Hyperreals and Their Applications

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Overview

Hyperreal numbers are an extension of the real numbers, which contain infinitesimals and infinite numbers. The set of hyperreal numbers is denoted by ${}^*\mathbb{R}$ or $\mathbb{R}^*$; in these notes, I opt for the former notation, as it allows us to read the *-symbol as the prefix 'hyper-'. Just like standard analysis (or calculus) is the theory of the real numbers, non-standard analysis (NSA) is the theory of the hyperreal numbers. NSA was developed by Robinson in the 1960's and can be regarded as giving rigorous foundations for intuitions about infinitesimals that go back to Leibniz (at least).

This document is prepared as a handout for two tutorial sessions on "Hyperreals and their applications", presented at the Formal Epistemology Workshop 2012 (May 29–June 2) in Munich. It is set up as an annotated bibliography about hyperreals. It does not aim to be exhaustive or to be formally precise; instead, its goal is to direct the reader to relevant sources in the literature on this fascinating topic. The document consists of two parts: sections 1–3 introduce NSA from different perspectives and sections 4–9 discuss applications, with an emphasis on topics that may be of interest to formal epistemologists and to philosophers of mathematics or science.

Part 1: Introducing the hyperreals

Abstract

NSA can be introduced in multiple ways. Instead of choosing one option, these notes include three introductions. Section 1 is best-suited for those who are familiar with logic, or who want to get a flavor of model theory. Section 2 focuses on some common ingredients of various axiomatic approaches to NSA, including the star-map and the Transfer principle. Section 3 explains the ultrapower construction of the hyperreals; it also includes an explanation of the notion of a free ultrafilter.

1 Existence proofs of non-standard models

1.1 Non-standard models of arithmetic

The second-order axioms for arithmetic are categoric: all models are isomorphic to the intended model $\langle \mathbb{N}, 0, +1 \rangle$. Dedekind was the first to prove this [Dedekind, 1888b]; his 'rules' for arithmetic were turned into axioms a year later by Peano, giving rise to what we now call "Peano Arithmetic" (PA) [Peano, 1889].

The first-order axioms for arithmetic are non-categoric: there exist non-standard models $\langle {}^*\mathbb{N}, {}^*0, {}^*\!+{}^*1 \rangle$ that are not isomorphic to $\langle \mathbb{N}, 0, +1 \rangle$. Skolem proved this based on the Compactness property of first-order logic (FOL) [Skolem, 1934]. With the Löwenheim-Skolem theorem, it can be proven that there exist models of any cardinality. ${}^*\mathbb{N}$ contains finite numbers as well as infinite numbers. We now call ${}^*\mathbb{N}$ a set of hypernatural numbers.

For a discussion of the order-type of countable non-standard models of arithmetic, see *e.g.* [Boolos et al., 2007, Chapter 25, p. 302–318] and [McGee, 2002]. More advanced topics can be found in this book: [Kossak and Schmerl, 2006].

1.2 Non-standard models of real closed fields

The second-order axioms for the ordered field of real numbers are categoric: all models are isomorphic to the intended model $\langle \mathbb{R}, +, \times, \leq \rangle$.

A "real closed field" (RCF) is a field which has the same first-order properties as $\mathbb{R}$. Robinson realized that Skolem's existence proof of non-standard models of arithmetic could be applied RCFs too. He thereby founded the field of non-standard analysis (NSA) [Robinson, 1966]. The axioms for RCFs (always in FOL) are non-categoric: there exist non-standard models $\langle {}^*\mathbb{R}, {}^*+, {}^*\times, {}^*\leq \rangle$ that are not isomorphic to $\langle \mathbb{R}, +, \times, \leq \rangle$. With the Löwenheim-Skolem theorem it can be proven that there exist models of any cardinality; in particular, there are countable models (*cf.* Skolem 'paradox' [Skolem, 1922]).

For the standard real numbers the Archimedean property holds:

$$\forall a \in \mathbb{R} \left(a > 0 \to \exists n \in \mathbb{N} \left(\frac{1}{n} < a \right) \right)$$

In particular, $\langle \mathbb{R}, +, \times, \leq \rangle$ is the only complete Archimedean field.

Non-standard models do *not* have such a property. $\langle {}^*\mathbb{R}, {}^*+, {}^*\times, {}^*\leq \rangle$ is a non-Archimedean ordered field. In other words: ${}^*\mathbb{R}$ contains infinitesimals. ${}^*\mathbb{R}$ contains finite, infinite and infinitesimal numbers; we call ${}^*\mathbb{R}$ a set of hyperreal numbers.

1.3 Other non-standard models

Analogous techniques can also be applied to the first-order formalization of the rational numbers to obtain non-standard models; see *e.g.* [Skyrms, 1980, Appendix 4, p. 177–187].

1.4 Meaning of non-standard models

On the one hand, non-standard models can be regarded as a sign of the relative weakness of FOL. On the other hand, they can be regarded as a fruitful new tool for mathematical discovery or interesting new objects of study in their own right. Some discussion along these lines can be found in [Gaifman, 2004] and [McGee, 2002].

2 Axiom systems

The original presentation of NSA by Robinson relied on advanced logic, which does not help to make the topic accessible to many. According to Luxemburg:

> "[F]rom the beginning Robinson was very interested in the formulation of an axiom system catching his non-standard methodology. Unfortunately he did not live to see the solution of his problem by E. Nelson presented in the 1977 paper entitled 'Internal Set Theory'." [Luxemburg, 2007, p. xi]

Nowadays, there exist many different axiom systems for NSA, although they differ in strength and scope. This is a non-exhaustive list of some of the approaches:

- Nelson's Internal Set Theory (IST) [Nelson, 1977];
- Hrbáček's axiomatic foundations [Hrbacek, 1978];
- Keisler's axioms for hyperreals [Keisler, 1963];

- Sommer & Suppes' elementary recursive nonstandard analysis (ERNA) [Sommer and Suppes, 1997];
- Benci & Di Nasso's alpha-theory [Benci and Di Nasso, 2003a];
- Hrbáček's relative or stratified analysis [Hrbacek, 2007, Hrbacek et al., 2010].

Various axiom systems may be suggestive of rather different applications and we will return to some of them in Part 2.

Instead of looking into the details of a particular axiom system, we look at some tools that are important in (nearly) all axiom systems. For this section, we follow a structure similar to that of [Benci et al., 2006a, section 1], which is very general and applies to many approaches to NSA. See also: [Cutland, 1983, section 1.2]. We do not mention saturation here, but check the final remark at the end of section 3.

2.1 Universe

By a universe, we mean a non-empty collection of mathematical objects, such as numbers, sets, functions, relations, *etc.*—all of which can be defined as sets by working in ZFC. This collection is assumed to be closed under the following relations and operations on sets: $\subseteq$, $\cup$, $\cap$, $\setminus$, $(\cdot,\cdot)$, $\times$, $\mathcal{P}(\cdot)$, $\therefore$. Furthermore, we assume that the universe contains $\mathbb{R}$ and that it obeys transitivity (*i.e.*, elements of an element of the universe are themselves elements of the universe).

In particular, we are interested in the standard universe, which is the superstructure $V(\mathbb{R})$, and a non-standard universe, ${}^*V(\mathbb{R})$.

2.2 Star-map

The star-map (or hyperextension) is a function from the standard universe to the non-standard universe.

$$\begin{aligned} * : V(\mathbb{R}) &\to {}^*V(\mathbb{R}) \\ A &\mapsto {}^*A \end{aligned}$$

We assume that $\forall n \in \mathbb{N}({}^*n = n)$ and that $\mathbb{N} \neq {}^*\mathbb{N}$.

2.3 Internal and external objects

It is important to realize that the star-map does *not* produce all the objects in the superstructure of ${}^*\mathbb{R}$; it only maps to the internal objects, which live in ${}^*V(\mathbb{R}) \subsetneq V({}^*\mathbb{R})$.

In the tutorial, I illustrate the internal-external distinction with a biological analogy (comparing caterpillar versus butterfly on the one hand, to

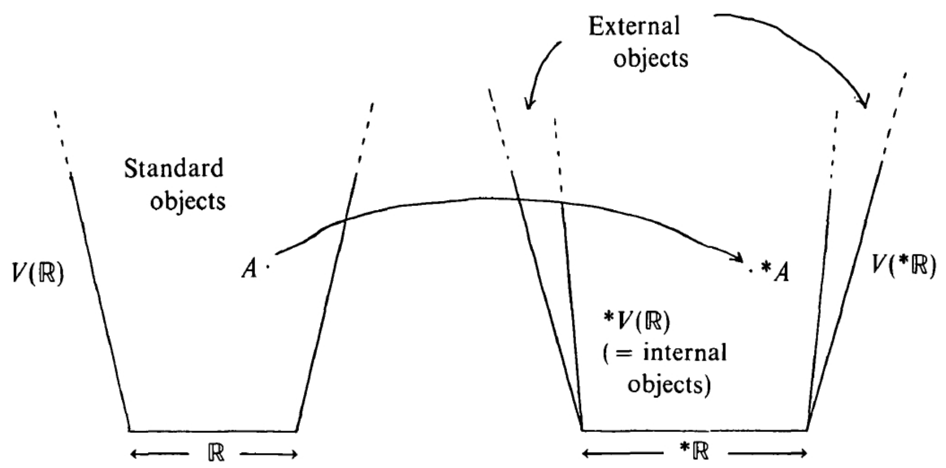

Figure 1: Schematic representation of the standard and non-standard superstructures considered in NSA. Image source: [Cutland, 1983, p. 538].

$\mathbb{N}$ versus $^*\mathbb{N}$ on the other hand), but I am too embarrassed to put this into writing. See Figure 1 instead.
Some examples of internal objects ($\in {}^*V(\mathbb{R})$):

- any element of $^*\mathbb{R}$, so in particular any element of $\mathbb{N}$ or $\mathbb{R}$;
- any hyperfinite set, such as $\{1, \ldots, N\}$ with $N \in {}^*\mathbb{N}$ (which can be obtained via the hyperextension of a family of finite sets);
- the hyperextensions of standard sets, such as $^*\mathbb{N}$ and $^*\mathbb{R}$;
- the hyperpowerset of a standard set, A: $^*\mathcal{P}(A)$, which is the collection of all *internal* subsets of $^*A$.

Some examples of external objects ($\in V(^*\mathbb{R}) \setminus {}^*V(\mathbb{R})$):

- elementwise copies of standard, infinite sets (notation for the elementwise copy of A in the nonstandard universe: $^\sigma A$), such as $^\sigma\mathbb{N}$ or $^\sigma\mathbb{R}$ (due to the embedding of $\mathbb{N}$ and $\mathbb{R}$ in $^*\mathbb{R}$, the $^\sigma$-prefix is often dropped);
- the complements of previous sets, such as $^*\mathbb{N} \setminus {}^\sigma N$ and $^*\mathbb{R} \setminus {}^\sigma R$;
- the 'halo' or 'monad' of any real number, r: $hal(r) = \{R \in {}^*\mathbb{R} \mid |r - R| \text{ is infinitesimal}\}$—in particular $hal(0)$, which is the set of all infinitesimals;
- the standard part function st (also known as the shadow), which maps a (bounded) hyperreal number to the unique real number that is infinitesimally close to it [Goldblatt, 1998, section 5.6];

- the full powerset of the hyperextension of a standard, infinite set, A: $\mathcal{P}({}^*A)$, which is the collection of *all* subsets of ${}^*A$, both internal and external.

2.4 Transfer principle

Consider some standard objects $A_1, \ldots, A_n$ and consider a property of these objects expressed as an 'elementary sentence' (a bounded quantifier formula in FOL): $P(A_1, \ldots, A_n)$. Then, the Transfer Principle says:

$$P(A_1, \ldots, A_n) \text{ is true } \Leftrightarrow P({}^*A_1, \ldots, {}^*A_n) \text{ is true.}$$

Observe: this is an implementation of Leibniz's "Law of continuity" in NSA (*cf.* section 5).

Example 1: well-ordering of $\mathbb{N}$ Consider the following sentence: "Every non-empty subset of $\mathbb{N}$ has a least element." Transfer does *not* apply to this, because the sentence is not 'elementary'. Indeed, we can find a counterexample in ${}^*\mathbb{N}$: the set of infinite hypernatural numbers, ${}^*\mathbb{N} \setminus \mathbb{N}$, does not have a least element. (Of course, this is an external object.)

If we rephrase the well-ordering of $\mathbb{N}$ as follows: "Every non-empty element of $\mathcal{P}(\mathbb{N})$ has a least element", then we *can* apply Transfer to this. The crucial observation to make here is that ${}^*\mathcal{P}(\mathbb{N}) \subsetneq \mathcal{P}({}^*\mathbb{N})$.

Example 2: completeness of $\mathbb{R}$ Consider the following sentence: "Every non-empty subset of $\mathbb{R}$ which is bounded above has a least upper bound." Again, Transfer does not apply to this, for the same reason as in Example 2. A counterexample in ${}^*\mathbb{R}$ is $hal(0)$, the set of infinitesimals. (Again, an external object.)

If we rephrase the completeness property of $\mathbb{R}$ as follows: "Every non-empty element of $\mathcal{P}(\mathbb{R})$ which is bounded above has a least upper bound", then we can apply Transfer to it. Similar as before, the crucial remark is that ${}^*\mathcal{P}(\mathbb{R}) \subsetneq \mathcal{P}({}^*\mathbb{R})$.

3 Ultrapower construction of the hyperreals

Before we consider the construction of ${}^*\mathbb{R}$, we first review the constructions of $\mathbb{R}$ and ${}^*\mathbb{Q}$, since these procedures are analogous to a large extent. To emphasize the analogy, we focus on the construction of $\mathbb{R}$ from Cauchy sequences of rational numbers. (This construction is due to Cantor [Cantor, 1872], whereas Dedekind constructed the reals based on cuts [Dedekind, 1888a].)

3.1 Construction of $\mathbb{R}$

$\mathbb{Q}^{\mathbb{N}}$ is the set of all functions $\mathbb{N} \to \mathbb{Q}$; in other words, it is the set of all rational-valued ω-sequences. We refer to a particular sequence in $\mathbb{Q}^{\mathbb{N}}$ by its initial elements $\langle q_1, q_2, q_3, \ldots \rangle$ or by its general element (at position n) $\langle q_n \rangle$.

We could try to regard each element of $\mathbb{Q}^{\mathbb{N}}$ as a new number. Suppose that we endow $\mathbb{Q}^{\mathbb{N}}$ with addition and multiplication, by defining these operations element-wise starting from the corresponding operations on $\mathbb{Q}$. Then, we would obtain, for instance:

$$\langle 0, 1, 1, 1, \ldots \rangle \times \langle 1, 0, 0, 0, \ldots \rangle = \langle 0, 0, 0, 0, \ldots \rangle,$$

which demonstrates that the notion of a multiplicative inverse is not well-defined on $\mathbb{Q}^{\mathbb{N}}$ and hence that $\mathbb{Q}^{\mathbb{N}}$ fails to form a field. Although $\mathbb{Q}^{\mathbb{N}}$ does form a ring, this is insufficient for it to be regarded as a collection of numbers.

The construction of $\mathbb{R}$ starts from a subset of $\mathbb{Q}^{\mathbb{N}}$ and proceeds in five steps:

1. Consider a particular subset of $\mathbb{Q}^{\mathbb{N}}$:

$$\begin{aligned} C =& \text{set of Cauchy sequences in } \mathbb{Q}^{\mathbb{N}} \\ =& \{\ \langle q_n \rangle \mid \forall n \in \mathbb{N}(q_n \in \mathbb{Q}) \wedge \\ & \forall \epsilon > 0 \in \mathbb{Q}, \exists N \in \mathbb{N}, \forall n, m > N \in \mathbb{N}\ (|q_m - q_n| < \epsilon)\}. \end{aligned}$$

2. Define the following equivalence relation $\sim$ on C:

$$\begin{aligned} \forall \langle q_n \rangle, \langle s_n \rangle \in C : & \langle q_n \rangle \sim \langle s_n \rangle \Leftrightarrow \\ & \forall \epsilon > 0 \in \mathbb{Q}, \exists N \in \mathbb{N}, \forall n > N \in \mathbb{N}\ (|q_n - s_n| < \epsilon). \end{aligned}$$

3. Define the equivalence classes on C based on equivalence relation $\sim$:

$$\forall \langle q_n \rangle \in C : [\langle q_n \rangle]_\sim = \{\ \langle s_n \rangle \in C \mid \langle q_n \rangle \sim \langle s_n \rangle\}.$$

4. Define $\mathbb{R}$ as the set of equivalence classes on C:

$$\mathbb{R} = \{\ [\langle q_n \rangle]_\sim \mid \langle q_n \rangle \in C\}.$$

   This can be written in quotient ring notation as follows: $\mathbb{R} = C/\sim$.

5. As a final step, we embed $\mathbb{Q}$ in $\mathbb{R}$:

$$\forall q \in \mathbb{Q} : q = [\langle q, q, q, \ldots \rangle]_\sim.$$

3.2 Construction of ${}^*\mathbb{Q}$

The previous construction of $\mathbb{R}$ may make us wonder whether it is possible to consider a different kind of equivalence relation, defined on all of $\mathbb{Q}^{\mathbb{N}}$. The answer is "yes", as we will now see in the construction of ${}^*\mathbb{Q}$.

The construction of ${}^*\mathbb{Q}$ proceeds in five steps, analogous to the construction of $\mathbb{R}$:

1. Consider *all* of $\mathbb{Q}^{\mathbb{N}}$.

2. Fix a free ultrafilter on $\mathbb{N}, \mathcal{U}$ (see section 3.2.1), and define the following equivalence relation $\sim_{\mathcal{U}}$ on $\mathbb{Q}^{\mathbb{N}}$:

$$\begin{aligned}\forall \langle q_n \rangle, \langle s_n \rangle \in \mathbb{Q}^{\mathbb{N}} : &\langle q_n \rangle \sim_{\mathcal{U}} \langle s_n \rangle \Leftrightarrow \\ &\{n \mid q_n = s_n\} \in \mathcal{U}.\end{aligned}$$

3. Define the equivalence classes on $\mathbb{Q}^{\mathbb{N}}$ based on equivalence relation $\sim_{\mathcal{U}}$:

$$\forall \langle q_n \rangle \in \mathbb{Q}^{\mathbb{N}} : [\langle q_n \rangle]_{\sim_{\mathcal{U}}} = \{\ \langle s_n \rangle \in \mathbb{Q}^{\mathbb{N}} \mid \langle q_n \rangle \sim_{\mathcal{U}} \langle s_n \rangle\}.$$

4. Define ${}^*\mathbb{Q}$ as the set of equivalence classes on $\mathbb{Q}^{\mathbb{N}}$:

$${}^*\mathbb{Q} = \{\ [\langle q_n \rangle]_{\sim_{\mathcal{U}}} \mid \langle q_n \rangle \in \mathbb{Q}^{\mathbb{N}}\}.$$

   This can be written in quotient ring notation as follows: ${}^*\mathbb{Q} = \mathbb{Q}^{\mathbb{N}} / \sim_{\mathcal{U}}$.

5. As a final step, we embed $\mathbb{Q}$ in ${}^*\mathbb{Q}$:

$$\forall q \in \mathbb{Q} : q = [\langle q, q, q, \ldots \rangle]_{\sim_{\mathcal{U}}}.$$

3.2.1 Free ultrafilter on $\mathbb{N}$

In the above construction, we needed a free (non-principal) ultrafilter on $\mathbb{N}$; we will define this important concept now.

Intuitively, a (proper) **filter** on a set X is a collection of subsets of X that are "large enough". The meaning of "large enough" is given by its formal definition:

$$\begin{aligned}
\mathcal{F} \text{ is a filter on } X \Leftrightarrow & \\
& \mathcal{F} \subset \mathcal{P}(X) \quad \wedge \\
& \emptyset \notin \mathcal{F} \quad \wedge && \text{('proper')} \\
& X \in \mathcal{F} \quad \wedge && \text{('non-empty')} \\
& A, B \in \mathcal{F} \Rightarrow A \cap B \in \mathcal{F} \quad \wedge && \text{('closed under finite meets')} \\
& (A \in \mathcal{F} \wedge B \supseteq A) \Rightarrow B \in \mathcal{F}. && \text{('upper set')}
\end{aligned}$$

Now we define an **ultrafilter**:

$$\begin{aligned}\mathcal{F} \text{ is an ultrafilter on } X \Leftrightarrow \\ & \mathcal{F} \text{ is a filter on } X \quad \wedge \\ & \forall A \subseteq X (A \notin \mathcal{F} \Rightarrow A^C (= X \setminus A) \in \mathcal{F}).\end{aligned}$$

A filter $\mathcal{F}$ is **principal** (or 'fixed') if: $\exists x_0 \in X, \forall A \in \mathcal{F}(x_0 \in A)$. A filter $\mathcal{F}$ is **free** if it is *not* principal, or equivalently: if the intersection of all the sets in $\mathcal{F}$ is empty.

For an infinite set X, its **Fréchet filter** is the filter that consists of all the cofinite subsets of X. Such a filter is free, but it is not an ultrafilter.

Finally, we define a **free ultrafilter**:

$$\begin{aligned}\mathcal{F} \text{ is a free ultrafilter on } X \Leftrightarrow \\ & \mathcal{F} \text{ is an ultrafilter on } X \quad \wedge \\ & \mathcal{F} \text{ is free.}\end{aligned}$$

Given the ultrafilter condition, the last condition implies that $\mathcal{F}$ contains no finite sets, which means that it does contains all cofinite sets; in other words: an ultrafilter is free if and only if it contains the Fréchet filter.

The existence of a free ultrafilter can be proven using Zorn's lemma; so the ultrapower construction crucially depends on the Axiom of Choice. For a further discussion of filters, including free ultrafilters, see *e.g.* [Goldblatt, 1998, p. 18–21] and [Cutland, 1983, section 1.1]. For an introduction to the meaning and application of ultrafilters, see [Komjáth and Totik, 2008].

3.3 Construction of $^*\mathbb{R}$

Replacing each instance of "$\mathbb{Q}$" in the preceding construction of $^*\mathbb{R}$ by "$\mathbb{R}$" yields the construction of $^*\mathbb{R}$.

The ultrapower construction of $^*\mathbb{R}$ is very important in NSA and there are many good sources for it. For this construction, including a discussing of free ultrafilters, see *e.g.* [Loeb and Wolff, 2000, p. 7–10]. The construction can also be found in: [Goldblatt, 1998, p. 23–27]. More basic introductions to this construction can be found in [Hoyle, 2007a,b] and in the handouts by Jensen [Jensen, 2007]. A more technical presentation can be found in [Luxemburg, 1973, section 3].

Some remarks:

- The ultrapower construction of $^*\mathbb{Q}$ or $^*\mathbb{R}$ can equivalently be expressed in terms of a Boolean prime ideal, which is the dual notion of a free ultrafilter.

- To see how the ultrapower construction is related to the existence proof of non-standard models using the Compactness theorem (section 1), observe that one way to prove the Compactness theorem is based on the notion of an ultraproduct (*cf.* [Goldblatt, 1998, p. 11]).

- The ultraproduct construction is a general method in model theory: see [Keisler, 2010] (including the references in the introduction) for more information.

- Not every hyperreal number system can be obtained from an ultrapower construction, but every hyperreal number system is isomorphic to a limit ultrapower [Keisler, 1963].

- The first construction in this style used a Fréchet filter on $\mathbb{N}$ rather than a free ultrafilter [Schmieden and Laugwitz, 1958]. Unlike a free ultrafilter, the existence of a Fréchet filter does not require any choice axiom. Therefore, it is still of interest for constructive approaches to NSA [Palmgren, 1998] (although in strictly constructivist approaches the framework of classical logic as used in [Schmieden and Laugwitz, 1958] also has to be replaced by intuitionist logic [Martin-Löf, 1990]). For an accessible introduction to a weak system of NSA based on Fréchet filters, see also: [Tao, 2012].

- In the construction of ${}^*\mathbb{Q}$ and ${}^*\mathbb{R}$, we have used a free ultrafilter on $\mathbb{N}$. This is sufficient to obtain a model with countable saturation. It is possible to fix a free ultrafilter on an infinite index set of higher cardinality. In particular, by choosing 'good' ultrafilters, it is possible to arrive at the desired level of saturation in a single step [Keisler, 2010, section 10]. See [Hurd and Loeb, 1985, p. 104–108] for more on saturation.

PART 2: SELECTED APPLICATIONS

Abstract

In section 4, we give an overview of the areas of applications of NSA. We return to a selection of them in the subsequent sections. Section 5: history of the calculus; section 6: intuitions about infinitesimals; section 7: paradoxes of infinity; section 8: probability and formal epistemology; and section 9: physics and philosophy of science.

4 Applications

Hyperreals can be applied in many branches of science, often with an interesting philosophical dimension. In this section, we give a brief overview of some of them; the next section looks into a selection of applications in more detail.

4.1 In mathematics

The first and still most important application of NSA is to make **proofs about standard analysis** shorter, easier, or both—mainly by alleviating epsilon-delta management [Tao, 2007]. An early expression of this can be found with Lagrange, as cited in [Błaszczyk et al., 2012, p. 29]. Recent examples are given by Terence Tao in his blog posts [Tao, 2007–2012].

Hyperreals can help us to reevaluate the **history** of the calculus: see section 5.

A related role of NSA is in the **didactics** of calculus. Keisler started using infinitesimals in beginning U.S. calculus courses in 1969 (source: [Stroyan, 2007, p. 369]) and he was the first to write a textbook for calculus based on NSA [Keisler, 1976a,b]. An empirical study with the teaching of high school students based on Keisler's book was conducted by Sullivan, who writes:

> "Any fears on the part of a would-be experimenter that students who learn calculus by way of infinitesimals will achieve less mastery of basic skills have surely been allayed. And it even appears highly probable that using the infinitesimal approach will make the calculus course a lot more fun both for the teachers and for the students." [Sullivan, 1976, p. 375]

Stroyan has a book-length treatise on NSA for teaching calculus is based on Keisler's earlier work [Stroyan, 2003]; see also [Stroyan and Luxemburg, 1976, Chapter 5] and [Stroyan, 2007]. Also the relative or stratified analysis approach to NSA [Hrbacek, 2007] is strongly motivated by didactical concerns [O'Donovan, 2007, Hrbacek et al., 2010]. Despite all the efforts of various authors, however, NSA still has very little impact on the current didactics of calculus [Artigue, 1991, section 1.4].

NSA can be used to analyze common intuitions concerning **infinitesimals**; see section 6. Whereas the history of the calculus is dominated by the concept of infinitesimals, NSA can also shed more light on problems related to the **infinitely large**. See section 7.

Another well-established application of NSA is that of **non-standard measure theory**. Seminal contributions were obtained by [Bernstein and Wattenberg, 1969] and [Loeb, 1975]. A good overview of this topic up to the 1980's can be found in [Cutland, 1983]. Since the subject of **probability** is of special interest to formal epistemologists, it is covered in further detail in section 8.

4.2 In physics

Physicists have continued to speak of infinitesimal quantities since the development of the calculus, seemingly not bothered by the foundational issues that were on the minds of the mathematicians. Therefore, the combination of physics and NSA seems to be a very natural one: it allows physicists to continue their appeal to the **intuitive notion** of infinitesimals, now knowing that there is a **rigorous** mathematical basis for this concept.

Many applications of NSA in physics are related to **differential equations and stochastic equations**. Examples covered in [Albeverio et al., 1986] include Lévy Brownian motion, Markov processes, and Sturm-Liouville problems. The applications are often embedded in the framework of non-standard measure theory, but this complication is not strictly necessary, as is illustrated by [Benci et al., 2010].

NSA has been applied to **quantum mechanics** in multiple ways, including Feynman path integrals and quantum field theory [Albeverio et al., 1986]. Moreover, it seems to be a very natural idea to reexamine the quantum-classical limit in this framework, by considering $\hbar$ as an infinitesimal, as has indeed been done in the literature [F.Werner and Wolff, 1995]. It is not known to me whether the relativistic-classical limit has also been studied in this way, *i.e.*, by taking $1/c$ to be an infinitesimal (indistinguishable from zero) in the classical theory.

Remark that many of the above applications do not require the full set of ${}^*\mathbb{Q}$ or ${}^*\mathbb{R}$; instead, they involve some kind of a **hyperfinite model**: a hyperfinite time line to study differential equations, a hyperfinite grid to perform integration, or a hyperfinite lattice to study Ising spin models. These aspects are interesting from the viewpoint of philosophy of science and we will come back to them in section 9.

4.3 In economics

Hyperreals have been used in mathematical economics: see for instance Kopp's chapter "Hyperfinite Mathematical Finance" in [Arkeryd et al., 1997,

p. 279–308], Sun's contribution in [Loeb and Wolff, 2000, Chapter IV], or Herzberg's [Herzberg, 2007b].

5 History of the calculus

Hyperreals can help us to understand why results that are now considered to be obtained in a non-rigorous way are nevertheless correct. One should keep in mind, however, that there is a difference in goal between developing an algorithm to solve a problem using numbers of a certain kind versus proving the existence of the totality of numbers of this kind [Błaszczyk et al., 2012, p. 9].

For the sake of illustration and brevity, we focus on a single notion from the calculus: the derivative. The derivative expresses the rate of change of the dependent variable with respect to the change in the independent variable. If the independent variable is interpreted as time, the derivative is the (instantaneous) velocity. If the independent variable is plotted as the abscissa (on the x-axis) and the dependent variable is plotted as the ordinate (on the y-axis), the derivative is the slope of a tangent curve to a point of the graph.

5.1 Leibniz and Newton: infinitesimals and fluxions

Leibniz's development of the calculus (around 1674) is characterized by three important ingredients:

Use of infinitesimals Leibniz's notation of the derivative as a quotient of infinitesimals ($e.g.$ $\frac{dx}{dy}$) is still in use in mathematics today, although in standard analysis the derivative cannot be interpreted as a quotient.

Law of continuity The law of continuity (also known with the French name '*souverain principe*') says that the rules of the finite also hold in the infinite, and vice versa. It is related to the Transfer Principle in NSA.

Law of transcendental homogeneity The law of transcendental homogeneity says that when comparing two quantities, quantities with a lower order of infinity can be ignored. It is related to the standard part function in NSA.

See for instance [Katz and Sherry, 2012b,a] for more on this topic.

Newton developed his version of the calculus (around 1666) with physics in mind, which led him to a dynamic concept of the derivative: he thought of the derivative of a continuous function—which he called the fluxion of a fluent—as a velocity, or rate of change. His dot notation of the derivative ($e.g.$ $\dot{x}$ for $\frac{dx}{dt}$) is still in use in physics today. Although Newton did not base his calculus on the notion of the infinitesimal, as Leibniz did, infinitesimals do appear in his work, too, both as infinitely small periods of time and as 'moments' of fluent quantities.

5.1.1 Criticism by Berkeley

Berkeley famously criticized the use of infinitesimals and evanescent quantities in his work "The Analyst":

> "And what are these Fluxions? The Velocities of evanescent Increments? And what are these same evanescent Increments? They are neither infinite Quantities nor Quantities infinitely small, nor yet nothing. May we not call them the Ghosts of departed Quantities?" [Berkeley, 1734, Section XXXV]

Berkeley's work was so influential, that many believed that infinitesimals had to be banned from mathematics once and for all. However, Katz and Sherry recently pointed out a flaw in Berkeley's criticism [Katz and Sherry, 2012a,a]: the evanescent quantity need not be treated as zero, merely discarded in certain contexts through an application of the law of transcendental homogeneity. They claim that Leibniz's system was consistent after all (albeit not rigorous to today's standards).

5.2 Weierstrass: standard analysis

The modern approach to standard analysis was developed by "the great triumvirate" [Boyer, 1949, p. 298]: Cantor, Dedekind, and Weierstrass. Weierstrass introduced the modern epsilon-delta definition of the limit (which goes back to Bolzano in 1817). This allows us to define the derivative as a limit of the quotient of differences.

$$\begin{aligned}\frac{dy}{dx} &= \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} \\ &= \lim_{\Delta x \to 0} \frac{y(x+\Delta x) - y(x)}{\Delta x},\end{aligned}$$

where:

$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = L \qquad \Leftrightarrow$$

$$\forall \epsilon > 0 \in \mathbb{R}, \exists \delta > 0 \in \mathbb{R}, \forall \Delta x \in \mathbb{R}(0 < |\Delta x| < \delta \Rightarrow |\frac{\Delta y}{\Delta x} - L| < \epsilon).$$

5.3 Robinson: non-standard analysis

In NSA, the classical derivative is defined as the standard part of a quotient of infinitesimals in ${}^*\mathbb{R}$:

$$\frac{dy}{dx} = st(\frac{{}^*y(x+\delta) - {}^*y(x)}{\delta}),$$

where ${}^*y(x) : {}^*\mathbb{R} \to {}^*\mathbb{R}$ is the hyperextension of the real function $y(x) : \mathbb{R} \to \mathbb{R}$ and δ is an infinitesimal in ${}^*\mathbb{R}$.

5.4 Further reading

A brief history of the concepts of infinitesimals and the infinitely large in the development of the calculus can be found in [Kleiner, 2001]. Some misconceptions on the history of the calculus, mainly centered on the role of infinitesimals therein, are discussed in the manuscript [Błaszczyk et al., 2012]. More on the general history of non-Archimedean number systems, not directly related to the development of the calculus, can be found in [Ehrlich, 2006].

6 Intuitions about infinitesimals

In standard analysis the sequence

$$\langle 0.1, 0.01, 0.001, \ldots, 10^{-n}, \ldots \rangle$$

converges to 0; it is a null-sequence. Can NSA teach us something about the common intuition that this sequence is a non-zero infinitesimal? Likewise, in standard analysis $0.999\ldots$ is exactly equal to (or just a different notation for) $1.000\ldots$. (See [Hart, 2012] for an amusing overview of various proofs.) Nevertheless, the intuition that $0.999\ldots$ is infinitesimally smaller than unity is a resilient one—not only among students in math class: every so often, it resurfaces in internet discussions. Can NSA teach us something about this common intuition?

By looking at the ultrapower construction of ${}^*\mathbb{R}$ in section 3, one may convince oneself that the equivalence class under a free ultrafilter of

$$\langle 0.1, 0.01, 0.001, \ldots, 10^{-n}, \ldots \rangle$$

is different from that of zero. (The former sequence is different from zero at every position, whereas the latter sequence is exactly zero at every position. Hence, the index set of positions where both sequences are exactly equal is the empty set, which is not in the free ultrafilter. Therefore, they correspond to different hyperreal numbers.) Likewise, the sequence

$$\langle 0.9, 0.99, 0.999, \ldots, 1 - 10^{-n}, \ldots \rangle$$

is different from the constant sequence $\langle 1, 1, 1, \ldots, 1, \ldots \rangle$ at every position and hence does not belong to the same equivalence class under a free ultrafilter.

Observe that this does *not* contradict $0.999\ldots = 1$, since the number

$$[\langle 0.9, 0.99, 0.999, \ldots, 1 - 10^{-n}, \ldots \rangle]_{\sim_{\mathcal{U}}}$$

is not equal to the real number $0.999\ldots$. In fact, by observing that the standard part of this number is 1, we could use it to prove that $0.999\ldots = 1$. The above number is a hyperreal number strictly smaller than 1, differing from it by the infinitesimal quantity

$$[\langle 0.1, 0.01, 0.001, \ldots, 10^{-n}, \ldots \rangle]_{\sim_{\mathcal{U}}}.$$

This topic is also taken up in [Katz and Katz, 2010].

7 Paradoxes of infinity

Consider the following two principles for comparing set sizes:

Euclidean part-whole principle If A is a proper subset of B, then A is strictly smaller than B.

Humean one-to-one correspondence If there is a one-to-one correspondence between A and B, then A and B are equal in size.

For finite sets, these principles lead to equivalent ways of measuring sets. Since an infinite set can be put into one-to-one correspondence with a proper subset of itself, these principles are incompatible for infinite sets. It led Galileo to puzzle over the number of squares, for it seemed as though there both had to be equally many squares as there are natural numbers and less squares than natural numbers [Mancosu, 2009, p. 613].

In Cantor's theory of cardinality, one-to-one correspondence is taken to be the guiding notion for determining set sizes and the part-whole principle is weakened.

One may wonder whether one can build an alternative theory of sizes of infinite sets, by keeping on board the part-whole principle and weakening the principle of one-to-one correspondence. The answer is "yes", as is demonstrated by Benci's theory of numerosity [Benci, 1995, Benci and Di Nasso, 2003b, Benci et al., 2006b]. 'Numerosity' is the term used to refer to set sizes based on the Euclidean part-whole principle. The numerosity of the natural numbers (taken to be the set $\{1, 2, 3, \ldots\}$) is defined to be α. Since numerosities—unlike cardinal or ordinal numbers—follow the usual algebra of finite numbers, it is easy to obtain the following results:

- $num(\mathbb{N} \setminus \{1\}) = \alpha - 1$;
- $num(\mathbb{Z}) = 2\alpha + 1$;
- $num(\mathbb{N} \times \mathbb{N}) = \alpha^2$;
- $num(\{1, \ldots, n\}) = n$.

As it turns out, Cantor's theory of cardinality is not the only consistent way to assign sizes to infinite sets [Mancosu, 2009].

Numerosity can be regarded as the ideal value of a real-valued ω-sequence, *i.e.*, some type of non-Archimedean limit). This idea is developed further in alpha-theory, an axiomatic approach to NSA [Benci and Di Nasso, 2003a, Benci et al., 2006a, Benci and Di Nasso, 2012].

8 Probability and formal epistemology

The field of non-standard measure theory and non-standard probability theory is among the most developed areas of application of NSA. In these notes, it is not possible to be exhaustive. Instead, we will focus on two topics that are of special interest to formal epistemologists.

8.1 Regularity and infinitesimal probabilities

A probability functions is regular if it only assigns probability zero to the impossible event (logical contradiction or empty set). Because of finite additivity, it is equivalent to only assigning probability one to the certain event (logical tautology or full sample space). It is well-known that standard probability functions, based on Kolmogorov's axioms for probability [Kolmogorov, 1933], can violate regularity in the case of countably infinite sample spaces and always do so in the uncountable case.

Regularity is often discussed in the context of subjective probability, where it is proposed as a norm for rationality, known as 'strict coherence' [Skyrms, 1995]. I think, however, that it is a desirable property also in the context of logical or objective probability. Assigning probabilities is a two-step process: first one determines all the elementary possible outcomes (atomic events), *i.e.*, one fixes a sample space. Then one assigns different weights to the elementary events and to combinations thereof, *i.e.*, one assigns probabilities to the events in the event space (an algebra over the sample space). It seems odd that some distinctions between possible and impossible outcomes, as established in the first step, should get lost in the second step. Yet, this is what happens if one practices probability theory within the framework of standard measure theory: on an infinite sample space, one may be forced to assign probability zero to a possible outcome, which is the exact same probability as that of the impossible event.

We are interested here in examples that are discussed in the philosophical literature, such as:

- drawing a random number from the natural numbers (sometimes called de Finetti's infinite lottery) and conditionalizing on an even number being drawn or on a finite subset;
- throwing darts at the interval $[0,1]_{\mathbb{R}}$ and conditionalizing on $[0,1]_{\mathbb{Q}}$;

Table 1: Various quantitative probability theories.

Domain: / Range:	Standard	Ideal
$\mathbb{R}$ (Archimedean field)	Kolmogorov	(b) Loeb
Non-Archimedean field	(c) NAP	(a) Nelson

- a fully specific outcome (*e.g.*, 'all heads') of an ω-sequence of tosses with a fair coin.

It has been suggested that regularity can be attained for such examples by considering infinitesimal probabilities, in particular in the context of NSA [Skyrms, 1980, Lewis, 1980, Skyrms, 1995]; but this suggestion has also been disputed [Williamson, 2007, Easwaran, 2010, Hájek, 2010]. The first problem—that of a fair lottery on $\mathbb{N}$—is of course directly related to the issue of countable additivity; see also [Kelly, 1996, p. 321–323].

Whereas the above problems may seem rather contrived, they are related to deeper issues in the philosophy of science, where they lead to the zero-fit problem: if all systems of laws assign probability zero to the present state, then one cannot select the best system based on a measure of goodness-of-fit. This problem has been discussed in relation to infinitesimal probabilities [Elga, 2004, Herzberg, 2007a].

Instead of trying to determine the feasibility of regular probability functions based on hyperreals *a priori*, I propose to take stock of the different approaches to hyperreal probabilities first: look at the details of how a theory of infinitesimal probability would look like and evaluate the proposal afterwards.

Multiple alternative approaches to probability theory have been discussed in the literature. We focus here on proposals that involve changing the domain or the range of the probability function to a non-standard set in the sense of NSA. We can thus distinguish three categories of alternatives, presented in Table 1: the probability function has (a) both a non-standard domain and a non-standard range, (b) only a non-standard domain, or (c) only a non-standard range.

Alternative (a) is easily obtained in the context of NSA by applying the Transfer principle to standard Kolmogorovian probability functions on finite domains. An example of this approach was developed by Nelson [Nelson, 1987]: his "Radically elementary probability theory" is based on non-Archimedean, hyperfinite sets as the domain and range of the probability function. This framework has the benefit of making probability theory on infinite sample spaces equally simple and straightforward as the corresponding theory on finite sample spaces; the appropriate additivity property is hyperfinite additivity. Nelson's theory is regular and, since it is obtained directly by Transfer, internal. However, this elegant theory does not apply *directly* to our current quest for finding regular probability functions on standard infinite sets, such as a fair lottery on $\mathbb{N}$, $\mathbb{R}$, or $2^{\mathbb{N}}$.

Alternative (b) is the dominant line of research in non-standard measure and integration theory; it is concerned mainly with finding new results in *standard* measure and integration theory [Cutland, 1983]. A measure with standard range and non-standard domain can be obtained in NSA by starting from (a) and applying the standard part function afterwards, which maps a hyperreal measure to the unique nearest real value. Probability measures of this type are known as Loeb measures [Loeb, 1975]; these functions are external objects (due to the use of the external standard map function). Although the well-developed theory of Loeb measures has proven fruitful in many applications, it too simply does not address the type of problems that concern us here.

To describe regular probability functions on standard infinite domains, we need to investigate the previously unexplored alternative (c). Together with Leon Horsten, I wrote about a probability function of this form for the special case of a fair lottery on $\mathbb{N}$ [Wenmackers and Horsten, 2010]. Together with Vieri Benci, we generalized this idea and developed our new approach to probability theory, which we call non-Archimedean probability (NAP) theory [Benci et al., 2012a, Wenmackers et al., 2012]. There is a philosophical companion paper to the mathematical results (work in progress [Benci et al., 2012b]). Remark: although McGee considered a sentential algebra rather than a set algebra, the approach in his demonstration of the equivalence between non-standard measures and Popper functions [McGee, 1994] is closely related to NAP.

Some properties of the theory:

- NAP theory is regular. Hence, it allows conditionalization on any possible event (*i.e.*, any subset of the sample space, except the empty set).
- Within NAP theory, the domain of the probability function can be the full powerset of any standard set from applied mathematics (*i.e.*, of any cardinality), whereas the general range is a non-Archimedean field. Hence, there are no non-measurable sets.
- Kolmogorov's countable additivity (which is a consequence of the use of standard limits) is replaced by a different type of infinite additivity (due to the use of a non-Archimedean limit concept).
- For fair lotteries, the probability assigned to an event by NAP theory is directly proportional to the numerosity of the subset representing that event.

- NAP functions are external objects: they cannot be obtained by taking a standard object and applying the star-map to it.

The price one has to pay for all this is that certain symmetries, which hold for standard measures, do not hold for NAP. Our theory is closely related to numerosity and has a similar Euclidean property: a strict subset has a smaller probability, as is necessary by regularity. Hence, for infinite sample spaces, NAP is bound to violate the Humean principle of one-to-one correspondence. Since translation symmetries amount to a particular type of one-to-one correspondence, they fail in NAP (*cf.* [Williamson, 2007]). In other words, as others have pointed out before [Bartha, 2004, Weintraub, 2008], these measures are strongly label-dependent.

8.2 Lockean thesis and relative infinitesimals

Whereas standard probability measures may seem too coarse-grained for some applications, where we would like to distinguish between possible and impossible events, they may not seem coarse-grained enough for other applications, as we will see now.

Suppose that you have detailed knowledge of the probabilities in a given situation. It has been argued that it may still be beneficial to hold some full (dis-)beliefs [Foley, 2009]. But when is it rational to believe something in this case? The Lockean thesis suggests that it is rational to believe a statement if the probability of that statement is sufficiently close to unity. This is usually modeled by means of a probability threshold. As is demonstrated by the Lottery Paradox [Kyburg, 1961], the threshold-based model is incompatible with the Conjunction Principle. Moreover, it can be objected that the actual probabilities are too vague to put a sharp threshold on them, and that a threshold should be context-dependent.

I have discussed the analogy between large and infinite lotteries elsewhere [Wenmackers, 2012b] (an earlier version can be found in [Wenmackers, 2011, Chapter 4]), and have suggested the use of NSA to introduce a form of vagueness or coarse-graining and context-dependence in the formal model of the Lockean thesis. My model is called Stratified belief [Wenmackers, 2012a] (an earlier version can be found in [Wenmackers, 2011, Chapter 3]) and it is based on the 'levels'—a formalization of the intuitive scales-of-magnitude concept—in Hrbáček's relative or stratified analysis [Hrbacek, 2007].

The basic idea is to interpret the Lockean thesis as follows: it is rational to believe a statement if the probability of that statement is indistinguishable from unity (in a given context). The context-dependent indistinguishability relation is then modeled using the notion of differences up to a level-dependent, ultrasmall number. These ultrasmall numbers, also called 'relative infinitesimals', are ordinary real numbers, which are merely unobservable, or do not have a unique name, in a given context. The aggregation rule for this model is the 'Stratified conjunction principle', which entails that the conjunction of a standard number of rational beliefs is rational, whereas the conjunction of an ultralarge number of rational beliefs is not necessarily rational. Remark: this model is intended to describe beliefs that are almost certain, but it can be used for weaker forms of belief by substituting a lower number instead of unity.

9 Physics and philosophy of science

9.1 Rethinking the continuum

We often use "the continuum" as a synonym for the standard reals. However, this is but one formalization of the concept of a physical continuum. Hyperreals form an alternative formalization of the concept. Katz *et al.* propose to refer to $^*\mathbb{R}$ as a "thick continuum", as opposed to the "thin continuum" $\mathbb{R}$ [Katz and Sherry, 2012a]. For a discussion of other rival continua, see [Błaszczyk et al., 2012, p. 23–24].

Like the standard reals, the hyperreals are infinitely divisible. In particular: infinitesimals are infinitely divisible. However, many applications make use of hyperfinite sets, which—like a finite set—do contain a smallest non-zero element. Therefore, such models are discrete or chunky, rather than continuous. Another distinction to be made here is that, besides the hyperreals, there are other systems that also contain infinitesimals, but which may have very different properties. Archimedes and Zeno, for instance, conceived of infinitesimals as dimensionless points (for a discussion of Zeno's paradoxes in relation to hyperreals, see [White, 1999]), and smooth infinitesimal analysis (SIA) describes nilsquare infinitesimals [Moerdijk and Reyes, 1991, Bell, 1998].

A more radical question is whether we need a continuum, be it modelled by reals or hyperreals, in the empirical sciences at all. Sommer and Suppes have proposed to dispense with the continuum in favor of $^*\mathbb{Q}$ [Sommer and Suppes, 1997]. They developed an axiomatic approach to NSA, called elementary recursive non-standard analysis (ERNA), and claimed that much of physics does not rely on the existence of a completed continuum ($\mathbb{R}$). They proposed to trade the axiom of completeness for an axiom that states the existence of infinitesimals. According to them, working in $^*\mathbb{Q}$ is more constructive and matches better with geometric intuitions than working in $\mathbb{R}$. Also in [Albeverio et al., 1986, p. 31], it is observed that although NSA is often claimed to be a non-constructive theory (due to the dependence on the Axiom of Choice), it is remarkably constructive in applications. And [Błaszczyk et al., 2012, p. 17] remark that proofs in NSA go *with* the flow of reasoning, whereas epsilon-delta constructions notoriously run against it (requiring specification of the change in the dependent variable prior to that of the independent one).

One may object that irrational numbers, such as $\sqrt{2}$ and π, are common in physics, but do not exist in ${}^*\mathbb{Q}$. However, there are elements in ${}^*\mathbb{Q}$ that have the same decimal places as these numbers. Hence, $\mathbb{R}$ and ${}^*\mathbb{Q}$ are observationally equivalent. This notion of empirical indistinguishability touches upon the very essence of 'infinitesimals', which have been described long before the advent of NSA as smaller than any assignable or measurable quantity.

9.2 Hyperfinite models and determinism

Hyperfinite models are models which are both infinite and discrete. As remarked before, they have a distinct constructive and intuitive flavor about them. Moreover, they allow for deterministic models, where the corresponding standard real-valued model may fail to do so. (Another advantage is that one often does not need very advanced techniques from NSA in order to use it to solve differential equations, as is demonstrated in [Benci et al., 2010].)

To illustrate this, we follow the example concerning Peano's existence theorem (PET) as discussed in [Albeverio et al., 1986, p. 31–33]. PET states that:

$$\begin{aligned}
&\forall f : [0,1] \times \mathbb{R} \to \mathbb{R} \text{ continuous and bounded,} \\
&\forall u_0 \in \mathbb{R}, \\
&\exists u : [0,1] \to \mathbb{R} \text{ such that :} \\
&\qquad\qquad \begin{cases} du(t)/dt & = f(t, u(t)) \\ \quad u(0) & = u_0. \end{cases}
\end{aligned}$$

The solution of this Cauchy problem typically is not unique (indeterminism). In the standard proof of PET, one such solution is constructed, but no hints are given about how to find the other solutions.

However, taken as functions of hyperreal numbers on a hyperfinite grid, the solution obtained in the proof of PET would be unique. This can be seen as follows: the corresponding difference equation, involving finite differences, does have a unique solution; by Transfer, the same holds for the hyperfinite equation.

Hyperreals can also be used to find all the solutions of the standard version of the Cauchy problem. First, perturb the initial condition and/or the differential equation by an infinitesimal. Then, find the unique solution to the hyperfinite difference equation using the construction in the proof of PET. Finally, take the standard part of the hyperreal solution.

In the example of Norton's dome [Norton, 2008], which is used to demonstrate a failure of determinism in classical mechanics, we encounter the same phenomenon as before, this time with a second-order differential equation. Giving a hyperfinite description of the dome yields a deterministic model. If we follow Sommer and Suppes' suggestion that nonstandard models and models based on the standard reals are empirically indistinguishable, we have to conclude that (in-)determinism is a model-dependent property. Observe that Werndl reaches the same conclusion for a different source of indeterminism [Werndl, 2009].

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