# Two Concepts of Plausibility in Default Reasoning

Hans Rott

Department of Philosophy, University of Regensburg,
93040 Regensburg, Germany
hans.rott@ur.de

## 1. Introduction

Friedman and Halpern's work on plausibility measures (Friedman and Halpern 1995, 1996, 1999, 2001, Halpern 1997, 2001, 2003) achieved a beautiful unification of several approaches to default reasoning. As Friedman and Halpern point out, plausibility measures can be seen as a generalization of various other semantics for default reasoning, such as preferential structures (Shoham; Kraus, Lehmann and Magidor, "KLM" in short), $\varepsilon$-semantics (Adams; Pearl; Geffner), possibilistic structures (Dubois and Prade) and $\kappa$-rankings (Spohn; Goldszmidt and Pearl). Friedman and Halpern claim that by focussing on such general measures, one can understand why the KLM-properties were bound to play the central role in the literature on default reasoning that they did in fact play.

On such a very general level, Friedman and Halpern use a fundamental property for plausibility measures they call *Qualitativeness*.

> If $A$, $B$ and $C$ are pairwise disjoint and $A < B \cup C$ and $B < A \cup C$, then $A \cup B < C$

This condition is very similar to a condition called "Choice" that features centrally in the entrenchment-based account to belief revision provided by Rott (1996, 2001, 2003).[1] Translated into the language of plausibilities, *Choice* looks like this:

$$A < B \cup C \text{ and } B < A \cup C \quad \text{if and only if} \quad A \cup B < C$$

Now default reasoning and belief revision have sometimes been called "two sides of the same coin" (Gärdenfors 1990), and thus this coincidence does not seem surprising. However, the property mentioned is very uncommon and slightly

[1] That $A \cup B <_{\mathrm{FH}} C$ implies $A <_{\mathrm{FH}} B \cup C$ follows from other conditions of Friedman and Halpern.

unintuitive. On closer inspection, there is actually a substantial difference between the set-ups of Friedman and Halpern on the one hand and Rott on the other. Friedman and Halpern's condition is restricted and thus considerably weaker than Rott's, and this restricted condition is used in a different context than the unrestricted condition. This paper explores the relationship between Friedman-Halpern plausibility measures and relations of "basic entrenchment." I argue that while basically equivalent, the latter way of structuring things has some clear advantages over that of Friedman and Halpern.

1. The central condition Choice is stronger and yet more natural than the central condition of Qualitativeness of FH.

2. The method of retrieving the plausibility relation from a given default reasoning system is better motivated than Qualitativeness, which is essentially a result of reverse engineering.[2]

3. There is a one-to-one correspondence between conditions for plausibility relations and conditions for inference operations, not just packages of conditions that are tied together "somewhat surprisingly."[3]

The plan of this paper is as follows. In section 2, we take the first steps to motivate the concept of plausibility. I argue that in thinking about plausibility, it makes sense to think of the concept of default reasoning as having methodological priority over that of plausibility, and I recommend taking default reasoning to be based on expectation revision. Section 3 recalls some arrangements of principles of default reasoning that are relevant to our discussion. Section 4 presents both Friedman-Halpern plausibility and "basic plausibility" in a way that facilitates the comparison. It is shown how the latter can be vindicated by the reconstructive view taken, and that the two kinds of plausibility relations are different. In the short section 5, we show that the two kinds of plausibility can be applied in the same way. In section 6, we first retrieve Friedman-Halpern plausibilities from default inference relations in a way similar to how basic plausibilities have been retrieved. Then we check whether there is a harmonic mapping between default inference relations and plausibility relations. This works fine for both FH plausibilities and basic plausibilities in one direction. In the other direction the mapping succeeds only for basic plausibilities. Section 7 shows how one can translate directly between the plausibility languages of Friedman-Halpern and of Rott. According to this translation (well-behaved) basic plausibility relations are subrelations of (qualitative) Friedman-Halpern relations, and they both agree on disjoint sets. Section 8 gives a more detailed philosophical motivation of basic plausibilities. Section 9 gives a short conclusion.

[2] Joseph Halpern, personal communication, July 2011.

[3] These are the words of Friedman and Halpern (1996, p. 1301, 2001, p. 658) and Halpern (2003, p. 303).

## 2. The primacy of default reasoning

Default reasoning is reasoning based on defaults. Defaults are assumptions about the normal or typical state of affairs. Like beliefs, defaults are in the mind of the agent. In fact, defaults can be thought of as being similar to beliefs, but they are weaker. A good term to indicate this relative weakness is *expectation* (Gärdenfors and Makinson 1994). Beliefs are based on ("hard") information, and information overrides ("soft") default assumptions. Information does not go against the agent's expectations as long as it is consistent with them. But what makes default reasoning interesting is that people have, and have to have, many expectations, and it is a frequent experience that the information people get conflicts with their default assumptions.

We need to be careful whether we are talking about sentences or propositions. Default reasoning will captured by a kind of logic, that is an inference relation $\mid\!\sim$ relating (single) premises to (single) conclusions. The relata thus are *sentences*. On the other hand, we shall proceed on the idea that beliefs or, more pertinently, expectations are *propositions*, where a proposition is what is said by an assertive sentence. In the present paper, propositions are thought of as sets of possible worlds. For the sake of simplicity, we presuppose a finitistic framework in which every set of possible worlds is expressible by a single sentence. Notation: For any sentence $\alpha$, $[\![\alpha]\!]$ denotes the proposition expressed by the sentence $\alpha$, i.e., the set of possible worlds at which $\alpha$ is true. For any proposition $A$, $\phi_A$ denotes a sentence that expresses $A$. Suppose that such a sentence $\phi_A$ has been chosen. Then clearly there are many sentences that express the same proposition, namely those that are logically equivalent with $\phi_A$. So $\phi_A$ is not uniquely determined. As will become clear soon, however, we will always work in contexts in which it is irrelevant which representative of the set of sentences expressing $A$ is chosen, so everything will be well-defined. And of course, $[\![\phi_A]\!] = A$, and $\phi_{[\![\alpha]\!]}$ is equivalent with $\alpha$.

What is the concept of plausibility in the context of default reasoning? The concept applies paradigmatically to propositions that are *surprising* in that they go against the agent's expectations. As a limiting case, one can also say that plausibility applies to propositions that are compatible with, or possible according to, one's expectations. Such propositions enjoy maximal plausibility. They are all equally plausible and more plausible than anything that contradicts the agent's expectations. But propositions that go against one's expectations may do so in varying degrees. That some proposition $A$ is *less plausible* than another proposition $B$ means that the expectations militating against $A$ are *stronger*, or *better entrenched*, than the expectations militating against $B$. It is easy to find representatives of such expectations: these are just the complements of the propositions in question. The representative expectation militating against $A$ is $-A$, and the representative expectation militating against $B$ is $-B$. So a potential piece of information $A$ is less plausible than another potential piece of information $B$ just in case that the expectation $-A$ is more entrenched than the expectation $-B$, in symbols

$$(<_{\mathrm{BP}} \text{ and } <_{\mathrm{BE}}) \quad A <_{\mathrm{BP}} B \quad \text{iff} \quad -B <_{\mathrm{BE}} -A$$

Here, $<_{\mathrm{BP}}$ and $<_{\mathrm{BE}}$ are strict relations of (basic) plausibility and (basic) entrenchment, respectively. The duality ($<_{\mathrm{BP}}$ and $<_{\mathrm{BE}}$) between "plausibility" and "entrenchment" is, of course, partly just an exercise in terminological stage-setting. But the terms and the bridge between them has been made use of for quite some time. Entrenchment relations are relations of comparative necessity, mainly over beliefs or expectations (defaults), plausibility relations are relations of comparative possibility, mainly over *dis*beliefs or *dis*expectations.[4]

I suggest to focus on a fundamental notion of plausibility that can be derived whenever a default inference relation is given. This is one of the most important ideological differences between the approach advocated in the present paper and that of Friedman and Halpern. It seems to me that the *reconstruction* of plausibility or entrenchment in terms of default reasoning can be clearly understood, in fact better than the *construction* of default reasoning in terms of entrenchment or plausibility.

We said that a proposition is less plausible if the expectations militating against it are stronger or more entrenched. What is the meaning of entrenchment? Entrenchment is always relative to a belief state or, and this is the interpretation that is relevant here, to a state of expectations. We use the covering term *doxastic state* and represent the core of a doxastic state by a proposition $X$. A proposition $A$ is an *expectation* or a *default proposition* in $X$, if $X \subseteq A$. A proposition $A$ is *less entrenched* (in a doxastic state $X$) than another proposition $B$ just in case the former is withdrawn while the latter is retained if at least one of them has to be withdrawn (from $X$). We can formalize this straightforward idea as follows:

$$(\text{From } \dot{-} \text{ to } <_{\mathrm{BE}}) \quad A <_{\mathrm{BE}} B \quad \text{iff} \quad X \dot{-} (A \cap B) \not\subseteq A \ \text{ and } \ X \dot{-} (A \cap B) \subseteq B$$

The symbol $\dot{-}$ signifies the operation of rational *withdrawal* of a proposition from the set of expectations $X$ (or, as many people say, the rational *contraction* of $X$ by a proposition). Withdrawing at least one of the expectations $A$ and $B$ just means, I submit, withdrawing the expectation $A \cap B$. In the principal case, then, saying that $A$ is less entrenched than $B$ just means that $B$ is retained, but $A$ is not retained.

The thesis that withdrawing $A \cap B$ just means withdrawing at least one of $A$ and $B$ looks problematic. There are examples in which withdrawing $A \cap B$ is clearly different from both withdrawing $A$ and withdrawing $B$, or any combination thereof.

[4]Both entrenchment and plausibility relations may be considered as representing degrees of belief. Compare Rott (2009), with references to the seminal work produced within the framework of Dubois and Prade's possibility theory.

*Example 1.* Suppose we have a plausibility relation $\prec$ on the worlds in $W = \{u, v_1, v_2, w_1, w_2\}$, and that the contraction of $X = \{u\}$ by a proposition is obtained by joining $X$ with the most plausible worlds not satisfying this proposition. Let $P = \{v_1, w_2\}$ and $Q = \{v_2, w_1\}$. Suppose that $u$ is more plausible than all the other worlds, i.e., $v_i \prec u$ and $w_i \prec u$ for all $i = 1, 2$, and that $v_2 \prec v_1$ and $w_2 \prec w_1$, and that no other pair of worlds is related by $\prec$. The result of giving up a proposition $A$ in $X$ is the proposition obtained by taking the union of $X$ with the set of most plausible worlds at which $A$ is not true. So, in our example, the result of giving up $-P$ is $X \dot{-} -P = \{u, v_1, w_2\}$ and the result of giving up $-Q$ is $X \dot{-} -Q = \{u, v_2, w_1\}$. But the result of giving up $-P \cap -Q$ is $X \dot{-} (-P \cap -Q) = \{u, v_1, w_1\}$ which is evidently not equal to either one of the former sets. Nor can it be obtained by their intersection or any other Boolean combination of them. So one might be tempted to think that "withdrawing $-P \cap -Q$" cannot mean "withdrawing at least one of $-P$ and $-Q$." But this would be a mistake. It is simply not the case that "withdrawing at least one of two propositions $A$ and $B$" means the same as "withdrawing $A$"-or-"withdrawing $B$"-or-"withdrawing $A$ and withdrawing $B$".[5]

For default reasoning, it is not enough to withdraw certain propositions. Such a step is made just in order to make possible the consistent expansion of the expectation set by another proposition—namely the premise of the default inference. Using the so-called "Harper identity" (Gärdenfors 1988, p. 70) linking belief/expectation contraction and belief/expectation revision, we get:

(From $*$ to $<_{\text{BE}}$) $\quad A <_{\text{BE}} B \quad$ iff $\quad X * -(A \cap B) \not\subseteq A$ and $X * -(A \cap B) \subseteq B$

Here the symbol $*$ signifies the operation of rational revision of the expectations by a proposition. The condition $X \subseteq B$, which is also required on the right-hand side by the Harper identity, is implied by $X * -(A \cap B) \subseteq B$, given some standard assumptions concerning consistent revisions.

Let us interpret default reasoning as expectation revision (in the style of Gärdenfors and Makinson 1994) and define

($\mid\!\sim$ and $*$) $\qquad \alpha \mid\!\sim \beta$ (at a doxastic state $X$) $\quad$ iff $\quad X * [\![\alpha]\!] \subseteq [\![\beta]\!]$

Putting the above pieces together yields:

(From $\mid\!\sim$ to $<_{\text{BE}}$) $\quad A <_{\text{BE}} B \quad$ iff $\quad \neg(\phi_A \wedge \phi_B) \not\mid\!\sim \phi_A$ and $\neg(\phi_A \wedge \phi_B) \mid\!\sim \phi_B$

and

[5] Alchourrón, Gärdenfors and Makinson (1985, pp. 525–526) discuss a postulate called "Ventilation" according to which "withdrawing $A \cap B$" means exactly the latter. But this is a very strong postulate that has no counterpart in the core of default reasoning as understood by Friedman and Halpern. As AGM show, Ventilation implies Rational monotony (see section 3 below).

(From $\mathrel{|\!\sim}$ to $<_{\mathrm{BP}}$) $A <_{\mathrm{BP}} B$ iff $\phi_A \vee \phi_B \mathrel{|\!\sim} \neg\phi_A$ and $\phi_A \vee \phi_B \not\mathrel{|\!\sim} \neg\phi_B$

Thus, saying that $A$ is less plausible than $B$ just means that $\neg\phi_A$, but not $\neg\phi_B$, is defeasibly derivable if one is provided with the sole premise $\phi_A \vee \phi_B$.[6] This gives a very clear meaning to the plausibility relation used, one that is directly connected with its use in belief revision and default reasoning. One of the goals of the present paper is to argue that this meaning of the term "plausibility" is the most adequate one.

Let us use the notation $\mathcal{P}_{\mathrm{B}}(\mathrel{|\!\sim})$ for the (basic) plausibility relation $<_{\mathrm{BP}}$ derived from a given inference relation $\mathrel{|\!\sim}$ by way of (From $\mathrel{|\!\sim}$ to $<_{\mathrm{BP}}$).[7]

## 3. Postulates for default inference relations

An *inference relation* is a set of elements of the form $\alpha \mathrel{|\!\sim} \beta$ that is closed under a certain set of conditions. Friedman and Halpern emphasize that the following set of conditions for default reasoning has proved to be of particular relevance. This "core" defines what has been called *preferential reasoning* since the seminal paper of Kraus, Lehmann and Magidor (1990).

| | | |
|---|---|---|
| (LLE) | If $\alpha \dashv\vdash \beta$, then $\alpha \mathrel{|\!\sim} \gamma$ iff $\beta \mathrel{|\!\sim} \gamma$ | (Left logical equivalence) |
| (RW) | If $\alpha \vdash \beta$ and $\gamma \mathrel{|\!\sim} \alpha$, then $\gamma \mathrel{|\!\sim} \beta$ | (Right weakening) |
| (REF) | $\alpha \mathrel{|\!\sim} \alpha$ | (Reflexivity) |
| (AND) | If $\alpha \mathrel{|\!\sim} \beta$ and $\alpha \mathrel{|\!\sim} \gamma$, then $\alpha \mathrel{|\!\sim} \beta \wedge \gamma$ | (And) |
| (OR) | If $\alpha \mathrel{|\!\sim} \gamma$ and $\beta \mathrel{|\!\sim} \gamma$, then $\alpha \vee \beta \mathrel{|\!\sim} \gamma$ | (Or) |
| (CM) | If $\alpha \mathrel{|\!\sim} \beta$ and $\alpha \mathrel{|\!\sim} \gamma$, then $\alpha \wedge \gamma \mathrel{|\!\sim} \beta$ | (Cumulative monotony) |

In Friedman and Halpern's presentation, these six postulates come in two packages of three elements each: The *general* case is defined by the set consisting of (RW), (LLE) and (REF), the *qualitative* case adds the set of (And), (Or) and (CM). From a belief revision point of view, it is the collection of the first four conditions, (LLE), (RW), (REF) and (AND) that has been called *basic.* These conditions correspond to the basic postulates of AGM (Alchourrón, Gärdenfors and Makinson 1985). While Friedman and Halpern's general case is designed to be open to probabilistic interpretations, AGM have been more qualitative to begin with. (OR) corresponds to the first, and (CM) is essentially a weakening

[6]Or if one just *assumes* $\phi_A \vee \phi_B$ as the sole premise. The same idea, by the way, already appears (with the ordering reversed) in Lewis (1973, p. 54). The point of the present paper is that this idea can be put to work in *much* weaker settings than the ones envisaged by Lewis.

[7]Essentially the same method of deriving a plausibility relation from a default inference relation was used by Lehmann and Magidor (1992, pp. 31, 45) and by Freund (1993, 1998).

of the second *supplementary* postulate of AGM.[8] Friedman and Halpern do not (and, I think, cannot) take apart the two packages in their analysis. We will be able handle the cases one by one in the present paper (except for the very basic conditions (REF) and (LLE)).

We also mention two more conditions.

(CP) If $\alpha \mid\!\sim \bot$, then $\alpha \vdash \bot$ (Consistency preservation)

(RM) If $\alpha \mid\!\sim \beta$ and $\alpha \not\mid\!\sim \neg\gamma$, then $\alpha \wedge \gamma \mid\!\sim \beta$ (Rational monotony)

The well-known condition of Consistency Preservation, (CP), makes it a lot easier to deal with some limiting cases, but it appears to be very strong from a philosophical point of view. So it would by preferable to work without (CP), were it not for the fact that this would incur considerable technical costs.[9] We will make use of it in the following presentation.

Rational monotony, (RM), is a condition made prominent by Lehmann and Magidor (1992), its counterpart for belief revision has been an essential part of the postulates of Alchourrón, Gärdenfors and Makinson (1985). However, in many contexts one may want to reject it as being excessively strong. I agree with Friedman and Halpern (and many others) that it should not be counted as belonging to the "core properties of default reasoning".

## 4. Two kinds of plausibility relations

### 4.1. Friedman-Halpern plausibility relations

Friedman and Halpern use the following set of axioms for their plausibility relations $<_{FH}$:

(FH0a) Not $A <_{FH} A$ (Irreflexivity)

(FH0b) If $A <_{FH} B$ and $B <_{FH} C$, then $A <_{FH} C$ (Transitivity)

[8] The supplementary postulates are labelled postulates number 7 and 8 in the usual AGM numbering. The second supplementary postulate of AGM corresponds to Rational monotony which is very strong (see below). Until some time in the 1990s, the research in nonmonotonic reasoning recognized finer distinctions than belief revision research.

[9] One could replace (CP) by the weaker conditions

($\bot$Cond) If $\alpha \wedge \beta \mid\!\sim \bot$, then $\alpha \mid\!\sim \neg\beta$ and

($\bot$CM) If $\alpha \mid\!\sim \bot$, then $\alpha \wedge \beta \mid\!\sim \bot$

($\bot$Cond) and ($\bot$CM) are very special cases of (Cond), If $\alpha \wedge \beta \mid\!\sim \gamma$, then $\alpha \mid\!\sim \beta \to \gamma$, and (CM). Taken together, they are still a lot weaker than (CP). (Note that($\bot$OR), If $\alpha \mid\!\sim \bot$ and $\beta \mid\!\sim \bot$, then $\alpha \vee \beta \mid\!\sim \bot$ would not be strong enough as a replacement of ($\bot$Cond). It is not (relatively) equivalent with ($\bot$Cond), even though (OR) is (relatively) equivalent with (Cond).) – Friedman and Halpern's condition (FH3) mentioned below, and also the conditions (E∅1) and (E∅2) discussed in Rott (2001, p. 235), are examples of similar technical conditions that are, I think, of little intrinsic interest.

(FH1$^{\leq}$) If $A \subseteq B$, then $A \leq_{\text{FH}} B$ (FH-Dominance)

(FH2) If $A$, $B$ and $C$ are pairwise disjoint and $A <_{\text{FH}} B \cup C$ and $B <_{\text{FH}} A \cup C$, then $A \cup B <_{\text{FH}} C$ (Qualitativeness)

(FH3) If $\emptyset <_{\text{FH}} A \cup B$, then either $\emptyset <_{\text{FH}} A$ or $\emptyset <_{\text{FH}} B$ (Bottom)

The most general concept of a *plausibility relation* in the sense of Friedman and Halpern is defined by (FH0a), (FH0b) and (FH1$^{\leq}$).[10] Much of the beauty of Friedman and Halpern's approach lies in the fact that the elementary conditions (FH0a), (FH0b) and (FH1$^{\leq}$) are suitable not only for the "qualitative" kinds of reasoning they focus on, but are at the same time valid for probabilistic approaches. We are interested in working with the strict relation $<$ rather with a non-strict relation $\leq$.[11] So I suggest to express the content of the non-strict relationship "$A \leq_{\text{FH}} B$" by the condition that for all propositions $C$, if $C <_{\text{FH}} A$, then $C <_{\text{FH}} B$, and if $B <_{\text{FH}} C$, then $A <_{\text{FH}} C$. The FH condition of Dominance is thus transformed into the following variant:

(FH1) If $A \subseteq B$ and $C <_{\text{FH}} A$, then $C <_{\text{FH}} B$;
if $A \subseteq B$ and $B <_{\text{FH}} C$, then $A <_{\text{FH}} C$ (FH-Dominance)

Despite its apparent complexity, this is a very natural condition. A smaller proposition $A$ is at most as plausible as a larger proposition $B$. Thus everything dominated by the former should be dominated by the latter, and everything dominating the latter should dominate the former.

Adding (FH2) and (FH3) gives what Friedman and Halpern call *qualitative* plausibility relations. To the best of my knowledge, the Qualitativeness condition was first mentioned by Friedman and Halpern (1995, p. 182) and Dubois and Prade (1995, p. 155).[12] (FH2) is the most interesting condition. There is in fact some magic to it. Friedman and Halpern discovered it when looking for a condition that is tailor-made for guaranteeing that the (AND) rule is satisfied in default reasoning. But then, "somewhat surprisingly",[13] it turned out that (FH2) at the same time yields cumulative monotony, (CM), and the principal case of the (OR) rule. On the one hand, this can be viewed as good news (and this is how Friedman and Halpern themselves have seen it): Only a single condition is needed to get essentially three central conditions of default reasoning. On the other hand, this may also be regarded as slightly unsettling: How

[10] They actually use plausibility *measures*, i.e., mappings that assign a member of a partially ordered set to each element of the domain. I will not attend to this difference in the present paper.

[11] Why? Because there is no "local" way of distinguishing between plausibility ties and plausibility incomparabilities. This can be seen from the basic idea of (From $\mid\!\sim$ to $<_{\text{BP}}$): If the agent's premise is $\phi_A \vee \phi_B$ and she then disbelieves neither $A$ nor $B$, this may be so either because $A$ and $B$ are tied or because they are incomparable in terms of plausibility. Also cf. Rott (1992, p. 50).

[12] Also see Dubois, Fargier and Prade (2004, pp. 34–35).

[13] See footnote 3.

exactly does this condition manage to achieve three tasks at the same time? Wouldn't it be more perspicuous if we had a modular approach that is capable of accounting for the three central conditions (AND), (CM) and (OR) one by one? This is what I want to offer with the present paper.

4.2. Basic plausibility relations

Basic plausibility relations $<_{\mathrm{BP}}$ are the duals to basic entrenchment relations as presented in Rott (2003). They are relations between propositions, too.[14] They are defined by the following four conditions:

(BP0) Not $A <_{\mathrm{BP}} A$ (Irreflexivity)

(BP1) If $A \neq \emptyset$, then $\emptyset <_{\mathrm{BP}} A$ (Minimality)

(BP2) If $A \cup B <_{\mathrm{BP}} C$, then $A <_{\mathrm{BP}} B \cup C$ (Choice easy)

(BP3) If $A <_{\mathrm{BP}} B \cup C$ and $B <_{\mathrm{BP}} A \cup C$, then $A \cup B <_{\mathrm{BP}} C$ (Choice hard)

Minimality (BP1) corresponds to (Consistency preservation) in default reasoning. The conjunction of the conditions (BP2) and (BP3), or rather the counterpart of this conjunction for entrenchment relations, was called *"Entrenchment and choice"* in (Rott 2001, p. 233) and simply *"Choice"* in (Rott 2003, p. 266). It played a crucial role in the choice-theoretic reinterpretation of belief revision and nonmonotonic reasoning offered in Rott (2001); we will briefly turn to this topic in section 8. (BP3) is an unrestricted variant of Friedman and Halpern's condition of Qualitativeness (FH3). This point was already mentioned in the introduction.[15]

Now we take down some of the properties of basic plausibility relations:

**Lemma 1.** Basic plausibility relations satisfy the following conditions.

(BP-i) If $A <_{\mathrm{BP}} B$, then not $B <_{\mathrm{BP}} A$ (Asymmetry)

(BP-ii) $A <_{\mathrm{BP}} B$ iff $A <_{\mathrm{BP}} A \cup B$ (Join right)

(BP-iii) $A <_{\mathrm{BP}} B$ iff $A <_{\mathrm{BP}} -A \cap B$ (Meet right)

(BP-iv) If $A <_{\mathrm{BP}} B$, then $B \not\subseteq A$ (GM-Dominance)

[14] Basic entrenchment relations were originally presented as relations between sentences rather than propositions. If we wanted to do the same here for the dual notion of plausibility, we would need a condition of Extensionality, like this one:

If $\alpha \dashv\vdash \beta$, then: $\alpha <_{\mathrm{BP}} \gamma$ iff $\beta <_{\mathrm{BP}} \gamma$, and $\gamma <_{\mathrm{BP}} \alpha$ iff $\gamma <_{\mathrm{BP}} \beta$

[15] However, somewhat confusingly, Halpern (1997, p. 5) uses the term "qualitative" for relations that satisfy the *unrestricted* condition (Choice hard). Since this usage was not maintained in (Friedman and) Halpern's later work, I will ignore it from now on.

(BP-v) If $A <_{\mathrm{BP}} C$ and $B <_{\mathrm{BP}} C$ and $A \cup B \subseteq C$, then $A \cup B <_{\mathrm{BP}} C$ (Weak join left)

(BP-vi) If $A <_{\mathrm{BP}} B$ and $A \cap -B \subseteq C \subseteq A$, then $C <_{\mathrm{BP}} B$ (Weak continuing down)

(BP-vii) If $A <_{\mathrm{BP}} B$ and $B \subseteq C \subseteq A \cup B$, then $A <_{\mathrm{BP}} C$ (Weak continuing up)[16]

*Proof.* (Asymmetry) Suppose for reductio that $A <_{\mathrm{BP}} B$ and $B <_{\mathrm{BP}} A$, i.e., $A \cup A <_{\mathrm{BP}} B$ and $B \cup B <_{\mathrm{BP}} A$. By (Choice easy), we get $A <_{\mathrm{BP}} A \cup B$ and $B <_{\mathrm{BP}} A \cup B$. This also means that $A <_{\mathrm{BP}} B \cup (A \cup B)$ and $B <_{\mathrm{BP}} A \cup (A \cup B)$. By (Choice hard), it follows that $A \cup B <_{\mathrm{BP}} A \cup B$, contradicting (Irreflexivity).
(Join right) Let $A <_{\mathrm{BP}} B$. This means that $A \cup A <_{\mathrm{BP}} B$. By (Choice easy) and (Choice hard), this is equivalent with $A <_{\mathrm{BP}} A \cup B$.
(Meet right) Let $A <_{\mathrm{BP}} B$. By (Join right), this is true just in case that $A <_{\mathrm{BP}} A \cup B$. Since $A \cup (-A \cup B) = A \cup B$, this means that $A <_{\mathrm{BP}} A \cup (-A \cup B)$. By (Choice easy) and (Choice hard), this is true just in case $A <_{\mathrm{BP}} -A \cup B$.
(GM-Dominance) Suppose for reductio that $A <_{\mathrm{BP}} B$ and $B \subseteq A$. Since $A = A \cup B$, we get $A \cup B <_{\mathrm{BP}} B$. So by (Join right), $A \cup B <_{\mathrm{BP}} (A \cup B) \cup B$, contradicting (Irreflexivity).
(Weak join left) Let $A <_{\mathrm{BP}} C$, $B <_{\mathrm{BP}} C$ and $A \cup B \subseteq C$. Since $A$ and $B$ are subsets of $C$, we have $A <_{\mathrm{BP}} B \cup C$ and $B <_{\mathrm{BP}} A \cup C$, so by (Choice hard) $A \cup B <_{\mathrm{BP}} C$.
(Weak continuing down) Let $A <_{\mathrm{BP}} B$ and $A \cap -B \subseteq C \subseteq A$. Since $A \cup C = A$, $A \cup C <_{\mathrm{BP}} B$. So by (Choice easy), $C <_{\mathrm{BP}} A \cup B$. But $A \cup B = C \cup B$, so this means that $C <_{\mathrm{BP}} C \cup B$. Thus by (Choice hard) $C <_{\mathrm{BP}} B$.
(Weak continuing up) Let $A <_{\mathrm{BP}} B$ and $B \subseteq C \subseteq A \cup B$. By (Choice easy), we get $A <_{\mathrm{BP}} A \cup B$. Since $A \cup C = A \cup B$, we get $A <_{\mathrm{BP}} A \cup C$. Thus by (Choice hard) $A <_{\mathrm{BP}} C$. QED

As we have seen, basic plausibility functions have quite a number of nice properties. Still they may be surprisingly ill-behaved. They are not in general acyclic, and thus, given irreflexivity, they are not in general transitive.[17] This defect can be remedied in an instructive way if the following two conditions are employed:

(BP4) If $A <_{\mathrm{BP}} B$, then $A <_{\mathrm{BP}} B \cup C$ (Continuing up)

(BP5) If $A \cup C <_{\mathrm{BP}} B$, then $A <_{\mathrm{BP}} B$ (Continuing down)

Let us call basic plausibility relations $<_{\mathrm{BP}}$ that satisfy (Continuing up) and (Continuing down) *well-behaved.*

[16] This condition could be strengthened to

If $A <_{\mathrm{BP}} B$ and $-A \cap B \subseteq C \subseteq A \cup B$, then $A <_{\mathrm{BP}} C$

with essentially the same proof, but then it would not be a weakened form of (Continuing up) (see below) any more.

[17] A counterexample for the dual notion of basic entrenchment is given in Rott (2003, p. 268, footnote 20).

Analogues of (BP4) and (BP5) have been known under the names "Continuing up" and "Continuing down" since Alchourrón & Makinson (1985). Taken together, they are essentially identical with (our rendering of) the Friedman-Halpern condition of Dominance (FH1), which we have called a natural condition. (BP4) and (BP5) immediately imply (Choice easy), and one half of each of (BP-ii) and (BP-iii). But they also give rise to substantially new properties which we collect in the next lemma.

**Lemma 2.** (a) Well-behaved basic plausibility relations satisfy the following conditions.

(BP-viii) If $A <_{\mathrm{BP}} B$ and $B <_{\mathrm{BP}} C$, then $A <_{\mathrm{BP}} C$ (Transitivity)

(BP-ix) If $A <_{\mathrm{BP}} C$ and $B <_{\mathrm{BP}} C$, then $A \cup B <_{\mathrm{BP}} C$ (Join left)

(BP-x) If $A <_{\mathrm{BP}} C$ and $B <_{\mathrm{BP}} D$, then $A \cup B <_{\mathrm{BP}} C \cup D$ (Join left&right)

(b) An alternative axiomatization of well-behaved basic plausibility relations can be given by Irreflexivity (BP0), Minimality (BP1), (Continuing up) (BP4), (Continuing down) (BP5), Join right (BP-ii) and Join left (BP-ix).[18]

*Proof.* (a) (BP-viii), transitivity: Let $A <_{\mathrm{BP}} B$ and $B <_{\mathrm{BP}} C$. Then, by (Continuing up), $A <_{\mathrm{BP}} B \cup C$ and $B <_{\mathrm{BP}} A \cup C$. By (Choice hard), it follows that $A \cup B <_{\mathrm{BP}} C$. So by (Continuing down), $A <_{\mathrm{BP}} C$.[19]
(BP-ix): Let $A <_{\mathrm{BP}} C$ and $B <_{\mathrm{BP}} C$. By (Continuing up), $A <_{\mathrm{BP}} B \cup C$ and $B <_{\mathrm{BP}} A \cup C$. So by (Choice hard), $A \cup B <_{\mathrm{BP}} C$.
(BP-x): Let $A <_{\mathrm{BP}} C$ and $B <_{\mathrm{BP}} D$. By (Continuing up), $A <_{\mathrm{BP}} B \cup C \cup D$ and $B <_{\mathrm{BP}} A \cup C \cup D$. So by (Choice hard), $A \cup B <_{\mathrm{BP}} C \cup D$.
(b) As already noted, (Choice easy) follows from (Continuing up) and (Continuing down). It remains to show that (Choice hard) is valid. Suppose that $A <_{\mathrm{BP}} B \cup C$ and $B <_{\mathrm{BP}} A \cup C$. Then, by (Continuing up), $A <_{\mathrm{BP}} A \cup B \cup C$ and $B <_{\mathrm{BP}} A \cup B \cup C$. So, by (BP-ix), $A \cup B <_{\mathrm{BP}} A \cup B \cup C$, and by (BP-ii), $A \cup B <_{\mathrm{BP}} C$. QED

The last condition of interest is Modularity (also known as Virtual connectivity or Negative transitivity):

[18] This in fact is essentially the axiomatization given by Freund (1993, pp. 237–238). It is essentially dual the to axiomatization of "generalized epistemic entrenchment" given in Rott (1992, p. 55).

[19] It is easy to show that the following condition corresponds precisely to the transitivity of $<_{\mathrm{BP}}$

(+) If $\alpha \vee \beta \mid\sim \neg\alpha$ and $\beta \vee \gamma \mid\sim \neg\beta$, then $\alpha \vee \gamma \mid\sim \neg\alpha$

This condition is valid in preferential reasoning. Proof: From $\alpha \vee \beta \mid\sim \neg\alpha$ and $\neg\alpha \wedge \gamma \mid\sim \neg\alpha$ it follows by (OR) that $\alpha \vee \beta \vee \gamma \mid\sim \neg\alpha$. Similarly, we get from $\beta \vee \gamma \mid\sim \neg\beta$ that $\alpha \vee \beta \vee \gamma \mid\sim \neg\beta$. By (REF), (AND) and (RW), we get $\alpha \vee \beta \vee \gamma \mid\sim \alpha \vee \gamma$. So by (CM) and (LLE), $\alpha \vee \gamma \mid\sim \neg\alpha$. Since apparently (+) cannot be nicely simplified and is not of any intrinsic interest, it will not be mentioned here any more.

(BP6) If $A <_{\text{BP}} B$, then $A <_{\text{BP}} C$ or $C <_{\text{BP}} B$ (Modularity)

It is well-known in the theory of belief revision and nonmonotonic reasoning that this condition corresponds to Rational Monotony.

4.3. The properties of basic plausibility vindicated

In the light of our retrieval idea embodied in (From $\mid\!\sim$ to $<_{\text{BP}}$), it turns out that the properties of basic entrenchment can be nicely vindicated.

**Lemma 3.** Let the basic plausibility relation $<_{\text{BP}}$ be derived from a given inference relation $\mid\!\sim$ by way of (From $\mid\!\sim$ to $<_{\text{BP}}$), that is, let $<_{\text{BP}} = \mathcal{P}_{\text{B}}(\mid\!\sim)$. Then

(i) (Irreflexivity) is trivially satisfied,

(ii) (Choice easy) follows from (RW),

(iii) (Choice hard) follows from (RW) and (AND),

(iv) (Continuing up) follows from (LLE), (RW), (REF), (AND), ($\perp$CM) and (OR),

(v) (Continuing down) follows from (LLE), (RW), (AND), ($\perp$Cond) and (CM).

*Proof.* Suppose that $<_{\text{BP}}$ is generated from $\mid\!\sim$ by (From $\mid\!\sim$ to $<_{\text{BP}}$). Because of (LLE) and (RW), $<_{\text{BP}}$ is well-defined. Since (LLE) is not contested we will use it without mentioning it explicitly.
(i) (Irreflexivity) is trivial.
(ii) (Choice easy) Let $A \cup B <_{\text{BP}} C$. We want to show that $A <_{\text{BP}} B \cup C$ and $B <_{\text{BP}} A \cup C$. Using (From $\mid\!\sim$ to $<_{\text{BP}}$), the supposition gives us
$(\phi_A \vee \phi_B) \vee \phi_C \mid\!\sim \neg(\phi_A \vee \phi_B)$ and $(\phi_A \vee \phi_B) \vee \phi_C \not\mid\!\sim \neg\phi_C$.
We need to show that
$\phi_A \vee (\phi_B \vee \phi_C) \mid\!\sim \neg\phi_A$ and $\phi_A \vee (\phi_B \vee \phi_C) \not\mid\!\sim \neg(\phi_B \vee \phi_C)$,
as well as
$\phi_B \vee (\phi_A \vee \phi_C) \mid\!\sim \neg\phi_B$ and $\phi_B \vee (\phi_A \vee \phi_C) \not\mid\!\sim \neg(\phi_A \vee \phi_C)$.
This, too, follows straightforwardly from the supposition by repeated applications of (LLE) and (RW).
(iii) (Choice hard) Let $A <_{\text{BP}} B \cup C$ and $B <_{\text{BP}} A \cup C$. We want to show that $A \cup B <_{\text{BP}} C$. Using (From $\mid\!\sim$ to $<_{\text{BP}}$), the supposition gives us
$\phi_A \vee \phi_B \vee \phi_C \mid\!\sim \neg\phi_A$ and $\phi_A \vee \phi_B \vee \phi_C \not\mid\!\sim \neg(\phi_B \vee \phi_C)$,
as well as
$\phi_A \vee \phi_B \vee \phi_C \mid\!\sim \neg\phi_B$ and $\phi_A \vee \phi_B \vee \phi_C \not\mid\!\sim \neg(\phi_A \vee \phi_C)$.
We need to show that
$\phi_A \vee \phi_B \vee \phi_\gamma \mid\!\sim \neg(\phi_A \vee \phi_B)$ and $\phi_A \vee \phi_B \vee \phi_\gamma \not\mid\!\sim \neg\phi_C$.

This follows straightforwardly from the supposition by repeated applications of (AND) and (RW).
(iv) (Continuing up) Let $A <_{\text{BP}} B$. We want to show that $A <_{\text{BP}} B \cup C$. Using (From $\mid\!\sim$ to $<_{\text{BP}}$), the supposition gives us
$\phi_A \vee \phi_B \mid\!\sim \neg\phi_A$ and $\phi_A \vee \phi_B \not\mid\!\sim \neg\phi_B$.
We need to show that
$\phi_A \vee \phi_B \vee \phi_C \mid\!\sim \neg\phi_A$ and $\phi_A \vee \phi_B \vee \phi_C \not\mid\!\sim \neg(\phi_B \vee \phi_C)$.
By (REF) and (RW), we have that
$\neg\phi_A \wedge \phi_C \mid\!\sim \neg\phi_A$.
So by (OR), we get
$(\phi_A \vee \phi_B) \vee (\neg\phi_A \wedge \phi_C) \mid\!\sim \neg\phi_A$, which simplifies to $\phi_A \vee \phi_B \vee \phi_C \mid\!\sim \neg\phi_A$.
Now suppose for reductio that also
$\phi_A \vee \phi_B \vee \phi_C \mid\!\sim \neg(\phi_B \vee \phi_C)$.
Then by (AND) and (RW), $\phi_A \vee \phi_B \vee \phi_C \mid\!\sim \bot$.
But then by ($\bot$CM) and (LLE), $\phi_A \vee \phi_B \mid\!\sim \bot$
and by (RW) $\phi_A \vee \phi_B \mid\!\sim \neg\phi_B$,
and we have a contradiction.
(v) (Continuing down) Let $A \cup C <_{\text{BP}} B$. We want to show that $A <_{\text{BP}} B$. Using (From $\mid\!\sim$ to $<_{\text{BP}}$), the supposition gives us
$\phi_A \vee \phi_B \vee \phi_C \mid\!\sim \neg(\phi_A \vee \phi_C)$ and $\phi_A \vee \phi_B \vee \phi_C \not\mid\!\sim \neg\phi_B$.
We need to show that
$\phi_A \vee \phi_B \mid\!\sim \neg\phi_A$ and $\phi_A \vee \phi_B \not\mid\!\sim \neg\phi_B$.
From the supposition, we get by two different applications of (RW) that both
$\phi_A \vee \phi_B \vee \phi_C \mid\!\sim \neg\phi_A$ and
$\phi_A \vee \phi_B \vee \phi_C \mid\!\sim \phi_A \vee \phi_B \vee \neg\phi_C$.
So by (CM) and (LLE) $\phi_A \vee \phi_B \mid\!\sim \neg\phi_A$.
Now suppose for reductio that also
$\phi_A \vee \phi_B \mid\!\sim \neg\phi_B$.
Then by (AND) and (RW), $\phi_A \vee \phi_B \mid\!\sim \bot$.
By (LLE) and ($\bot$Cond), we get
$(\phi_A \vee \phi_B \vee \phi_C) \wedge (\phi_A \vee \phi_B) \mid\!\sim \bot$ and $\phi_A \vee \phi_B \vee \phi_C \mid\!\sim \neg(\phi_A \vee \phi_B)$.
So finally, by (RW)
$\phi_A \vee \phi_B \vee \phi_C \mid\!\sim \neg\phi_B$,
and we have a contradiction. QED

Lemma 3 does not give a full 1-1 mapping between the properties of the default inference relation $\mid\!\sim$ and the properties of the derived plausibility relation $<_{\text{BP}} = \mathcal{P}_{\text{B}}(\mid\!\sim)$. But even in the more problematic cases (iv) and (v), it seems justified to call (OR) the property that essentially yields (Continuing up) and (CM) the property that essentially yields (Continuing down).

### 4.4. Basic plausibility relations are different from FH plausibility relations

#### 4.4.1. Not all basic plausibility relations are FH plausibility relations

As we have pointed out, basic plausibility relations need not be transitive.

But all basic plausibility relations that satisfy (Continuing up) and (Continuing down) are FH plausibility relations. (FH0a) is (BP0), (FH0b) is (BP-viii), (FH1) is the conjunction of (Continuing up) and (Continuing down), (FH2) is a restricted form of (Choice hard), and (FH3) follows from (BP0) and (BP1).

4.4.2. Not all FH plausibility relations are basic plausibility relations

It turns out that (Choice) is stronger than (Qualitativeness), even in the presence of Friedman and Halpern's other conditions One might have suspected that Friedman and Halpern's restricted condition of (Qualitativeness) is sufficient to imply the (Choice) condition, if the more basic properties of FH plausibilities are satisfied as well. This, however, is not true. Joseph Halpern has come up with the following counterexample.

*Example 2.*[20] Let $W = \{u, v, w\}$ and assign plausibility values to the subsets of $W$ as follows: $plaus(\emptyset) = 0$, $plaus(\{u\}) = 2$, $plaus(\{v\}) = 1$, $plaus(\{w\}) = 1$, $plaus(\{u, v\}) = 2$, $plaus(\{u, w\}) = 2$, $plaus(\{v, w\}) = 1$ and $plaus(\{u, v, w\}) = 3$. The plausibility relation $<$ between the subsets of $W$ is supposed to derive from these numbers in the obvious way: $A < B$ iff $plaus(A) < plaus(B)$. The only non-trivial comparison for Qualitativeness concerns the three disjoint sets $\{u\}$, $\{v\}$ and $\{w\}$, and we can indeed verify that

$$\{w\} < \{v\} \cup \{u\} \text{ and } \{v\} < \{w\} \cup \{u\} \text{ taken together imply } \{v\} \cup \{w\} < \{u\}$$

(This is just $1 < 2$ everywhere.) Thus the condition Qualitativeness is satisfied in this example. However, for sets that are not disjoint, the analogous implication does not longer hold. We find that

$$\{v, w\} < \{u, w\} \cup \{u, v\} \text{ and } \{u, w\} < \{v, w\} \cup \{u, v\}, \quad \text{but not } \{u, w\} \cup \{v, w\} < \{u, v\}.$$

(This is $1 < 3$ and $2 < 3$ on the left-hand side, but $3 \not< 2$ on the right-hand side.) Thus the condition Choice does not hold here.

The counterexample shows that Qualitativeness does not imply Choice, even in the presence of Friedman and Halpern's remaining conditions (FH0a), (FH0b), ($FH1^{\leq}$) and (FH3).

## 5. Using plausibility relations for default reasoning

FH use the following condition for the construction of a default inference relation on the basis of their plausibility relations[21]

[20] Personal communication, July 2011. Dubois, Fargier and Prade (2004, pp. 34) remark that Choice is much stronger than Qualitativeness, but they do not present a counterexample.

[21] See Friedman and Halpern (2001, p. 655) and Halpern (2003, p. 302).

(From $<_{\text{FH}}$ to $\mid\!\sim$) $\alpha \mid\!\sim \beta$ iff $[\![\alpha \wedge \neg\beta]\!] <_{\text{FH}} [\![\alpha \wedge \beta]\!]$ or $\emptyset \not<_{\text{FH}} [\![\alpha]\!]$

The condition for basic plausibility uses the dual of the entrenchment-based method:[22]

(From $<_{\text{BP}}$ to $\mid\!\sim$) $\alpha \mid\!\sim \beta$ iff $[\![\alpha \wedge \neg\beta]\!] <_{\text{BP}} [\![\alpha]\!]$ or $[\![\alpha]\!] = \emptyset$.

We seem to have different suggestions here. However, given (BP-iii), the main condition $[\![\alpha \wedge \neg\beta]\!] <_{\text{BP}} [\![\alpha]\!]$ is equivalent with $[\![\alpha \wedge \neg\beta]\!] <_{\text{BP}} [\![\alpha \wedge \beta]\!]$.[23] The limiting cases are equivalent given (Minimality). We want to endorse (Minimality) for the sake of simplicity. So there is no real difference here, and we can use the following equation generically, for both kinds of plausibility relations.

(From $<$ to $\mid\!\sim$) $\quad \alpha \mid\!\sim \beta$ iff $[\![\alpha \wedge \neg\beta]\!] < [\![\alpha \wedge \beta]\!]$ or $[\![\alpha]\!] = \emptyset$.

Let us use the notation $\mathcal{I}(<)$ for the default inference relation $\mid\!\sim$ based on a given plausibility relation $<$ by way of (From $<$ to $\mid\!\sim$).

**Lemma 4.** Let $\mid\!\sim$ be generated from $<$ by (From $<$ to $\mid\!\sim$). Then

$\mid\!\sim$ satisfies (LLE).

If $<$ satisfies (Irreflexivity), then $\mid\!\sim$ satisfies (REF).

If $<$ satisfies (Choice easy), then $\mid\!\sim$ satisfies (RW).

If $<$ satisfies (Choice easy) and (Choice hard), then $\mid\!\sim$ satisfies (AND).

If $<$ satisfies (Choice easy), (Choice hard) and (Continuing up), then $\mid\!\sim$ satisfies (OR).

If $<$ satisfies (Choice easy), (Choice hard) and (Continuing down), then $\mid\!\sim$ satisfies (CM).

*Proof.* (LLE) Let $\alpha \dashv\vdash \beta$ and $\alpha \mid\!\sim \gamma$, that is, $[\![\alpha \wedge \neg\gamma]\!] < [\![\alpha \wedge \gamma]\!]$ or $[\![\alpha]\!] = \emptyset$. For $\beta \mid\!\sim \gamma$ we have to show that $[\![\beta \wedge \neg\gamma]\!] < [\![\beta \wedge \gamma]\!]$ or $[\![\beta]\!] = \emptyset$. But this is immediate since $\alpha \dashv\vdash \beta$ implies that $[\![\beta]\!] = [\![\alpha]\!]$, $[\![\beta \wedge \gamma]\!] = [\![\alpha \wedge \gamma]\!]$ and $[\![\beta \wedge \neg\gamma]\!] = [\![\alpha \wedge \neg\gamma]\!]$.

(REF) For $\alpha \mid\!\sim \alpha$ we need to show that $[\![\alpha \wedge \neg\alpha]\!] < [\![\alpha \wedge \alpha]\!]$ or $[\![\alpha]\!] = \emptyset$. But the former reduces to $\emptyset < [\![\alpha]\!]$. This is fulfilled if $[\![\alpha]\!] \neq \emptyset$, by (BP1).

(RW) Let $\alpha \mid\!\sim \beta$ and $\beta \vdash \gamma$. Then $[\![\alpha \wedge \neg\beta]\!] < [\![\alpha \wedge \beta]\!]$ or $[\![\alpha]\!] = \emptyset$. If the latter, we have $\alpha \mid\!\sim \gamma$ by definition. So assume the former. We need to show

[22] Used in Rott (2001, pp. 254, 264) and in Rott (2003, p. 264).
[23] This equivalence is not valid for FH plausibility relations.

that $[\![\alpha \wedge \neg\gamma]\!] < [\![\alpha \wedge \gamma]\!]$. Since $[\![\alpha \wedge \neg\beta]\!] = [\![\alpha \wedge \neg\beta \wedge \gamma]\!] \cup [\![\alpha \wedge \neg\gamma]\!]$, we have $[\![\alpha \wedge \neg\beta \wedge \gamma]\!] \cup [\![\alpha \wedge \neg\gamma]\!] < [\![\alpha \wedge \beta]\!]$. So by (Choice easy), $[\![\alpha \wedge \neg\gamma]\!] < [\![\alpha \wedge \neg\beta \wedge \gamma]\!] \cup [\![\alpha \wedge \beta]\!]$. Equivalently, $[\![\alpha \wedge \neg\gamma]\!] < [\![\alpha \wedge \gamma]\!]$, and this is what we wanted to show.

(AND) Let $\alpha \mid\!\sim \beta$ and $\alpha \mid\!\sim \gamma$. Then $[\![\alpha\wedge\neg\beta]\!] < [\![\alpha\wedge\beta]\!]$ and $[\![\alpha\wedge\neg\gamma]\!] < [\![\alpha\wedge\gamma]\!]$, or $[\![\alpha]\!] = \emptyset$. If the latter, we have $\alpha \mid\!\sim \beta \wedge \gamma$ by definition. So suppose the former. We have to show that $[\![\alpha \wedge \neg(\beta \wedge \gamma)]\!] < [\![\alpha \wedge \beta \wedge \gamma]\!]$. By (Choice easy), $[\![\alpha\wedge\neg\beta]\!] < [\![\alpha]\!]$ and $[\![\alpha\wedge\neg\gamma]\!] < [\![\alpha]\!]$. So by (BP-v), $[\![\alpha\wedge\neg\beta]\!] \cup [\![\alpha\wedge\neg\gamma]\!] < [\![\alpha]\!]$, or equivalently, $[\![\alpha \wedge \neg(\beta \wedge \gamma)]\!] < [\![\alpha]\!]$. By (BP-iii), $[\![\alpha \wedge \neg(\beta \wedge \gamma)]\!] < -([\![\alpha \wedge \neg(\beta \wedge \gamma)]\!]) \cap [\![\alpha]\!]$, or equivalently, $[\![\alpha \wedge \neg(\beta \wedge \gamma)]\!] < [\![\alpha \wedge \beta \wedge \gamma]\!]$, which is what we wanted to show. (Note that (BP-iii) and (BP-v) follow from (Choice easy) and (Choice hard).)

(OR) Let $\alpha \mid\!\sim \gamma$ and $\beta \mid\!\sim \gamma$. Then $[\![\alpha\wedge\neg\gamma]\!] < [\![\alpha\wedge\gamma]\!]$ or $[\![\alpha]\!] = \emptyset$, and $[\![\beta\wedge\neg\gamma]\!] < [\![\beta\wedge\gamma]\!]$ or $[\![\beta]\!] = \emptyset$. If $[\![\alpha]\!] = \emptyset$ or $[\![\beta]\!] = \emptyset$, then $\beta$ or $\alpha$ is logically equivalent with $\alpha \vee \beta$, and we get $\alpha \vee \beta \mid\!\sim \gamma$ by (LLE). So suppose that $[\![\alpha]\!] \neq \emptyset$ and $[\![\beta]\!] \neq \emptyset$. We have to show that $[\![(\alpha\vee\beta)\wedge\neg\gamma)]\!] < [\![(\alpha\vee\beta)\wedge\gamma]\!]$. By (Continuing up), we get $[\![\alpha\wedge\neg\gamma]\!] < [\![\beta\wedge\neg\gamma]\!] \cup [\![\alpha\wedge\gamma]\!] \cup [\![\beta\wedge\gamma]\!]$ and $[\![\beta\wedge\neg\gamma]\!] < [\![\alpha\wedge\neg\gamma]\!] \cup [\![\alpha\wedge\gamma]\!] \cup [\![\beta\wedge\gamma]\!]$. So by (Choice hard), $[\![\alpha \wedge \neg\gamma]\!] \cup [\![\beta \wedge \neg\gamma]\!] < [\![\alpha \wedge \gamma]\!] \cup [\![\beta \wedge \gamma]\!]$, and this is equivalent with $[\![(\alpha \vee \beta) \wedge \neg\gamma)]\!] < [\![(\alpha \vee \beta) \wedge \gamma]\!]$, which is what we wanted to show.

(CM) Let $\alpha \mid\!\sim \beta$ and $\alpha \mid\!\sim \gamma$. Then $[\![\alpha\wedge\neg\beta]\!] < [\![\alpha\wedge\beta]\!]$ and $[\![\alpha\wedge\neg\gamma]\!] < [\![\alpha\wedge\gamma]\!]$, or $[\![\alpha]\!] = \emptyset$. If the latter, we have $[\![\alpha \wedge \gamma]\!] = \emptyset$, and we get $\alpha \wedge \gamma \mid\!\sim \beta$ by definition. So suppose the former. We have to show that $[\![\alpha \wedge \neg\beta \wedge \gamma]\!] < [\![\alpha \wedge \beta \wedge \gamma]\!]$. We first copy the proof of (AND) and derive $[\![\alpha \wedge \neg(\beta \wedge \gamma)]\!] < [\![\alpha \wedge \beta \wedge \gamma]\!]$. Now we note that $[\![\alpha \wedge \neg\beta \wedge \gamma]\!] \subseteq [\![\alpha \wedge \neg(\beta \wedge \gamma)]\!]$, apply (Continuing down) and get $[\![\alpha \wedge \neg\beta \wedge \gamma]\!] < [\![\alpha \wedge \beta \wedge \gamma]\!]$, as desired. QED

Like Lemma 3, Lemma 4 does not give a full 1-1 mapping between the properties of plausibility relations and properties of default inference relation. But again, even in the last two cases which appear to be most problematic, (Continuing up) is quite definitely the property corresponding to (OR), and (Continuing down) is the property corresponding to (CM).

## 6. Plausibility-based default reasoning

### 6.1. FH plausibilities retrieved from default reasoning

In section 2, we mentioned that it has been a central idea of basic entrenchment to interpret it as retrieved from a belief/expectation withdrawal operation. We showed how basic plausibility can equally well be retrieved from a given default inference relation and obtained the condition (From $\mid\!\sim$ to $<_{\text{BP}}$). There is a comparable move in Friedman and Halpern's work (2001, pp. 657, 669–70, Lemma 4.1) that starts from sets of defaults satisfying the rules of preferential reasoning.

(From $\mid\!\sim$ to $\leq_{\text{FH}}$) $A \leq_{\text{FH}} B$ iff $\phi_A \vee \phi_B \mid\!\sim \phi_B$

This retrieval method was used by Kraus, Lehmann and Magidor (1990, Definition 5.9),[24] who observed that the relation $\leq_{\text{FH}}$ thus defined is transitive, provided that $\mid\!\sim$ is a preferential system (Kraus et al. 1990, Lemma 5.5). This observation was used by Friedman and Halpern (2001, Proof of Lemma 4.1). It should be noted, however, that this was a mainly technical matter for Friedman and Halpern, while we have obtained (From $\mid\!\sim$ to $<_{\text{BP}}$) as a result of our attempt to gain an intuitive understanding of the concept of plausibility.

As already pointed out on page 8, we are interested in working with a strict relation $<_{\text{FH}}$ rather than with the non-strict relation $\leq_{\text{FH}}$. It is natural to assume that we obtain the appropriate strict relation just by taking the asymmetric part of Friedman and Halpern's non-strict relation $\leq_{\text{FH}}$. This is captured by the following definition:

(From $\mid\!\sim$ to $<_{\text{FH}}$) $A <_{\text{FH}} B$ iff $\phi_A \vee \phi_B \mid\!\sim \phi_B$ and $\phi_A \vee \phi_B \not\mid\!\sim \phi_A$

Let us use the notation $\mathcal{P}_{\text{FH}}(\mid\!\sim)$ for the plausibility relation $<_{\text{FH}}$ so derived from a given inference relation $\mid\!\sim$.

What is the relation between $\mathcal{P}_{\text{FH}}(\mid\!\sim)$ and $\mathcal{P}_{\text{B}}(\mid\!\sim)$? Provided that $\mid\!\sim$ satisfies (REF), (AND) and (RW), $\phi_A \vee \phi_B \mid\!\sim \neg\phi_A$ and $\phi_A \vee \phi_B \not\mid\!\sim \neg\phi_B$ taken together imply the conjunction of $\phi_A \vee \phi_B \mid\!\sim \phi_B$ and $\phi_A \vee \phi_B \not\mid\!\sim \phi_A$. So $<_{\text{BP}} = \mathcal{P}_{\text{B}}(\mid\!\sim)$ is a subrelation of $<_{\text{FH}} = \mathcal{P}_{\text{FH}}(\mid\!\sim)$.[25]

But the two relations are in general different. The recipes (From $\mid\!\sim$ to $<_{\text{BP}}$) and (From $\mid\!\sim$ to $<_{\text{FH}}$) come apart just in case $\phi_A \vee \phi_B \mid\!\sim \phi_B$, $\phi_A \vee \phi_B \not\mid\!\sim \phi_A$ and $\phi_A \vee \phi_B \not\mid\!\sim \neg\phi_A$. This is a consistent state of affairs. For instance, if the plausibility of a proposition is "measured by" the plausibility of its most plausible worlds (as is common in many semantics), then the two relations are divergent if the set of most plausible worlds in $A \cup B$ is contained in $B$, but intersects both $A$ and $-A$. In such cases, we have $A <_{\text{FH}} B$ but not $A <_{\text{BP}} B$.[26] Here is an example:

*Example 3.* Consider the plausibility relation between three worlds $u$, $v$ and $w$ with $u$ being more plausible than both $v$ and $w$, and $v$ and $w$ being unrelated. Suppose that the inference operation $\mid\!\sim$ is based on this pre-ordering: $\alpha \mid\!\sim \beta$ holds just in case all the maximally plausible worlds in $[\![\alpha]\!]$ are in $[\![\beta]\!]$. Let $A$ be $\{v\}$ and $B$ be $\{v, w\}$. Here we have in fact $\phi_A \vee \phi_B \mid\!\sim \phi_B$, $\phi_A \vee \phi_B \not\mid\!\sim \phi_A$ and $\phi_A \vee \phi_B \not\mid\!\sim \neg\phi_A$. Thus $A <_{\text{FH}} B$ but not $A <_{\text{BP}} B$.

[24] As mentioned in footnote 7, Lehmann and Magidor later changed to the method of (From $\mid\!\sim$ to $<_{\text{BP}}$) in the context of "rational" systems of nonmonotonic reasoning, unfortunately without commenting on this change.

[25] For this reason, whenever $\mathcal{P}_{\text{B}}(\mid\!\sim)$ fails to be acyclic, so does $\mathcal{P}_{\text{FH}}(\mid\!\sim)$.

[26] Suppose we interpret $<$ abstractly as a relation that makes distinctions: $A < B$ just means that $A$ and $B$ can be told apart. Then the basic plausibility relation is *coarser* than the Friedman-Halpern plausibility relation. (But we should not put too much strain on this interpretation.)

Notice that for *disjoint* propositions $A$ and $B$, $\phi_A <_{\mathrm{FH}} \phi_B$ implies $\phi_A <_{\mathrm{BP}} \phi_B$. The reason is that if $\phi_B \vdash \neg\phi_A$, then $\phi_A \vee \phi_B \mathrel{|\!\sim} \phi_B$ implies $\phi_A \vee \phi_B \mathrel{|\!\sim} \neg\phi_A$, by (RW), and that $\phi_A \vee \phi_B \mathrel{|\!\sim} \phi_B$ and $\phi_A \vee \phi_B \not\mathrel{|\!\sim} \phi_A$ taken together imply $\phi_A \vee \phi_B \not\mathrel{|\!\sim} \neg\phi_B$, by (AND) and (RW). So the two relations $<_{\mathrm{BP}} = \mathcal{P}_{\mathrm{B}}(\mathrel{|\!\sim})$ and $<_{\mathrm{FH}} = \mathcal{P}_{\mathrm{FH}}(\mathrel{|\!\sim})$ derived from a given inference relation $\mathrel{|\!\sim}$ agree on all pairs of disjoint propositions.

## 6.2. The harmony between default reasoning and plausibility relations

The following observation makes sure that the Friedman-Halpern relation retrieved from an inference relation for default reasoning $\mathrel{|\!\sim}$ is suitable for reconstructing this very relation. This result is, of course, due to Friedman and Halpern. Within this section, we suppose that $\mathrel{|\!\sim}$ satisfies (LLE), (REF), (AND), (RW) and (CP).

**Observation 1 (Friedman and Halpern).** Let $\mathrel{|\!\sim}$ be an inference relation and $<_{\mathrm{FH}} = \mathcal{P}_{\mathrm{FH}}(\mathrel{|\!\sim})$ be the Friedman-Halpern plausibility relation retrieved from $\mathrel{|\!\sim}$. Then the inference relation $\mathrel{|\!\sim}' = \mathcal{I}(<_{\mathrm{FH}})$ is identical with $\mathrel{|\!\sim}$. In short, $\mathcal{I}(\mathcal{P}_{\mathrm{FH}}(\mathrel{|\!\sim})) = \mathrel{|\!\sim}$.

*Proof.* Let $\alpha \mathrel{|\!\sim}' \beta$. Then by (From $<_{\mathrm{FH}}$ to $\mathrel{|\!\sim}$),

$[\![\alpha \wedge \neg\beta]\!] <_{\mathrm{FH}} [\![\alpha \wedge \beta]\!]$ or $\emptyset \not<_{\mathrm{FH}} [\![\alpha]\!]$.

Thus, by (From $\mathrel{|\!\sim}$ to $<_{\mathrm{FH}}$)

$(\alpha \wedge \beta) \vee (\alpha \wedge \neg\beta) \mathrel{|\!\sim} \alpha \wedge \beta$ and $(\alpha \wedge \beta) \vee (\alpha \wedge \neg\beta) \not\mathrel{|\!\sim} \alpha \wedge \neg\beta$, or $\alpha \not\mathrel{|\!\sim} \alpha$ or $\alpha \mathrel{|\!\sim} \bot$

That is, by (LLE) and (Ref)

$\alpha \mathrel{|\!\sim} \alpha \wedge \beta$ and $\alpha \not\mathrel{|\!\sim} \alpha \wedge \neg\beta$, or $\alpha \mathrel{|\!\sim} \bot$

By the reflexivity condition (REF), (AND) and (RW), this means

$\alpha \mathrel{|\!\sim} \beta$ and $\alpha \not\mathrel{|\!\sim} \neg\beta$, or $\alpha \mathrel{|\!\sim} \bot$

Finally, by (AND) and (RW), this reduces to

$\alpha \mathrel{|\!\sim} \beta$ or $\alpha \mathrel{|\!\sim} \bot$

and finally, by (RW) again, to

$\alpha \mathrel{|\!\sim} \beta$

But this means that $\mathrel{|\!\sim}'$ is identical with $\mathrel{|\!\sim}$. QED

Now we want to make sure that the basic plausibility relations retrieved from a default inference relation $\mathrel{|\!\sim}$ are in exactly the same way suitable for the reconstruction of $\mathrel{|\!\sim}$ as the Friedman-Halpern relations are. The following two observations essentially reproduce results from Rott (2001, section 8.7) and Rott (2003, section 3).

**Observation 2.** Let $\mid\!\sim$ be an inference relation and $<_{\text{BP}} = \mathcal{P}_{\text{B}}(\mid\!\sim)$ be the basic plausibility relation retrieved from $\mid\!\sim$. Then the inference relation $\mid\!\sim' = \mathcal{I}(<_{\text{BP}})$ is identical with $\mid\!\sim$. In short, $\mathcal{I}(\mathcal{P}_{\text{B}}(\mid\!\sim)) = \mid\!\sim$.

*Proof.* Let $\alpha \mid\!\sim' \beta$. Then by (From $<$ to $\mid\!\sim$),
$[\![\alpha \wedge \neg\beta]\!] <_{\text{BP}} [\![\alpha \wedge \beta]\!]$, or $[\![\alpha]\!] = \emptyset$
Thus, by (From $\mid\!\sim$ to $<_{\text{BP}}$),
$(\alpha \wedge \beta) \vee (\alpha \wedge \neg\beta) \mid\!\sim \neg(\alpha \wedge \neg\beta)$ and $(\alpha \wedge \beta) \vee (\alpha \wedge \neg\beta) \not\mid\!\sim \neg(\alpha \wedge \beta)$, or $\alpha \vdash \bot$
that is,
$\alpha \mid\!\sim \neg\alpha \vee \beta$ and $\alpha \not\mid\!\sim \neg\alpha \vee \neg\beta$, or $\alpha \vdash \bot$
By the reflexivity condition (REF), (AND) and (RW), this is equivalent with
$\alpha \mid\!\sim \beta$ and $\alpha \not\mid\!\sim \neg\beta$, or $\alpha \vdash \bot$.
By (REF), (AND), (RW) and (CP), this is equivalent with
$\alpha \mid\!\sim \beta$
But this means that $\mid\!\sim'$ is identical with $\mid\!\sim$. QED

Now we verify that we get a similar harmony result between basic plausibility relations $<_{\text{BP}}$ and default inference relations $\mid\!\sim$ if we start from the former rather than from the latter.

**Observation 3.** Let $<_{\text{BP}}$ be a basic plausibility relation and $\mid\!\sim = \mathcal{I}(<_{\text{BP}})$ be the inference relation based on $<_{\text{BP}}$. Then the basic plausibility relation $<_{\text{BP}}' = \mathcal{P}_{\text{B}}(\mid\!\sim)$ retrieved from $\mid\!\sim$ is identical with $<_{\text{BP}}$. In short, $\mathcal{P}_{\text{B}}(\mathcal{I}(<_{\text{BP}})) = <_{\text{BP}}$.

*Proof.* Let $A <_{\text{BP}}' B$. Then by (From $\mid\!\sim$ to $<_{\text{BP}}$),
$\phi_A \vee \phi_B \mid\!\sim \neg\phi_A$ and $\phi_A \vee \phi_B \not\mid\!\sim \neg\phi_B$
Thus, by (From $<_{\text{BP}}$ to $\mid\!\sim$)
$(A \cup B) \cap --A <_{\text{BP}} (A \cup B) \cap -A$ or $A \cup B = \emptyset$, and $(A \cup B) \cap --B \not<_{\text{BP}} (A \cup B) \cap -B$ and $A \cup B \neq \emptyset$
This condition reduces to
$A <_{\text{BP}} -A \cap B$ and $B \not<_{\text{BP}} A \cap -B$ and $A \cup B \neq \emptyset$
which, by (BP-iii) and (Asymmetry) is equivalent to
$A <_{\text{BP}} B$, and $A \neq \emptyset$ or $B \neq \emptyset$
and by (Irreflexivity) this reduces to
$A <_{\text{BP}} B$. QED

The same thing does not work equally well with $<_{\text{FH}}$.

**Observation 4.** Let $<_{\text{FH}}$ be a Friedman-Halpern plausibility relation and $\mid\!\sim = \mathcal{I}(<_{\text{FH}})$ be the inference relation based on $<_{\text{FH}}$. Then the Friedman-Halpern plausibility relation $<_{\text{FH}}' = \mathcal{P}_{\text{FH}}(\mid\!\sim)$ is in general *not* identical with $<_{\text{FH}}$. In short, generally $\mathcal{P}_{\text{FH}}(\mathcal{I}(<_{\text{FH}})) \neq <_{\text{FH}}$.

*Proof.* Let $A <_{\text{FH}}{}' B$. Then by (From $\mid\!\sim$ to $<_{\text{FH}}$),
$\phi_A \vee \phi_B \mid\!\sim \phi_B$ and $\phi_A \vee \phi_B \not\mid\!\sim \phi_A$
Thus, by (From $<_{\text{FH}}$ to $\mid\!\sim$)
$(A \cup B) \cap -B <_{\text{FH}} (A \cup B) \cap B$ or $\emptyset \not<_{\text{FH}} A \cup B$, and
$(A \cup B) \cap -A \not<_{\text{FH}} (A \cup B) \cap A$ and $\emptyset <_{\text{FH}} A \cup B$
This condition reduces to
(+) $\quad A \cap -B <_{\text{FH}} B$ and $-A \cap B \not<_{\text{FH}} A$ and $\emptyset <_{\text{FH}} A \cup B$
This is not reducible to
$A <_{\text{FH}} B$.
Both directions fail, as we will show with the help of two counterexamples. Let in the following $W = \{u, v, w\}$ be the set of possible worlds.
*Example 2* above shows that (+) does not imply $A <_{\text{FH}} B$. Remember that in this example the following plausibility values were assigned to the subsets of $W$: $plaus(\emptyset) = 0$, $plaus(\{u\}) = 2$, $plaus(\{v\}) = 1$, $plaus(\{w\}) = 1$, $plaus(\{u, v\}) = 2$, $plaus(\{u, w\}) = 2$, $plaus(\{v, w\}) = 1$ and $plaus(\{u, v, w\}) = 3$. The plausibility relation $<$ between subsets of $W$ is again derived from these numbers by putting $A < B$ iff $plaus(A) < plaus(B)$. We verified before that (Qualitativeness) is satisfied and thus we have an FH plausibility relation. But now consider $A = \{u, v\}$ and $B = \{u, w\}$. We have $A \cap -B <_{\text{FH}} B$ and $-A \cap B \not<_{\text{FH}} A$ and $\emptyset <_{\text{FH}} A \cup B$, so (+) is satisfied, but not $A <_{\text{FH}} B$.
*Example 4*, which we are going to define now, shows that $A <_{\text{FH}} B$ does not imply (+). Assign plausibility values to the subsets of $W$ as follows (the only value that differs from Example 2 is that of $\{u, w\}$): $plaus(\emptyset) = 0$, $plaus(\{u\}) = 2$, $plaus(\{v\}) = 1$, $plaus(\{w\}) = 1$, $plaus(\{u, v\}) = 2$, $plaus(\{u, w\}) = 3$, $plaus(\{v, w\}) = 1$ and $plaus(\{u, v, w\}) = 3$. The plausibility relation $<$ between subsets of $W$ is again derived from these numbers in the obvious way. The only non-trivial comparison for (Qualitativeness) concerns the three disjoint sets $\{u\}$, $\{v\}$ and $\{w\}$, and we can indeed verify that

$$\{w\} < \{u\} \cup \{v\} \text{ and } \{v\} < \{u\} \cup \{w\} \text{ taken together imply } \{v\} \cup \{w\} < \{u\}$$

(This is $1 < 2$ and $1 < 3$ on the left-hand side, and $1 < 2$ on the right-hand side.) Thus the condition of Qualitativeness is satisfied here, we do in fact have an FH plausibility relation. Consider again $A = \{u, v\}$ and $B = \{u, w\}$. Clearly, we have $A <_{\text{FH}} B$, but not $-A \cap B \not<_{\text{FH}} A$, violating (+). QED

### 7. Direct bridges between FH plausibilities and basic plausibilities

We can go directly from FH plausibilities to basic plausibilities, and back again. Given an FH plausibility relation $<_{\text{FH}}$, we define the corresponding basic plausibility relation $<_{\text{BP}} = \mathcal{P}_{\text{B}}(<_{\text{FH}})$ by

(From $<_{\text{FH}}$ to $<_{\text{BP}}$) $\quad A <_{\text{BP}} B \quad$ iff $\quad A <_{\text{FH}} -A \cap B$

Conversely, given a basic plausibility relation $<_{\text{BP}}$, we define the corresponding FH plausibility relation $<_{\text{FH}} = \mathcal{P}_{\text{FH}}(<_{\text{BP}})$ by

$$(\text{From} <_{\text{BP}} \text{ to } <_{\text{FH}}) \qquad A <_{\text{FH}} B \quad \text{iff} \quad A \cap -B <_{\text{BP}} B \text{ and } -A \cap B \not< _{\text{BP}} A$$

Now it becomes clear that the meanings of plausibility judgements in Friedman and Halpern's language are in general different from those in the language of basic plausibility. For a correct understanding of the other camp's statements, an interpreter is necessary. However, the meanings are not too far apart. We can show that in either translation, $<_{\text{BP}}$ is a subrelation of $<_{\text{FH}}$, and as long FH plausibilists and basic plausibilists restrict themselves to talking about disjoint sets, no misunderstanding is possible, because they agree about such sets.

**Lemma 5.** (a) If $<_{\text{FH}}$ is an FH plausibility relation, $<_{\text{BP}} = \mathcal{P}_{\text{B}}(<_{\text{FH}})$ and $A <_{\text{BP}} B$, then $A <_{\text{FH}} B$.

(b) If $<_{\text{BP}}$ is a basic plausibility relation, $<_{\text{FH}} = \mathcal{P}_{\text{FH}}(<_{\text{BP}})$ and $A <_{\text{BP}} B$, then $A <_{\text{FH}} B$.

(c) If $<_{\text{FH}}$ is an FH plausibility relation, $<_{\text{BP}} = \mathcal{P}_{\text{B}}(<_{\text{FH}})$ and $A$ and $B$ are disjoint propositions, then: $A <_{\text{BP}} B$ iff $A <_{\text{FH}} B$.

(d) If $<_{\text{BP}}$ is a basic plausibility relation, $<_{\text{FH}} = \mathcal{P}_{\text{FH}}(<_{\text{BP}})$ and $A$ and $B$ are disjoint propositions, then: $A <_{\text{FH}} B$ iff $A <_{\text{BP}} B$.

*Proof.* (a) Let $<_{\text{FH}}$ be an FH plausibility relation, $<_{\text{BP}} = \mathcal{P}_{\text{B}}(<_{\text{FH}})$ and $A <_{\text{BP}} B$. By definition, this means that $A <_{\text{FH}} -A \cap B$. So by FH-Dominance, $A <_{\text{FH}} B$.
(b) Let $<_{\text{BP}}$ be a basic plausibility relation, $<_{\text{FH}} = \mathcal{P}_{\text{FH}}(<_{\text{BP}})$ and $A <_{\text{BP}} B$. We want to show that $A <_{\text{FH}} B$, that is, by definition, $A \cap -B <_{\text{BP}} B$ and $-A \cap B \not<_{\text{BP}} A$. But the former follows from $A <_{\text{BP}} B$ by (Weak Continuing down), (BP-vi). From $A <_{\text{BP}} B$, we also get $A <_{\text{BP}} -A \cap B$ by (BP-iii). So $-A \cap B \not<_{\text{BP}} A$, by (Asymmetry), (BP-i), which is what we needed.
(c) Let $A$ and $B$ be disjoint propositions. Then $-A \cap B$ is identical with $B$, so $<_{\text{BP}} = \mathcal{P}_{\text{B}}(<_{\text{FH}})$ agrees by definition with $<_{\text{FH}}$ on disjoint propositions.
(d) Let $A$ and $B$ be disjoint propositions. Then $A \cap -B$ is identical with $A$, and $-A \cap B$ is identical with $B$. So the clause for the definition of $A <_{\text{FH}} B$ with $<_{\text{FH}} = \mathcal{P}_{\text{FH}}(<_{\text{BP}})$ reduces to $A <_{\text{BP}} B$ and $B \not<_{\text{BP}} A$. Since $<_{\text{BP}}$ satisfies (Asymmetry), this reduces to $A <_{\text{BP}} B$. Thus $<_{\text{FH}} = \mathcal{P}_{\text{FH}}(<_{\text{BP}})$ agrees with $<_{\text{BP}}$ on disjoint propositions. QED

Lemma 5 teaches an interesting lesson for plausibility-based default reasoning. The specific condition that (From $<$ to $\mid\!\sim$) uses for the definition of $\alpha \mid\!\sim \beta$ refers to $[\![\alpha \wedge \beta]\!]$ and $[\![\alpha \wedge \neg\beta]\!]$. Obviously, these propositions are disjoint. Lemma 5 tells us that if we translate from the language of Friedman and Halpern to the language of basic plausibilities or vice versa, using either one of the direct bridges $\mathcal{P}_{\text{FH}}$ and $\mathcal{P}_{\text{B}}$ between $<_{\text{FH}}$ and $<_{\text{BP}}$, and then construct a default inference relation using (From $<$ to $\mid\!\sim$), the result will be the same on both sides. Although Friedman and Halpern have a different concept from the concept of a proponent of basic plausibility, the construction recipe both parties use, viz., (From $<$ to $\mid\!\sim$), gives the same result for both. Within the specific context of the construction of a plausibility-based inference relation, it does not matter that the meanings of the term "plausibility" differ.

Next we show that (Qualitativeness) of Friedman-Halpern relations corresponds roughly to the well-behavedness of basic plausibility relations.

**Observation 5.** If $<_{\mathrm{BP}}$ is a well-behaved basic plausibility relation, then $<_{\mathrm{FH}} = \mathcal{P}_{\mathrm{FH}}(<_{\mathrm{BP}})$ is a qualitative FH-relation.

*Proof.* (FH0a) First we need to show (Irreflexivity).
(FH0b) Next we show *Transitivity.*
Let $A <_{\mathrm{FH}} B$ and $B <_{\mathrm{FH}} C$, that is
(i) $A \cap -B <_{\mathrm{BP}} B$ and $-A \cap B \not<_{\mathrm{BP}} A$
and
(ii) $B \cap -C <_{\mathrm{BP}} C$ and $-B \cap C \not<_{\mathrm{BP}} B$
In order to show that $A <_{\mathrm{FH}} C$, we need to show that
(iii) $A \cap -C <_{\mathrm{BP}} C$ and $-A \cap C \not<_{\mathrm{BP}} A$.
Let us first prove the first conjunct of (iii). From (i) and (ii) we get, using (BP-x),
$(A \cap -B) \cup (B \cap -C) <_{\mathrm{BP}} B \cup C$
By Meet right, (BP-iii), we get
$(A \cap -B) \cup (B \cap -C) <_{\mathrm{BP}} -((A \cap -B) \cup (B \cap -C)) \cap (B \cup C)$
or equivalently,
$(A \cap -B) \cup (B \cap -C) <_{\mathrm{BP}} (-A \cup B) \cap (-B \cup C) \cap (B \cup C)$
The right-hand side is a subset of $C$, so we get, using (Continuing up),
$(A \cap -B) \cup (B \cap -C) <_{\mathrm{BP}} C$
The left-hand side is a superset of $A \cap -C$, so by (Continuing down),
$A \cap -C <_{\mathrm{BP}} C$, which is the first conjunct of (iii).
Let us now turn to the second conjunct of (iii). Suppose for reductio that also $-A \cap C <_{\mathrm{BP}} A$. Taken together with $A \cap -C <_{\mathrm{BP}} C$, which we have just established, this gives us, with the help of (BP-x),
$(A \cap -C) \cup (-A \cap C) <_{\mathrm{BP}} A \cup C$
Now we use Meet right, (BP-iii), and get
$(A \cap -C) \cup (-A \cap C) <_{\mathrm{BP}} -((A \cap -C) \cup (-A \cap C)) \cap (A \cup C)$
which is equivalent with
$(A \cap -C) \cup (-A \cap C) <_{\mathrm{BP}} (-A \cup C) \cap (A \cup -C) \cap (A \cup C)$
or simply
$(A \cap -C) \cup (-A \cap C) <_{\mathrm{BP}} A \cap C$
Now we join this with $A \cap -B <_{\mathrm{BP}} B$ from (i), using (BP-x) and get
$(A \cap -B) \cup (A \cap -C) \cup (-A \cap C) <_{\mathrm{BP}} B \cup (A \cap C)$
Applying Meet right, (BP-iii) once more, we get
$(A\cap-B)\cup(A\cap-C)\cup(-A\cap C) <_{\mathrm{BP}} -((A\cap-B)\cup(A\cap-C)\cup(-A\cap C))\cap(B\cup(A\cap C))$
which is equivalent with
$(A\cap-B)\cup(A\cap-C)\cup(-A\cap C) <_{\mathrm{BP}} ((-A\cup B)\cap(-A\cup C)\cap(A\cup-C))\cap(B\cup(A\cap C))$
The right-hand side of this inequality is a subset of $B$, so by (Continuing up), we get

$(A \cap -B) \cup (A \cap -C) \cup (-A \cap C) <_{BP} B$
The left-hand side of the inequality includes $-B \cap C$, so by (Continuing down), we infer
$-B \cap C <_{BP} B$
But this violates (ii). So we have found a contradiction and proved the second conjunct of (iii). Therefore $<_{FH} = \mathcal{P}_{FH}(<_{BP})$ is transitive.

(FH1) Secondly, we need to show that $<_{FH} = \mathcal{P}_{FH}(<_{BP})$ satisfies FH-Dominance. So suppose that $A \subseteq B$.
First, we show that if $C <_{FH} A$, then $C <_{FH} B$. That is, if $C \cap -A <_{BP} A$ and $-C \cap A \not<_{BP} C$, then $C \cap -B <_{BP} B$ and $-C \cap B \not<_{BP} C$. But $C \cap -A <_{BP} A$ entails $C \cap -B <_{BP} B$ due to (Continuing up) and (Continuing down), since $C \cap -B \subseteq C \cap -A$. Similarly, $-C \cap A \not<_{BP} C$ and (Continuing down) entail that $-C \cap B \not<_{BP} C$.
Second, we show that if $B <_{FH} C$, then $A <_{FH} C$. That is, if $B \cap -C <_{BP} C$ and $-B \cap C \not<_{BP} B$, then $A \cap -C <_{BP} C$ and $-A \cap C \not<_{BP} A$. But again, $B \cap -C <_{BP} C$ implies $A \cap -C <_{BP} C$ due to (Continuing down), and $-B \cap C \not<_{BP} B$ implies $-A \cap C \not<_{BP} A$ due to (Continuing up) and (Continuing down).

(FH2) Thirdly, we verify that $<_{FH} = \mathcal{P}_{FH}(<_{BP})$ satisfies (Qualitativeness). Since $<_{BP}$ satisfies the unrestricted condition (Choice hard) and $<_{FH}$ agrees with $<_{BP}$ over disjoint sets, by Lemma 5(d), it follows immediately that $<_{FH}$ satisfy Choice over disjoint sets, and that just means that it satisfies Qualitativeness.

(FH3) Lastly, we need to show that $\emptyset <_{FH} A \cup B$ implies that either $\emptyset <_{FH} A$ or $\emptyset <_{FH} B$. Since $<_{FH}$ agrees with $<_{BP}$ over disjoint sets, by Lemma 5(d), it is sufficient to show that $\emptyset <_{BP} A \cup B$ implies that either $\emptyset <_{BP} A$ or $\emptyset <_{BP} B$. But the former implies that $A \cup B \neq \emptyset$, by (Irreflexivity). Hence either $A \neq \emptyset$ or $B \neq \emptyset$. So by (Minimality), either $\emptyset <_{BP} A$ or $\emptyset <_{BP} B$. QED

Above (in subsection 4.4.1) we noted that the condition (Choice hard) does not follow from Friedman and Halpern's condition (Qualitativeness), which is a restriction of (Choice hard) to disjoint propositions, even if (Irreflexivity), (Transitivity) and (FH-Dominance) are present as background conditions. But if we construct basic plausibilities from FH plausibilities properly, that is, by using the equation $<_{BP} = \mathcal{P}_{B}(<_{FH})$, we understand how the former can satisfy the much stronger condition of (Choice hard), provided that latter satisfy the weaker condition (Qualitativeness).

The more general result is this:

**Observation 6.** If $<_{FH}$ is a qualitative FH relation, then $<_{BP} = \mathcal{P}_{B}(<_{FH})$ satisfies the axioms for well-behaved basic plausibility relations except (Minimality). If $<_{FH}$ is a qualitative FH relation satisfying (Minimality), then $<_{BP} = \mathcal{P}_{B}(<_{FH})$ is a well-behaved basic plausibility relation.

*Proof.* Let $<_{FH}$ be a qualitative FH-relation and $<_{BP} = \mathcal{P}_{B}(<_{FH})$.
(Irreflexivity). Suppose $A <_{BP} A$, that is, by definition, $A <_{FH} -A \cap A$. This means that $A <_{FH} \emptyset$, contradicting (FH1) and (Irreflexivity) for $<_{FH}$.

(Minimality). Let $A \neq \emptyset$. We need to show that $\emptyset <_{\text{BP}} A$, that is, by definition $\emptyset <_{\text{FH}} -\emptyset \cap A$, which is $\emptyset <_{\text{FH}} A$. This is guaranteed if, and only if, $<_{\text{FH}}$ satisfies (Minimality).
(Continuing up). Let $A <_{\text{BP}} B$. By definition, this means that $A <_{\text{FH}} -A \cap B$. By (FH1), this implies $A <_{\text{FH}} -A \cap (B \cup C)$, which means, by definition again, $A <_{\text{BP}} B \cup C$.
(Continuing down). Let $A \cup C <_{\text{BP}} B$. By definition, this means that $A \cup C <_{\text{FH}} -(A \cup C) \cap B$. By (FH1), this implies $A <_{\text{FH}} -A \cap B$, which means, by definition again, $A <_{\text{BP}} B$.
(Choice easy) follows from (Continuing up) and (Continuing down).
(Choice hard). Suppose that $A <_{\text{BP}} B \cup C$ and $B <_{\text{BP}} A \cup C$. We need to show that $A \cup B <_{\text{BP}} C$.
By the definition of $\mathcal{P}_{\text{B}}(<_{\text{FH}})$, our supposition means that
(i) $A <_{\text{FH}} -A \cap (B \cup C)$ and
(ii) $B <_{\text{FH}} -B \cap (A \cup C)$.
Condition (i) can be equivalently expressed as
(i′) $A <_{\text{FH}} (-A \cap B) \cup (-A \cap -B \cap C)$
Consider also the condition
(ii′) $-A \cap B <_{\text{FH}} A \cup (-A \cap -B \cap C)$
This condition is entailed by (ii), since $-A \cap B$ is a subset of $B$ and $-B \cap (A \cup C)$ is a subset of $A \cup (-A \cap -B \cap C)$, and $<_{\text{FH}}$ satisfies FH-Dominance.
Clearly, the sets $A$, $-A \cap B$ and $-A \cap -B \cap C$ are pairwise disjoint. Now we can apply the (Qualitativeness) of $<_{\text{FH}}$ and conclude from (i′) and (ii′) that
(iii′) $A \cup (-A \cap B) <_{\text{FH}} -A \cap -B \cap C$
But $A \cup (-A \cap B)$ is identical with $A \cup B$, and $-A \cap -B \cap C$ is identical with $-(A \cup B) \cap C$, so (iii′) just means
(iii) $A \cup B <_{\text{FH}} -(A \cup B) \cap C$
By the definition of $\mathcal{P}_{\text{B}}(<_{\text{FH}})$, this just means that $A \cup B <_{\text{BP}} C$, which is what we needed to show. QED

Due to the difference in the central concept of plausibility, it makes sense to say that an advocate of basic plausibility speaks a language that is different from the language spoken by Friedman and Halpern. We can then view the bridge principles between the two concepts of plausibility discussed in this paper as translations. Translations are supposed to preserve meanings across different languages. The only way to capture this idea in the present framework is to translate back and forth and see, within a single language, whether there is any change to what has been said. The result is that this idea works well in one direction, starting from the language of basic plausibility, but it does not work in the other direction that starts from the language of Friedman and Halpern.

**Observation 7.** (a) Let $<_{\text{BP}}$ be a basic plausibility relation and $<_{\text{BP}}' = \mathcal{P}_{\text{B}}(\mathcal{P}_{\text{FH}}(<_{\text{BP}}))$. Then $<_{\text{BP}}' = <_{\text{BP}}$.

(b) Let $<_{\text{FH}}$ be an FH plausibility relation and $<_{\text{FH}}' = \mathcal{P}_{\text{FH}}(\mathcal{P}_{\text{B}}(<_{\text{FH}}))$. Then in general $<_{\text{FH}}' \neq <_{\text{FH}}$.

*Proof.* (a) Assume that $A <_{\mathrm{BP}}{}' B$. This means, by (From $<_{\mathrm{FH}}$ to $<_{\mathrm{BP}}$),
$A <_{\mathrm{FH}} -A \cap B$
By Lemma 5(d), this is equivalent to
$A <_{\mathrm{BP}} -A \cap B$
which by (BP-iii) just means
$A <_{\mathrm{BP}} B$
as desired.
(b) Assume that $<_{\mathrm{FH}}{}' = \mathcal{P}_{\mathrm{FH}}(\mathcal{P}_{\mathrm{B}}(<_{\mathrm{FH}}))$, and assume that
$A <_{\mathrm{FH}}{}' B$.
This means, by (From $<_{\mathrm{BP}}$ to $<_{\mathrm{FH}}$),
$A \cap -B <_{\mathrm{BP}} B$ and $-A \cap B \not<_{\mathrm{BP}} A$,
By Lemma 5(c), this is equivalent to
$A \cap -B <_{\mathrm{FH}} B$ and $-A \cap B \not<_{\mathrm{FH}} A$.
By (FH1), the former conjunct of this is implied by $A <_{\mathrm{FH}} B$, and the latter conjunct implies $B \not<_{\mathrm{FH}} A$. But this is all we can say. As counterexamples against both directions we can use Example 2 and Example 4, exactly in the same way as they are used in the proof of Observation 4. Thus there is no reduction to the target sentence $A <_{\mathrm{FH}} B$. QED

Observation 7 shows that there is no perfect intertranslatability between the language of Friedman and Halpern and the language of basic plausibility. The observation does not tell us, though, how this slightly disappointing result is to be interpreted. One direction works, the other does not, but where is the problem? Is it due to the translation methods suggested in (From $<_{\mathrm{FH}}$ to $<_{\mathrm{BP}}$) or (From $<_{\mathrm{BP}}$ to $<_{\mathrm{FH}}$)? I do not think so, since the methods have been so designed that $\mathcal{P}_{\mathrm{B}}(<_{\mathrm{FH}}) = \mathcal{P}_{\mathrm{B}}(\mathcal{I}(<_{\mathrm{FH}}))$ and $\mathcal{P}_{\mathrm{FH}}(<_{\mathrm{BP}}) = \mathcal{P}_{\mathrm{FH}}(\mathcal{I}(<_{\mathrm{BP}}))$. Is it due to a defect in one of the theories? If we consider the design just mentioned and compare Observations 3 and 4, we find reason to say that the problem showing up in the second part of Observation 7 is caused by the FH theory rather than the theory of basic plausibility. This is not a very serious defect of the former theory, but it is not as harmoniously built up as the latter.

## 8. How to motivate the unrestricted Choice condition

It is nice that the properties of basic plausibility, in contrast to those of FH plausibility, can account for properties of the default reasoning operation in a (more or less) modular way. But doubts remain. The strong condition (Choice hard) is *much* stronger than the FH-condition (Qualitativeness). So shouldn't it be *much* harder to motivate it? Fortunately, this is not the case. In fact I think it is fair to claim that it has been *much* better motivated than the latter.

The first steps for the motivation of the Choice conditions were already made in section 2. But this is only part of the story. Rott (2001, chapter 8) gives a detailed and systematic motivation.[27] Here we have to be satisfied with a brief sketch of the argument. The whole approach is best understood in terms of

[27]This work also underlies the presentation of "basic entrenchment" in Rott (2003).

*withdrawals* (*contractions*) rather than *revisions*. This is the reason why (From $\dot{-}$ to $<_{\text{BE}}$) features prominently in section 2.

The unrestricted Choice condition, i.e., the conjunction of (Choice easy) and (Choice hard), reads

$$A <_{\text{BP}} B \cup C \text{ and } B <_{\text{BP}} A \cup C \quad \text{if and only if} \quad A \cup B <_{\text{BP}} C$$

Using ($<_{\text{BP}}$ and $<_{\text{BE}}$) and swapping negations, this can be transformed into talk about entrenchments:

(Choice$^{\text{ent}}$) $B \cap C <_{\text{BE}} A$ and $A \cap C <_{\text{BE}} B$ if and only if $C <_{\text{BE}} A \cap B$

Concerning *the left-hand side* of (Choice$^{\text{ent}}$), $B \cap C <_{\text{BE}} A$ means

> If one is to withdraw at least one of $B \cap C$ and $A$, then one will withdraw $B \cap C$ and keep $A$

Because "withdrawing $B \cap C$" means "withdrawing at least one of $B$ and $C$" (see section 2), this in turn means:

> If one is to withdraw at least one of $A$ and $B$ and $C$, then one will withdraw at least one of $B$ and $C$ and keep $A$.

Similarly, $A \cap C <_{\text{BE}} B$ means

> If one is to withdraw at least one of $A$ and $B$ and $C$, then one will withdraw at least one of $A$ and $C$ and keep $B$.

Concerning *the right-hand side* of (Choice$^{\text{ent}}$), $C <_{\text{BE}} A \cap B$ means

> If one is to withdraw at least one of $C$ and $A \cap B$, then one will withdraw $C$ and keep $A \cap B$

Because "withdrawing $A \cap B$" means "withdrawing at least one of $A$ and $B$", and "keeping $A \cap B$" means "keeping both $A$ and $B$", this in turn means:

> If one is to withdraw at least one of $A$ and $B$ and $C$, then one will withdraw $C$ and keep both $A$ and $B$.

This choice-theoretic interpretation shows that the left-hand side of (Choice$^{\text{ent}}$) and the right-hand side of (Choice$^{\text{ent}}$) *mean exactly the same thing*.

No such motivation is available for Friedman and Halpern's condition of Qualitativeness.

## 9. Conclusion

Friedman and Halpern's work on *plausibility measures* is beautiful because it starts from the very general conditions: Transitivity and Dominance (what we called FH-Dominance). Being well below the level of preferential reasoning, this set-up is suitable for covering both probabilistic and qualitative interpretations. By adding the condition of Qualitativeness, Friedman and Halpern get exactly what they declare to be the "core" of default reasoning, namely, preferential reasoning.

A striking fact about the present author's earlier work on basic entrenchment is that the approach works in very general contexts—much more general than AGM originally envisaged and, if suitably translated into the context of default reasoning, clearly more general than preferential reasoning. *Basic plausibility*, the concept introduced in the present paper, is the dual of basic entrenchment. It has been compared to (the relational perspective on) Friedman and Halpern's work on plausibility measures.

The two approaches were developed independently (and roughly at the same time), but have now turned out to be similar. Concerning the crucial and unfamiliar conditions (Qualitativeness) and (Choice), some considerable differences have been identified. I have argued that the approach using basic plausibilities is preferable because it offers a more modular and better motivated account of default reasoning.

A more general, and presumably more important, lesson from the exercise performed in this paper is that one can mean different things when speaking about plausibility in the context of default reasoning.

**References**


Alchourrón, C. E., Gärdenfors, P. & Makinson, D. (1985), 'On the logic of theory change: Partial meet contraction and revision functions', *Journal of Symbolic Logic* **50**, 510–530.

Alchourrón, C. E. & Makinson, D. (1985), 'On the logic of theory change: Safe contraction', *Studia Logica* **44**, 405–422.

Dubois, D., Fargier, H. & Prade, H. (2004), 'Ordinal and probabilistic representations of acceptance', *Journal of Artificial Intelligence Research* **22**, 23–56.

Dubois, D. & Prade, H. (1995), Numerical representations of acceptance, *in* 'Proceedings of the 11th Conference on Uncertainty in Artificial Intelligence', pp. 149–156.

Freund, M. (1993), 'Injective models and disjunctive relations', *Journal of Logic and Computation* **3**, 231–247.

Freund, M. (1998), 'Preferential orders and plausibility measures', *Journal of Logic and Computation* **8**, 147–158.

Friedman, N. & Halpern, J. Y. (1995), Plausibility measures: A user's guide, *in* 'Proceedings of the 11th Conference on Uncertainty in Artificial Intelligence (UAI'95)', pp. 175–184.

Friedman, N. & Halpern, J. Y. (1996), Plausibility measures and default reasoning, *in* 'Proceedings of the 13th National Conference on Artificial Intelligence (AAAI'96)', Portland, OR,, pp. 1297–1304.

Friedman, N. & Halpern, J. Y. (1999), Plausibility measures and default reasoning: An overview, *in* 'Proceedings of the 14th Symposium on Logic in Computer Science (LICS'99)', pp. 130–135.

Friedman, N. & Halpern, J. Y. (2001), 'Plausibility measures and default reasoning', *Journal of the ACM* **48**, 648–685.

Gärdenfors, P. (1988), *Knowledge in Flux: Modeling the Dynamics of Epistemic States*, Bradford Books, MIT Press, Cambridge, Mass.

Gärdenfors, P. (1990), Belief revision and nonmonotonic logic: Two sides of the same coin?, *in* L. C. Aiello, ed., '9th European Conference on Artificial Intelligence (ECAI'90)', Pitman, London, pp. 768–773.

Gärdenfors, P. & Makinson, D. (1994), 'Nonmonotonic inference based on expectations', *Artificial Intelligence* **65**, 197–245.

Halpern, J. Y. (1997), 'Defining relative likelihood in partially-ordered preferential structures', *Journal of Artificial Intelligence Research* **7**, 1–24.

Halpern, J. Y. (2001), Plausibility measures: A general approach for representing uncertainty, *in* 'Proceedings of the 17th Joint Conference on Artificial Intelligence (IJCAI'2001)', Morgan Kaufmann, San Francisco, pp. 1474–1483.

Halpern, J. Y. (2003), *Reasoning about Uncertainty*, MIT Press, Cambridge, MA.

Kraus, S., Lehmann, D. & Magidor, M. (1990), 'Nonmonotonic reasoning, preferential models and cumulative logics', *Artificial Intelligence* **44**, 167–207.

Lehmann, D. & Magidor, M. (1992), 'What does a conditional knowledge base entail?', *Artificial Intelligence* **55**, 1–60.

Lewis, D. (1973), *Counterfactuals*, Blackwell, Oxford. Second edition 1986.

Rott, H. (1992), 'Preferential belief change using generalized epistemic entrenchment', *Journal of Logic, Language and Information* **1**, 45–78.

Rott, H. (1996), *Making Up One's Mind. Foundations, Coherence, Nonmonotonicity*, unpublished Habilitationsschrift, University of Konstanz.

Rott, H. (2001), *Change, Choice and Inference: A Study in Belief Revision and Nonmonotonic Reasoning*, Oxford University Press, Oxford.

Rott, H. (2003), 'Basic entrenchment', *Studia Logica* **73**, 257–280.

Rott, H. (2009), Degrees all the way down: Beliefs, non-beliefs and disbeliefs, *in* F. Huber & C. Schmidt-Petri, eds, 'Degrees of Belief', Springer, Dordrecht, pp. 301–354.