb{R}$

Non-Archimedean Probability (NAP)

NAP functions:

- Are regular;
- Allow conditionalization on any possible event (*i.e.*, not on $\emptyset$);
- Are defined on the full sample space of any standard set of any cardinality (*i.e.*, no non-measurable sets);
- Obey an infinite additivity principle (not CA);
- Are external objects.



Non-Archimedean Probability (NAP)

In the special case of a fair lottery,
NAP theory is closely related to numerosity:

If

$\forall \omega_1, \omega_2 \in \Omega$, $P(\{\omega_1\}) = P(\{\omega_2\})$, "fair"

then

$\forall A \subseteq \Omega$, $P(A) = \mathit{num}(A) \,/\, \mathit{num}(\Omega)$

What's the cost?

We claim that we can assign infinitesimal probabilities to fully specific outcomes of an infinite sequence of coin tosses $= 1/\mathit{num}(2^{\mathbb{N}})$.

What about Williamson's argument?

What's the cost?

Remember these two principles?

Euclidean part-whole principle

If A is a proper subset of B, then A is strictly smaller than B. $\Rightarrow$ Regularity

Humean one-to-one correspondence

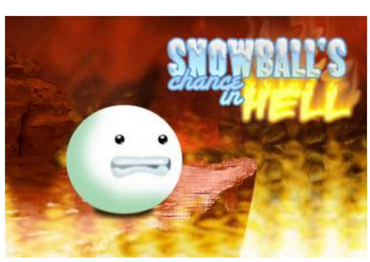
If there is a 1-1 correspondence between A and B, then A and B are equal in size.

$\Rightarrow$ Translation symmetry

☹ For infinite sets, you can't have both

Non-Archimedean Probability (NAP)

Many problems in the foundations of probability theory can be solved (or at least better understood), if we allow the probability function to have a non-Archimedean range.

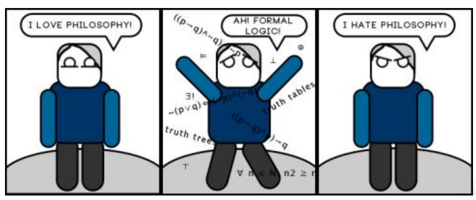


$= \varepsilon$

Applications

Formal epistemology

Can infinitesimals help us to formalize the Lockean Thesis?



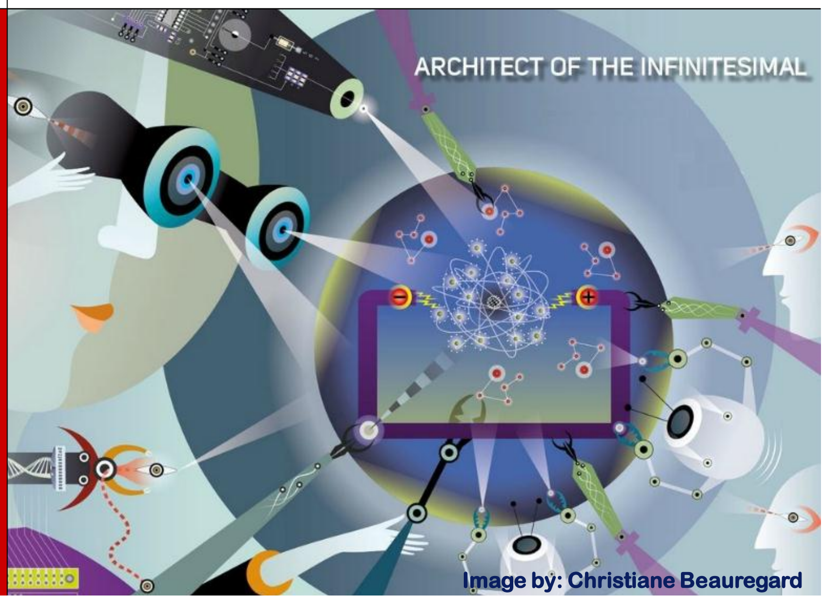
Lockean Thesis

"It is rational to believe statement x if the probability of that statement P(x) is sufficiently close to unity"

Usual formalization:
thresholds (not compatible with CP)

My idea, interpret LT as follows:
"It is rational to believe x if P(x) is indistinguishable from 1 (in a given context)"
+ Formalize this using 'relative analysis'

Relative analysis



Relative analysis

8 axioms introduce new predicate 'level' on the domain of the real numbers:

(1) $\forall$ A $\in \mathcal{P}_{fin}(\mathbb{R})$, $\exists$ coarsest level V,
s.t. $\forall$ x $\in$ A, x is observable at level V

(2) $\forall$ two levels (V_1, V_2), we can say which level is *at least as fine as* the other ('$V_1 \supseteq V_2$' or '$V_2 \supseteq V_2$')

(3) $\forall$ level V, $\exists$ x $\mathbb{R}\setminus\{0\}$: x is ultrasmall compared to V

(4) Neighbor Principle

(5) Closure Principle

(6) Stability Principle

(7) Definition Principle

(8) Density of levels

Relative analysis

Example: level of a bucket of sand

Grain	Bucket	Beach
< 1 mm	~ 1 dm	> 1 km
Microscopic	Mesoscopic	Macroscopic

Negligibly small	Appreciable size	Inconceivably large
Ultrasmall		Ultralarge
Relative infinitesimal	Standard	Relatively infinite

Lockean Thesis

In relative analysis it is easy to define this indistinguishability relation:

$$\forall\, r, s \in \mathbb{R}: \boxed{r \approx_V s} \Leftrightarrow$$
$$\exists\, u \in \mathbb{R} \text{ such that } r = s + u \text{ and}$$
$$u \text{ is ultrasmall compared to } V$$

We can use this relation to formalize LT in a soritic, context-dependent way:

LT formalized with levels:

$$B(x) \in R_V \Leftrightarrow P(x) \approx_V 1$$

Lockean Thesis

LT formalized with levels:

$$B(x) \in R_V \Leftrightarrow P(x) \approx_V 1$$

This model is called "Stratified belief"

The aggregation rule for this model is the "Stratified conjunction principle":

- the conjunction of a standard number of rational beliefs is rational;
- not necessarily so for an ultralarge number of conjuncts.

university of groningen
faculty of philosophy

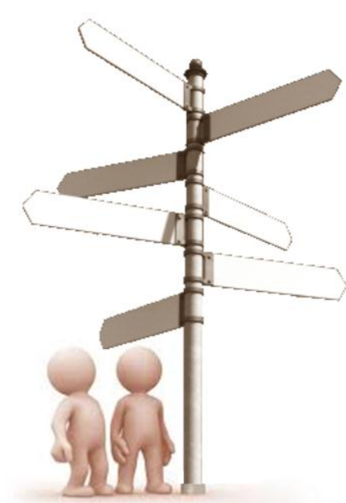
The importance of relative infinitesimals

The importance of relative infinitesimals

Applications

Philosophy of science 1/2

Rethinking the continuum



Infinitesimals and the continuum

We often use “the continuum” as a synonym for the standard reals.

However, this is but one formalization of the concept of a continuum.

Hyperreals form an alternative formalization of the concept.
Katz *et al.* propose to refer to $^*\mathbb{R}$ as a “thick continuum” ($\mathbb{R}$).

Are infinitesimals chunky?

Like the standard reals, hyperreals are infinitely divisible.
In particular: infinitesimals are infinitely divisible.

Important distinction 1:
Like a finite set, hyperfinite grids do contain a smallest non-zero element. $\Rightarrow$ Chunky

Are infinitesimals chunky?

Important distinction 2:

Besides the hyperreals, there are other systems to model infinitesimals, but which have different properties.

- Archimedes, Zeno, *et al*:
  Infinitesimals as dimensionless points.
- SIA (Bell):
  Nilsquare infinitesimals

$^{(*)}\mathbb{R}$?

Infinitesimals and the continuum

But do we need the continuum – be it $\mathbb{R}$ or $^*\mathbb{R}$ – at all?

In particular, do we need it in the empirical sciences?

*Q

Dispensing with the continuum

Sommer and Suppes, 1997

ERNA: an axiomatic approach to NSA.

Much of physics does not rely on the existence of a completed continuum; for this, the structure of *$\mathbb{Q}$ suffices.

Let's trade the axiom of completeness for an axiom that states the existence of infinitesimals: more constructive + better match to geometric intuitions.

*Q

Dispensing with the continuum

Possible objection:
Irrational numbers, such as $\sqrt{2}$ and π, are common in physics, but do not exist in *$\mathbb{Q}$.

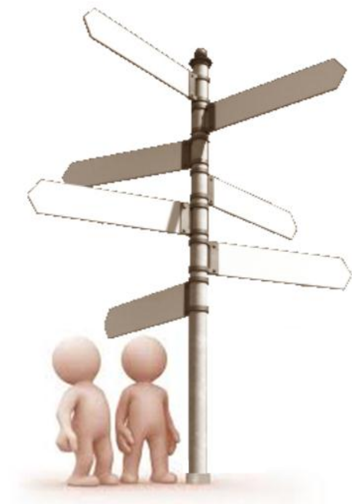
However, there are elements in *$\mathbb{Q}$ that have the same decimal places as these numbers. Hence, $\mathbb{R}$ and *$\mathbb{Q}$ are observationally equivalent.

This notion of empirical indistinguishability touches upon the very essence of 'infinitesimals'.

Applications

Philosophy of science 2/2

Hyperfinite models and determinism



Hyperfinite models

Differential equations and stochastic analysis with a hyperfinite time line.

$\mathbb{T} = \{0, \Delta t, 2\Delta t, \ldots, 1-\Delta t, 1\}$
with Δt a positive infinitesimal

Typically use non-standard measure theory (Albeverio *et al.*), but this is not really necessary (Benci *et al.*):
"In many applications of NSA, only elementary facts and techniques seem necessary."

(In-)determinism

Peano's existence theorem (PET)

$\forall$ f:[0,1]×$\mathbb{R} \to \mathbb{R}$ continuous & bounded
$\forall$ $u_0 \in \mathbb{R}$
$\exists$ u:[0,1] $\to \mathbb{R}$
such that: $\begin{cases} du(t)/dt = f(t, u(t)) \\ u(0) = u_0 \end{cases}$

Observe:

- The solution typically is not unique: indeterminism;
- in the standard proof one such solution is constructed.

(In-)determinism

Peano's existence theorem (PET)

$\forall$ f:[0,1]×$\mathbb{R} \to \mathbb{R}$ continuous & bounded
$\forall$ $u_0 \in \mathbb{R}$
$\exists$ u:[0,1] $\to \mathbb{R}$
such that: $\begin{cases} du(t)/dt = f(t, u(t)) \\ u(0) = u_0 \end{cases}$

Observe:

- Taken as functions of hyperreal numbers on a hyperfinite grid, the solution obtained in the proof of PET would be unique.

(In-)determinism

Example $\begin{cases} du(t)/dt = 3u^{2/3} \\ u(0) = 0 \end{cases}$

This generates a family of solutions ("Peano broom"):

$\forall a \in [0,1]$ $\begin{cases} u(t)=0 & 0 \leq t < a \\ u(t)=(t-a)^3 & a \leq t \leq 1 \end{cases}$

One way to obtain all the solutions:

(1) Allow for infinitesimal perturbations of the initial condition and/or the ODE.
(2) For each perturbation, follow the construction of a solution in the PET proof.
(3) Take the standard part.

(In-)determinism

Norton's dome: a failure of determinism in classical mechanics.
Same phenomenon as before, here with a 2nd order ODE.

Hyperfinite model of the dome:
is deterministic!

Hyperfinite model and standard model are empirically indistinguishable
$\Rightarrow$ Determinism is model-dependent
Cf. Werndl, 2009

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Thank you for your attention!

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