Formal Epistemology Workshop
May 29 – June 2, 2012 München

Tutorial 1
Hyperreals & Their Applications

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Overview

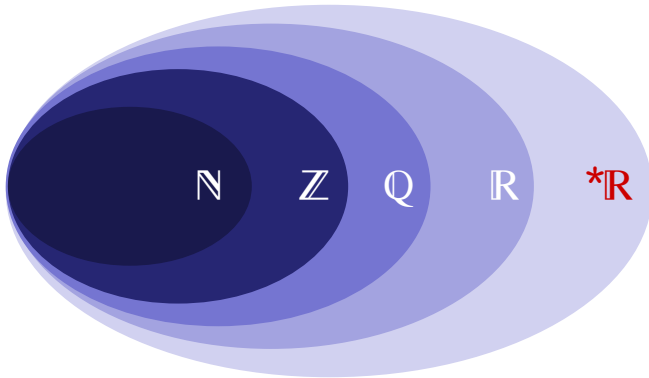
Hi, I am and an amateur…

I ♥ Infinitesimals

…and I hope that you will be, too.

Introduction

Hyperreal numbers are an extension of the real numbers, which contain infinitesimals and infinite numbers.



Introduction

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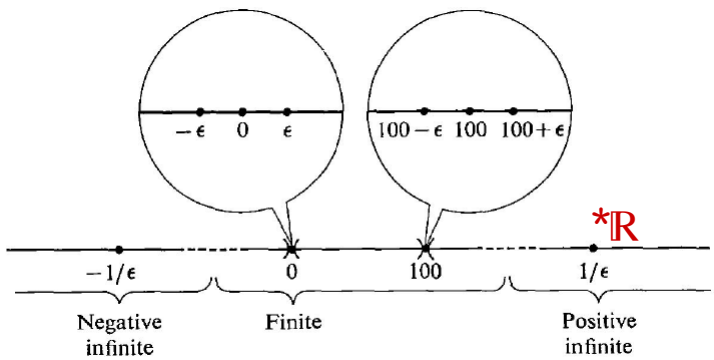


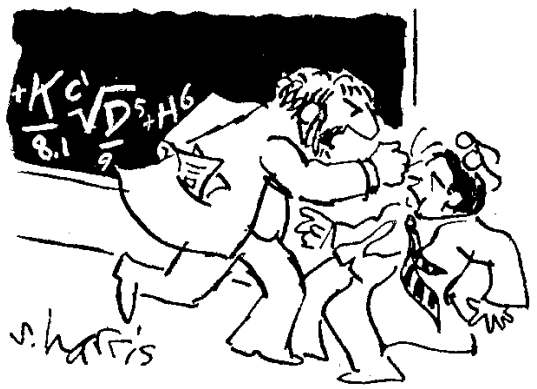
Source: Keisler (1986) Elementary calculus

Introduction

Just like standard analysis (or calculus) is the theory of the real numbers,
non-standard analysis (NSA) is the theory of the hyperreal numbers.

NSA was developed by Robinson in the 1960's and can be regarded as giving rigorous foundations for intuitions about infinitesimals that go back (at least) to Leibniz.

About this tutorial



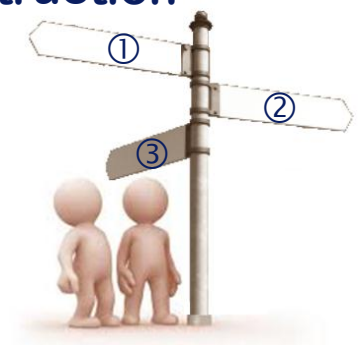
You want proof? I'll give you proof!

For references to source material, please consult the handout.

Overview

Three ways to introduce hyperreals:

① Existence proof (Model Theory)
② Axiom systems
③ Ultrapower construction



Overview

Many ways to motivate hyperreals:

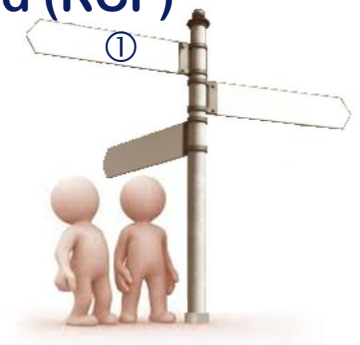
① History of calculus
② Infinitesimal intuitions
③ Paradoxes of infinity
④ Formal epistemology
⑤ Philosophy of science
⑥ & Much more

Introduction ①

Existence proofs (Model theory)

Non-standard models of:

- Peano Arithmetic (PA)
- Real Closed Field (RCF)



Peano Arithmetic (PA)

Axioms:

P1) $\forall n \neg(0=Sn)$

P2) $\forall n \forall m (\neg(n=m) \rightarrow \neg(Sn=Sm)$

P3) Induction Principle:

$$(0 \in F \wedge \forall n (n \in F \rightarrow Sn \in F)) \rightarrow \forall n (n \in F)$$

F is a set of numbers: second-order notion

A model of PA is a triple $\langle N, 0, S \rangle$:

- N an infinite set,
- $0 \in N$,
- $S : N \rightarrow N$ satisfies the axioms.

Peano Arithmetic (PA)

Intended model of PA = $\langle \mathbb{N}, 0, +1 \rangle$.

In second-order logic, any two models of PA are isomorphic to the intended model.

Richard Dedekind (1888) "*Was sind und was sollen die Zahlen*".

Peano Arithmetic (PA)

Axioms:

P1) $\forall n \neg(0=Sn)$

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P3) Induction Principle:

$$(0 \in F \wedge \forall n (n \in F \rightarrow Sn \in F)) \rightarrow \forall n (n \in F)$$

F is a set of numbers: second-order notion

Rephrase P3 in First-Order Logic (FOL):

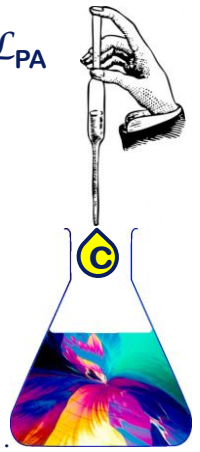
$$(\varphi(0) \wedge \forall n (\varphi(n) \rightarrow \varphi(Sn))) \rightarrow \forall n \, \varphi(n)$$

The synthesis of a non-standard model

We start from the axioms in FOL and add a new variable to $\mathcal{L}_{PA}$: c

PA = set of all true sentences of $\mathcal{L}_{PA}$

PA′ = PA $\cup$ {c>0, c>1, c>2, …}



Each finite subset of PA′ has a model

+ Compactness of FOL

$\Rightarrow$ PA′ has a model $\langle {}^*\mathbb{N}, {}^*0, {}^*+{}^*1\rangle$.

Skolem (1934) "Über die Nicht-charakterisierbarkeit der Zahlenreihe…", *Fundamenta Mathematicae* **23** p. 150–161.

The synthesis of a non-standard model

Intended model of PA = $\langle \mathbb{N}, 0, +1\rangle$.

In second-order logic, any two models of PA are isomorphic to the intended model.

Richard Dedekind (1888) "Was sind und was sollen die Zahlen".

Compactness theorem

+ Löwenheim-Skolem theorem:

In FOL, there are non-isomorphic models of PA of any cardinality.

Th. Skolem (1934) "Über die Nicht-charakterisierbarkeit der Zahlenreihe…", *Fundamenta Mathematicae* **23** p. 150–161.

$\mathbb{R}$ Real Closed Field (RCF)

Axioms for the real numbers:

R1) $\langle \mathbb{R}, +, \times \rangle$ is a field

R2) $\langle \mathbb{R}, +, \times; \leq \rangle$ is an ordered field

R3) The order is Dedekind-complete:

Every non-empty subset F of $\mathbb{R}$
with an upper bound in $\mathbb{R}$
has a least upper bound in $\mathbb{R}$.

F is a set of numbers:
second-order notion

An RCF is a field which has the same first-order properties as $\mathbb{R}$.

$\mathbb{R}$ Real Closed Field (RCF)

Construction of a non-standard model for RCF, analogous to PA:

RCF′ = RCF $\cup$ {c>0, c>1, c>2, …}

Each finite subset of RCF′ has a model
+ Compactness of FOL
$\Rightarrow$ RCF′ has a model $\langle {}^*\mathbb{R}, {}^*+, {}^*\times; {}^*\leq \rangle$.

$\mathbb{R}$ Real Closed Field (RCF)

Analogous to PA:

- The second-order axioms for $\mathbb{R}$ are categorical.
  'The' real numbers $\langle \mathbb{R}, +, \times; \leq \rangle$
- Compactness theorem
  + Löwenheim-Skolem theorem:
  The FOL axioms for an RCF have non-isomorphic models of any cardinality.
  Non-standard models $\langle {}^*\mathbb{R}, {}^*+, {}^*\times; {}^*\leq \rangle$

$\mathbb{R}$ Real Closed Field (RCF)

Archimedean property of the real numbers:

$$\forall a \in \mathbb{R}\ (a>0 \rightarrow \exists n \in \mathbb{N}\ (1/n < a))$$

$\langle \mathbb{R}, +, \times; \leq \rangle$ is the only complete Archimedean field.

Non-standard models do *not* have such a property.

$\langle {}^*\mathbb{R}, {}^*+, {}^*\times; {}^*\leq \rangle$ is a non-Archimedean totally ordered field:

${}^*\mathbb{R}$ contains infinitesimals.

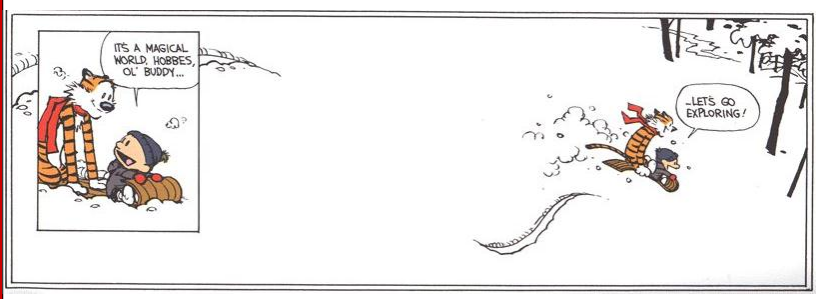
I ♥ Infinitesimals

Meaning of NS-models?

Weakness of FOL

or

New structures to explore?



Introduction ②

Axiom systems

User interface for NSA;
A tale of two universes

Axiom systems

[F]rom the beginning Robinson was very interested in the formulation of an axiom system catching his non-standard methodology.
Unfortunately he did not live to see the solution of his problem by E. Nelson presented in the 1977 paper entitled "Internal Set Theory".

[Quote from Luxemburg, 2007]

Axiom systems

Axiom systems

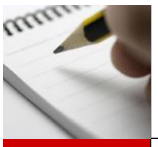
Check consistency:

Build a model
(May not look nice,
but now it works)

Axiom systems

Now it's safe to apply
the axiom system

Enjoy the ride!



Axiom systems

This is a non-exhaustive list:

- Nelson's Internal Set Theory
- Keisler's axioms for hyperreals
- Sommer & Suppes' ERNA
- Benci & Di Nasso's α-theory
- Hrbácek's relative analysis

So, you can shop around depending on the application you have in mind.

A tale of two universes

Common aspects of axiom systems

Star-map
Transfer principle
Internal/external distinction

A tale of two universes

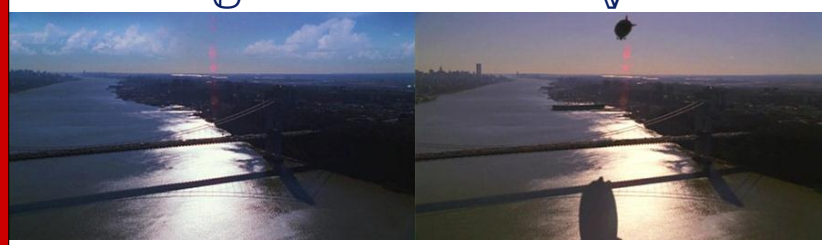
Universe: non-empty collection of mathematical objects, such as numbers, sets, functions, relations, *etc.* (all defined as sets)
Closed under: $\subseteq$, $\cup$, $\cap$, $\setminus$, $(\cdot,\cdot)$, $\times$, $\mathscr{P}(\cdot)$, $\cdots$

We further assume:
- The universe contains $\mathbb{R}$
- Transitivity (*i.e.* elements of an element of the universe are themselves elements of the universe)

A tale of two universes

Now consider two universes

$\mathbb{U}$ $\mathbb{V}$



In particular, we are interested in:

$\mathbb{U} = V(\mathbb{R})$ and $\mathbb{V} = {}^*V(\mathbb{R})$

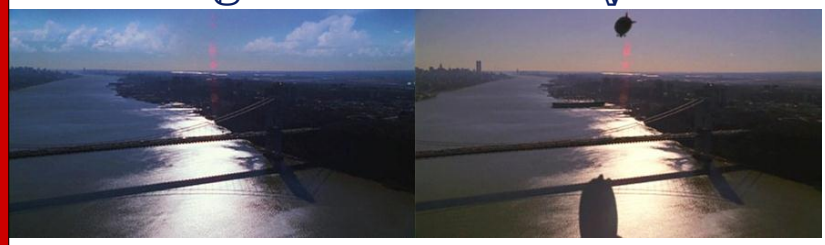
standard universe non-standard universe

(Superstructures)

Star-map

Now consider two universes

$\mathbb{U}$ $\mathbb{V}$



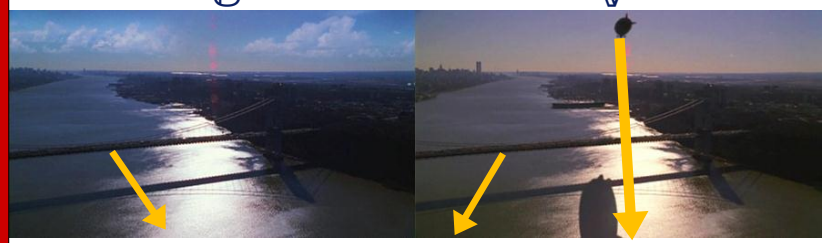
Star-map = function from $\mathbb{U}$ to $\mathbb{V}$

$$*: \mathbb{U} \to \mathbb{V}$$
$$A \mapsto *A$$

Assumptions: $\forall n \in \mathbb{N}\ (*n=n)$ and $\mathbb{N} \neq *\mathbb{N}$

Star-map

$\mathbb{U}$ $\mathbb{V}$



A,B,C,… *A,*B,*C,X,Y,Z,…

"standard" "non-standard"

Star-map

A

"standard"

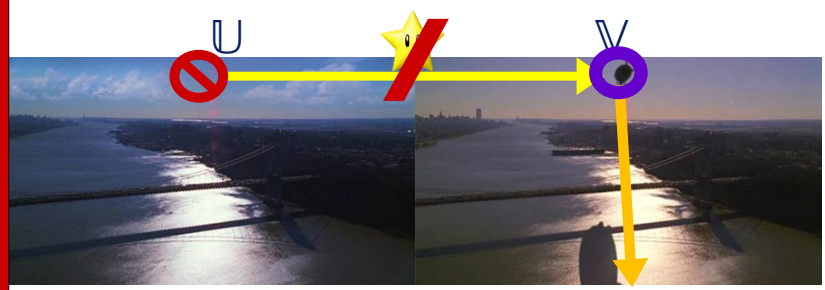
*A

"non-standard"

"internal"

E.g., $\mathbb{N}$ $\quad$ $^*\mathbb{N}$

Star-map



X

"standard"

"non-standard"

"external"

E.g., set of all infinitesimals; st; $\mathbb{N}$

Transfer principle

Consider a property of standard objects $A_1,\ldots,A_n$ expressed as an 'elementary sentence' (a bounded quantifier formula in FOL), P

Transfer Principle:

$$P(A_1,\ldots,A_n) \text{ is true}$$
$$\Leftrightarrow P(^*A_1,\ldots,^*A_n) \text{ is true}$$

Cf. Leibniz's "Law of continuity"

Transfer principle

Example 1: well-ordering of $\mathbb{N}$

"Every non-empty subset of $\mathbb{N}$ has a least element"

☹ Transfer does *not* apply to this!
(Failure due to unbounded quantifier)
Counterexample in $^*\mathbb{N}$: $^*\mathbb{N}\setminus\mathbb{N}$

"Every non-empty element of $\mathscr{P}(\mathbb{N})$ has a least element"

☺ Transfer does apply to this
Remark: $^*\mathscr{P}(\mathbb{N}) \subset \mathscr{P}(^*\mathbb{N})$

Transfer principle

Example 2: completeness of $\mathbb{R}$

"Every non-empty subset of $\mathbb{R}$ which is bounded above has a least upper bound"

☹ Transfer does *not* apply to this!

(Failure due to unbounded quantifier)

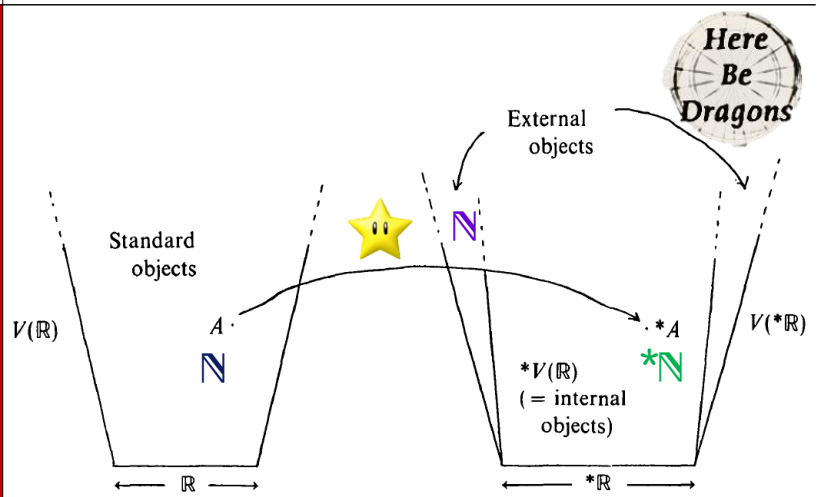
Counterexample in $^*\mathbb{R}$: halo around 0

"Every non-empty element of $\mathscr{P}(\mathbb{R})$ which is bounded above has a least upper bound"

☺ Transfer does apply to this

Remark: $^*\mathscr{P}(\mathbb{R}) \subset \mathscr{P}(^*\mathbb{R})$

Internal / external

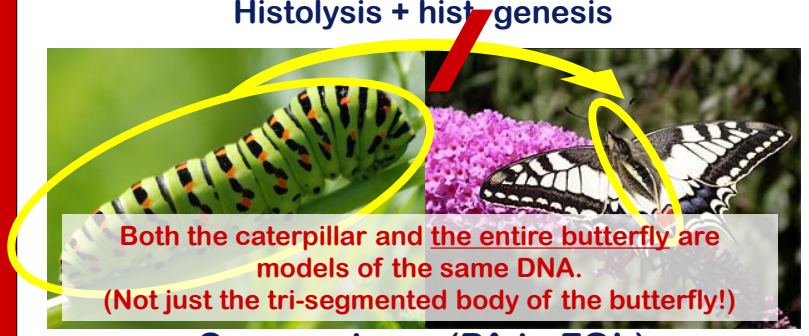


Source: Cutland (1983) p. 538

Parable of the butterfly

Same DNA

Histolysis + histogenesis



Same axioms (PA in FOL)

Standard model	Non-standard model (countable)
Order type: ω	Order type: $\omega + (^*\omega + \omega) \cdot \eta$
$\mathbb{N}$ within itself is a model of PA	$\mathbb{N}^*$ has an initial segment isomorphic to $\mathbb{N}$, but this $^\sigma\mathbb{N}$ is not a model of PA; $\mathbb{N}^*$ is.

Parable of the butterfly

Moral:

The butterfly does not remember that it once was a caterpillar.

Likewise, a non-standard model does not 'know' that it is anything other than the standard model.

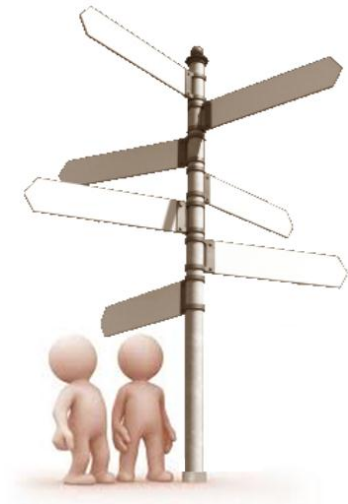
(Its non-isomorphic structure cannot be distinguished from the standard one by any elementary sentence.)

Applications

What are hyperreals used for?

Applications
in many branches
of science;

often interesting
philosophical
dimension!



Applications of hyperreals

The earliest application of $^*\mathbb{R}$:
Making proofs about $\mathbb{R}$
easier and/or shorter.

Lagrange (1811) *Méchanique Analytique*:
"Once one has duly captured the spirit of this system [*i.e.*, infinitesimal calculus], and has convinced oneself of the correctness of its results [...], one can then exploit the infinitely small as a reliable and convenient tool so as to shorten and simplify proofs."

Applications of hyperreals

The earliest application of $^*\mathbb{R}$:
Making proofs about $\mathbb{R}$
easier and/or shorter.

This still is a major application.
See for instance the blog by Field-medalist Terence Tao.

But, it is far from the only one!

Applications of hyperreals

Related to Mathematics

- History of mathematics
  How could results, now considered to be obtained in a non-rigorous way, nevertheless be correct?
- Didactics
- Non-standard measure theory
  Special case: probability theory

Applications of hyperreals

Related to Physics

- Intuitive use of infinitesimals
  Can be given a rigorous basis
- Understanding 'classical limits'
  quantum vs class.: infinitesimal $\hbar$
  relativity vs class.: infinitesimal 1/c
- Deterministic models
  Hyperfinite model of Norton's dome is deterministic

Applications of hyperreals

Two important observations

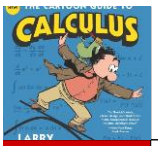
(1) Nearly no application of NSA uses technically advanced parts of the theory ☺

(2) Nearly all applications of NSA use hyperfinite models

Applications

History of mathematics

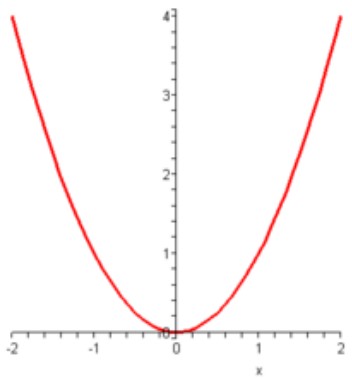
Calculus:
from Leibniz
to NSA

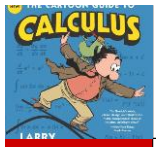


Calculus

Consider a function $y = x^2$

Q: What is dy/dx?
A: $dy/dx = 2x$



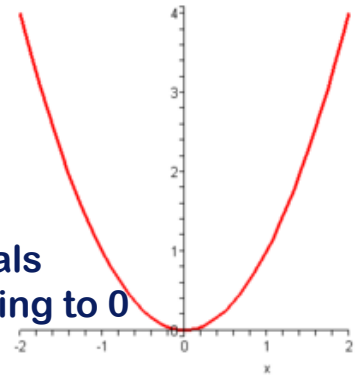


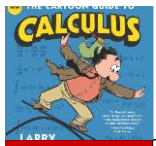
Calculus

Consider a function $y = x^2$

Q: What does dy/dx mean?

Speed
Derivative
Rate of change
Slope of tangent
Quotient of infinitesimals
Limit of $\Delta y/\Delta x$ for Δx going to 0



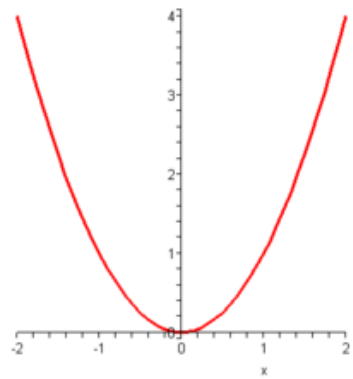


Calculus

Consider a function $y = x^2$

Q: What does dy/dx mean?

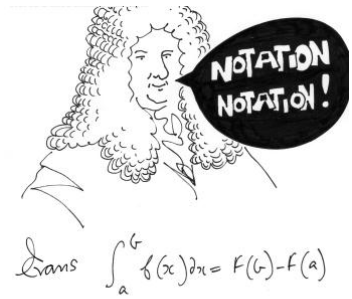
A1: Leibniz
A2: Newton
A3: Weierstrass
A4: Robinson



Leibniz's calculus

Consider a function $y = x^2$

Q: What does dy/dx mean?
A1: Quotient of infinitesimals



...as Leibniz said to Newton...

Leibniz's calculus

Consider a function $y = x^2$

Q: What does dy/dx mean?
A1: Quotient of infinitesimals

Quotient of finite differences

$$\Delta y/\Delta x = \frac{y(x+\Delta x) - y(x)}{\Delta x} = \frac{(x+\Delta x)^2 - x^2}{\Delta x} = 2x + \Delta x$$

"Law of continuity" ("As above, so below")

Quotient of infinitesimal differences

$dy/dx = 2x + dx$

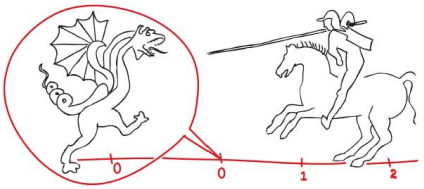
$= 2x$

"Law of transcendental homogeneity"

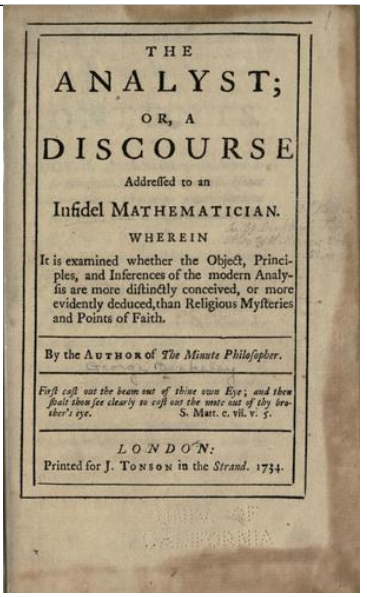
Criticism by Berkeley

The Analyst (1734) Criticism of infinitesimals

"Ghosts of departed quantities"



Source: Blaszczyk *et al.* (2012)



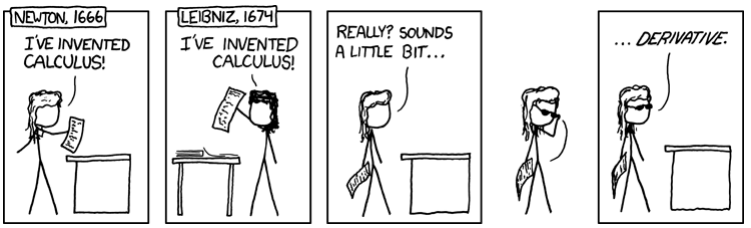
THE
ANALYST;
OR, A
DISCOURSE
Addressed to an
Infidel MATHEMATICIAN.
WHEREIN
It is examined whether the Object, Principles, and Inferences of the modern Analysis are more distinctly conceived, or more evidently deduced, than Religious Mysteries and Points of Faith.

By the AUTHOR of *The Minute Philosopher*.

First cast out the beam out of thine own Eye; and then shalt thou see clearly to cast out the mote out of thy brother's eye. S. Matt. c. vii. v. 5.

LONDON:
Printed for J. TONSON in the *Strand*. 1734.

Newton's calculus



Source: http://xkcd.com/626/

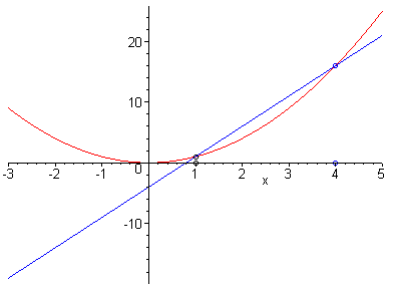
Newton's calculus

Consider a function $y = x^2$

Q: What does dy/dx mean?
A2: Fluxion of fluent quantity

Kinetic notion
of limit

Limit of a quotient
of finite differences



Standard calculus

Consider a function $y = x^2$

Q: What does dy/dx mean?
A3: Classical limit

$$dy/dx = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{y(x+\Delta x) - y(x)}{\Delta x}$$

Recall: $\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = L$

$$\Leftrightarrow \forall \varepsilon > 0 \in \mathbb{R}, \exists \delta > 0 \in \mathbb{R},$$
$$\forall \Delta x \in \mathbb{R}\ (0 < |\Delta x| < \delta \Rightarrow |\Delta y/\Delta x - L| < \varepsilon)$$

Non-standard analysis

Consider a function $y = x^2$

Q: What does dy/dx mean?
A4: Standard part of a quotient
of infinitesimals in $^*\mathbb{R}$

$$\frac{dy}{dx} = \operatorname{st} \frac{{}^*y(x+\delta) - {}^*y(x)}{\delta} = \operatorname{st}(2x + \delta) = 2x$$

(where δ is an infinitesimal in $^*\mathbb{R}$)

I ♥ Infinitesimals

Applications

Our last topic of today:
Hyperreals and 0.999…

Hyperreals and 0.999…

In classical analysis:

"0.999…" is exactly equal to (or just a different notation for) "1.000…"

Can NSA teach us something about the common intuition that 0.999…is infinitesimally smaller than unity?

Vi Hart:

http://www.youtube.com/watch?v=TINfzxSnnIE