First-Order Extensions of Classical Modal Logic

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June 2, 2012


Plan

1. Background
   - Neighborhood Semantics for Propositional Modal Logic
   - First-Order Modal Logic
   - The Barcan Formula

2. Neighborhood Models for First-Order Modal Logic


4. General Frames for First-Order Modal Logic
Plan

1. Background
   - Neighborhood Semantics for Propositional Modal Logic
   - First-Order Modal Logic
   - The Barcan Formula

2. Neighborhood Models for First-Order Modal Logic


4. General Frames for First-Order Modal Logic

\[ w \models \square \varphi \quad \text{if the truth set of } \varphi \text{ is a neighborhood of } w \]
$w \models \Box \varphi$ if the truth set of $\varphi$ is a \textit{neighborhood} of $w$

What does it mean to be a \textit{neighborhood}?
\[ w \models \Box \varphi \] if the truth set of \( \varphi \) is a neighborhood of \( w \)

neighborhood in some topology.

$w \models \Box \varphi$ if the truth set of $\varphi$ is a neighborhood of $w$

neighborhood in some topology.


contains all the immediate neighbors in some graph

\[ w \models \Box \varphi \quad \text{if the truth set of } \varphi \text{ is a neighborhood of } w \]

neighborhood in some topology.


contains all the immediate neighbors in some graph


an element of some distinguished collection of sets


To see the necessity of the more general approach, we could consider probability operators, conditional necessity, or, to invoke an especially perspicuous example of Dana Scott, the present progressive tense.... Thus $N$ might receive the awkward reading ‘it is being the case that’, in the sense in which ‘it is being the case that Jones leaves’ is synonymous with ‘Jones is leaving’. (Montague, 1970)
**Brief History**

**PC** Propositional Calculus

\[ E \quad \square \varphi \leftrightarrow \neg \lozenge \neg \varphi \]

\[ M \quad \square (\varphi \land \psi) \rightarrow (\square \varphi \land \square \psi) \]

\[ C \quad (\square \varphi \land \square \psi) \rightarrow \square (\varphi \land \psi) \]

\[ N \quad \square \top \]

\[ K \quad \square (\varphi \rightarrow \psi) \rightarrow (\square \varphi \rightarrow \square \psi) \]

\[ RE \quad \frac{\varphi \leftrightarrow \psi}{\square \varphi \leftrightarrow \square \psi} \]

\[ Nec \quad \frac{\varphi}{\square \varphi} \]

\[ MP \quad \frac{\varphi \quad \varphi \rightarrow \psi}{\psi} \]
PC  Propositional Calculus

\[ E \quad \Box \varphi \leftrightarrow \neg \Diamond \neg \varphi \]
\[ M \quad \Box(\varphi \land \psi) \rightarrow (\Box \varphi \land \Box \psi) \]
\[ C \quad (\Box \varphi \land \Box \psi) \rightarrow \Box(\varphi \land \psi) \]
\[ N \quad \Box \top \]
\[ K \quad \Box(\varphi \rightarrow \psi) \rightarrow (\Box \varphi \rightarrow \Box \psi) \]

RE  \[ \varphi \leftrightarrow \psi \quad \frac{\Box \varphi \leftrightarrow \Box \psi}{\Box \varphi \leftrightarrow \Box \psi} \]

Nec  \[ \varphi \quad \frac{\Box \varphi}{\Box \varphi} \]

MP  \[ \varphi \quad \varphi \rightarrow \psi \quad \frac{\psi}{\psi} \]

A modal logic \( L \) is classical if it contains all instances of \( E \) and is closed under \( RE \).
A modal logic $L$ is classical if it contains all instances of $E$ and is closed under $RE$.

$E$ is the smallest classical modal logic.
**PC** Propositional Calculus

- **E** $\square \varphi \leftrightarrow \neg \Diamond \neg \varphi$
- **M** $\square (\varphi \land \psi) \rightarrow (\square \varphi \land \square \psi)$
- **C** $(\square \varphi \land \square \psi) \rightarrow \square (\varphi \land \psi)$
- **N** $\square \top$
- **K** $\square (\varphi \rightarrow \psi) \rightarrow (\square \varphi \rightarrow \square \psi)$

**RE**

\[
\begin{array}{c}
\varphi \leftrightarrow \psi \\
\hline
\square \varphi \leftrightarrow \square \psi
\end{array}
\]

**Nec**

\[
\frac{\varphi}{\square \varphi}
\]

**MP**

\[
\begin{array}{c}
\varphi \varphi \rightarrow \psi \\
\hline
\psi
\end{array}
\]

$\mathbf{E}$ is the smallest classical modal logic.

In $\mathbf{E}$, $\mathbf{M}$ is equivalent to

(\text{Mon}) \quad \frac{\varphi \rightarrow \psi}{\square \varphi \rightarrow \square \psi}$
**Brief History**

**PC** Propositional Calculus

\[ E \quad \square \varphi \leftrightarrow \neg \diamond \neg \varphi \]

**Mon**

\[
\begin{align*}
&\frac{\varphi \to \psi}{\square \varphi \to \square \psi} \\
&\frac{\square \varphi \land \square \psi}{\square (\varphi \land \psi)} \\
&\square T \\
&K \quad \square (\varphi \to \psi) \to (\square \varphi \to \square \psi)
\end{align*}
\]

**RE**

\[
\begin{align*}
&\frac{\varphi \leftrightarrow \psi}{\square \varphi \leftrightarrow \square \psi}
\end{align*}
\]

**Nec**

\[
\begin{align*}
&\frac{\varphi}{\square \varphi}
\end{align*}
\]

**MP**

\[
\begin{align*}
&\frac{\varphi \quad \varphi \to \psi}{\psi}
\end{align*}
\]

**E** is the smallest *classical* modal logic.

**EM** is the logic **E** + **Mon**
**Brief History**

**PC** 6. Propositional Calculus

- **E** \( \square \varphi \iff \neg \Diamond \neg \varphi \)
- **Mon** \( \frac{\varphi \rightarrow \psi}{\square \varphi \rightarrow \square \psi} \)
- **C** \( (\square \varphi \land \square \psi) \rightarrow \square (\varphi \land \psi) \)
- **N** \( \square \top \)
- **K** \( \square (\varphi \rightarrow \psi) \rightarrow (\square \varphi \rightarrow \square \psi) \)
- **RE** \( \frac{\varphi \leftrightarrow \psi}{\square \varphi \leftrightarrow \square \psi} \)
- **Nec** \( \frac{\varphi}{\square \varphi} \)
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**E** is the smallest **classical** modal logic.

**EM** is the logic **E + Mon**

**EC** is the logic **E + C**
**Brief History**

**PC** Propositional Calculus

\[
\begin{align*}
E & \quad \square \varphi \leftrightarrow \neg \Diamond \neg \varphi \\
Mon & \quad \frac{\varphi \rightarrow \psi}{\square \varphi \rightarrow \square \psi} \\
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N & \quad \square \top \\
K & \quad \square (\varphi \rightarrow \psi) \rightarrow (\square \varphi \rightarrow \square \psi) \\
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\end{align*}
\]

**E** is the smallest **classical** modal logic.

**EM** is the logic E + Mon

**EC** is the logic E + C

**EMC** is the smallest regular modal logic
**Brief History**

**PC**  Propositional Calculus

**E**  \( \square \varphi \leftrightarrow \neg \lozenge \neg \varphi \)

**Mon**  

\[
\frac{\varphi \rightarrow \psi}{\square \varphi \rightarrow \square \psi}
\]

**C**  \( (\square \varphi \land \square \psi) \rightarrow \square (\varphi \land \psi) \)

**N**  \( \square \top \)

**K**  \( \square (\varphi \rightarrow \psi) \rightarrow (\square \varphi \rightarrow \square \psi) \)

**RE**  

\[
\frac{\varphi \leftrightarrow \psi}{\square \varphi \leftrightarrow \square \psi}
\]

**Nec**  

\[
\frac{\varphi}{\square \varphi}
\]

**MP**  

\[
\frac{\varphi \varphi \rightarrow \psi}{\psi}
\]

**E** is the smallest **classical** modal logic.

**EM** is the logic **E** + **Mon**

**EC** is the logic **E** + **C**

**EMC** is the smallest **regular** modal logic

A logic is **normal** if it contains all instances of **E**, **C** and is closed under **Mon** and **Nec**
**Brief History**

**PC** Propositional Calculus

$$E \quad \Box \varphi \leftrightarrow \neg \Diamond \neg \varphi$$

**Mon**

$$\frac{\varphi \rightarrow \psi}{\Box \varphi \rightarrow \Box \psi}$$

$$C \quad (\Box \varphi \land \Box \psi) \rightarrow \Box (\varphi \land \psi)$$

**N**

$$\Box \top$$

**K**

$$\Box (\varphi \rightarrow \psi) \rightarrow (\Box \varphi \rightarrow \Box \psi)$$

**RE**

$$\frac{\varphi \leftrightarrow \psi}{\Box \varphi \leftrightarrow \Box \psi}$$

**Nec**

$$\frac{\varphi}{\Box \varphi}$$

**MP**

$$\frac{\varphi \varphi \rightarrow \psi}{\psi}$$

**E** is the smallest classical modal logic.

**EM** is the logic \( E + \text{Mon} \)

**EC** is the logic \( E + C \)

**EMC** is the smallest regular modal logic

**K** is the smallest normal modal logic
**Brief History**

**PC** Propositional Calculus

\[ E \quad \square \varphi \leftrightarrow \neg \diamond \neg \varphi \]

**Mon**

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**K**

\[ \square (\varphi \rightarrow \psi) \rightarrow (\square \varphi \rightarrow \square \psi) \]

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\[ E \text{ is the smallest classical modal logic.} \]

\[ \textbf{EM} \text{ is the logic } E + \text{Mon} \]

\[ \textbf{EC} \text{ is the logic } E + C \]

\[ \textbf{EMC} \text{ is the smallest regular modal logic} \]

\[ K = \text{EMCN} \]
**Brief History**

<table>
<thead>
<tr>
<th>Logic</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
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**E** is the smallest **classical** modal logic.

**EM** is the logic **E** + **Mon**

**EC** is the logic **E** + **C**

**EMC** is the smallest **regular** modal logic

**K** = **PC**$(+E)$ + **K** + **Nec** + **MP**
Motivating Examples

Logics of High Probability

\(\□\phi\) means "\(\phi\) is assigned ‘high’ probability”, where high means above some threshold \(r \in (\frac{1}{2}, 1]\).
Logics of High Probability

□ϕ means “ϕ is assigned ‘high’ probability”, where high means above some threshold $r \in (\frac{1}{2}, 1]$.

Claim: Mon is a valid rule of inference.
Motivating Examples

Logics of High Probability

□φ means “φ is assigned ‘high’ probability”, where high means above some threshold \( r \in \left( \frac{1}{2}, 1 \right] \).

**Claim:** Mon is a valid rule of inference.

**Claim:** C is not valid.
Motivating Examples

Logics of High Probability

□φ means “φ is assigned ‘high’ probability”, where high means above some threshold \( r \in (\frac{1}{2}, 1] \).

**Claim:** Mon is a valid rule of inference.

**Claim:** C is not valid.


Motivating Examples

Neighborhood Frames

Let $W$ be a non-empty set of states.

Any map $N : W \rightarrow \mathcal{P}(\mathcal{P}(W))$ is called a neighborhood function.

A pair $\langle W, N \rangle$ is called a neighborhood frame if $W$ is a non-empty set and $N$ is a neighborhood function.

A neighborhood model is a tuple $\langle W, N, V \rangle$ where $V : At \rightarrow \mathcal{P}(W)$ is a valuation function and $\langle W, N \rangle$ is a neighborhood frame.
Motivating Examples

Truth in a Model

- $\mathcal{M}, w \models p$ iff $w \in V(p)$

- $\mathcal{M}, w \models \neg \varphi$ iff $\mathcal{M}, w \not\models \varphi$

- $\mathcal{M}, w \models \varphi \land \psi$ iff $\mathcal{M}, w \models \varphi$ and $\mathcal{M}, w \models \psi$
Motivating Examples

Truth in a Model

- $M, w \models p$ iff $w \in V(p)$

- $M, w \models \neg \varphi$ iff $M, w \not\models \varphi$

- $M, w \models \varphi \land \psi$ iff $M, w \models \varphi$ and $M, w \models \psi$

- $M, w \models \Box \varphi$ iff $(\varphi)^M \subseteq N(w)$

- $M, w \models \Diamond \varphi$ iff $W - (\varphi)^M \not\subseteq N(w)$

where $(\varphi)^M = \{w \mid M, w \models \varphi\}$. 

Motivating Examples

Validities

(Dual) $\square \varphi \leftrightarrow \neg \lozenge \neg \varphi$ is valid in all neighborhood models.

(Re) If $\varphi \leftrightarrow \psi$ is valid then $\square \varphi \leftrightarrow \square \psi$ is valid.
Motivating Examples

Horacio’s Possibility

\[ \mathcal{M}, w \models \square \varphi \iff (\varphi)^m \in N(w) \]

\[ \mathcal{M}, w \models \diamond \varphi \iff W - (\varphi)^m \not\in N(w) \]

\[ \mathcal{M}, w \models \square \varphi \iff \text{there is an } X \in N(w) \text{ such that for all } v \in W, v \in X \iff \mathcal{M}, v \models \varphi \]

\[ \mathcal{M}, w \models \diamond \varphi \iff \exists X \in N(w) \text{ such that } \exists v \in X, \mathcal{M}, v \models \varphi \]

\[ \mathcal{M}, w \models \square \varphi \iff \forall X \in N(w) \text{ such that } \forall v \in X, \mathcal{M}, v \models \varphi \]

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Horacio’s Possibility

\( M, w \models \Box \varphi \text{ iff } (\varphi)^M \in N(w) \)

\( M, w \models \Diamond \varphi \text{ iff } W - (\varphi)^M \notin N(w) \)

\( M, w \models \Box \varphi \text{ iff there is an } X \in N(w) \text{ such that for all } v \in W, v \in X \text{ iff } M, v \models \varphi \)

\( M, w \models \langle \rangle \varphi \text{ iff } \exists X \in N(w) \text{ such that } \exists v \in X, M, v \models \varphi \)

\( M, w \models [ ] \varphi \text{ iff } \forall X \in N(w) \text{ such that } \forall v \in X, M, v \models \varphi \)

\( M, w \models \langle \rangle \varphi \text{ iff } \exists X \in N(w) \text{ such that } \forall v \in X, M, v \models \varphi \)

\( M, w \models [ ] \varphi \text{ iff } \forall X \in N(w) \text{ such that } \exists v \in X, M, v \models \varphi \)
Horacio’s Possibility

\[ \mathcal{M}, w \models \square \varphi \iff (\varphi)^\mathcal{M} \in N(w) \]

\[ \mathcal{M}, w \models \Diamond \varphi \iff W - (\varphi)^\mathcal{M} \not\in N(w) \]

\[ \mathcal{M}, w \models \square \varphi \iff \text{there is an } X \in N(w) \text{ such that for all } v \in W, v \in X \iff \mathcal{M}, v \models \varphi \]

- \[ \mathcal{M}, w \models \langle \rangle \varphi \iff \exists X \in N(w) \text{ such that } \exists v \in X, \mathcal{M}, v \models \varphi \]
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Motivating Examples

Horacio’s Possibility

\[ M, w \models \square \varphi \iff (\varphi)_M \in N(w) \]

\[ M, w \models \Diamond \varphi \iff W - (\varphi)_M \not\in N(w) \]

\[ M, w \models \square \varphi \iff \text{there is an } X \in N(w) \text{ such that for all } v \in W, \]
\[ v \in X \iff M, v \models \varphi \]

- \[ M, w \models \langle \rangle \varphi \iff \exists X \in N(w) \text{ such that } \exists v \in X, M, v \models \varphi \]
- \[ M, w \models [\ ] \varphi \iff \forall X \in N(w) \text{ such that } \forall v \in X, M, v \models \varphi \]

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Motivating Examples

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\[ M, w \models \square \varphi \text{ iff } (\varphi)^m \in N(w) \]

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Motivating Examples

Other Examples, I

Reasoning about abilities


Reasoning about games


Reasoning about coalitions

Motivating Examples

Other Examples, II

Epistemic Logic: the logical omniscience problem.

Motivating Examples

Other Examples, II

Epistemic Logic: the logical omniscience problem.


Reasoning about evidence and beliefs (and how they change over time)

J. van Benthem and EP. *Dynamics of Evidence-Based Beliefs*. Studia Logica, 2011.

Motivating Examples

Other Examples, III

Program logics: modeling concurrent programs


The Logic of deduction


Deontic logics...
Motivating Examples

Constraints on neighborhood frames

- **Monotonic or Supplemented:** If $X \in N(w)$ and $X \subseteq Y$, then $Y \in N(w)$
Motivating Examples

Constraints on neighborhood frames

- **Monotonic or Supplemented:** If $X \in N(w)$ and $X \subseteq Y$, then $Y \in N(w)$

- **Closed under finite intersections:** If $X \in N(w)$ and $Y \in N(w)$, then $X \cap Y \in N(w)$
Motivating Examples

Constraints on neighborhood frames

- **Monotonic or Supplemented:** If $X \in N(w)$ and $X \subseteq Y$, then $Y \in N(w)$

- **Closed under finite intersections:** If $X \in N(w)$ and $Y \in N(w)$, then $X \cap Y \in N(w)$

- **Contains the unit:** $W \in N(w)$
Motivating Examples

Constraints on neighborhood frames

- **Monotonic or Supplemented:** If $X \in N(w)$ and $X \subseteq Y$, then $Y \in N(w)$

- **Closed under finite intersections:** If $X \in N(w)$ and $Y \in N(w)$, then $X \cap Y \in N(w)$

- **Contains the unit:** $W \in N(w)$

- **Augmented:** Supplemented plus for each $w \in W$, $\bigcap N(w) \in N(w)$
Coherent Neighborhoods

A neighborhood of \( w \) is “perfectly coherent” provided
\[
\bigcap N(w) \neq \emptyset,
\]
A neighborhood of $w$ is “perfectly coherent” provided \( \bigcap N(w) \neq \emptyset \), how should we measure the “level of coherence” when \( N(w) \) is not closed under conjunction?

Motivating Examples

From Kripke Frames to Neighborhood Frames

Let \( R \subseteq W \times W \), define a map \( R^\to : W \to \wp(W) \):

for each \( w \in W \), let \( R^\to(w) = \{ v \mid wRv \} \)
From Kripke Frames to Neighborhood Frames

Let $R \subseteq W \times W$, define a map $R^\rightarrow : W \rightarrow \wp(W)$:

for each $w \in W$, let $R^\rightarrow(w) = \{v \mid wRv\}$

Definition
Given a relation $R$ on a set $W$ and a state $w \in W$. A set $X \subseteq W$ is $R$-necessary at $w$ if $R^\rightarrow(w) \subseteq X$. 
Motivating Examples

From Kripke Frames to Neighborhood Frames

Let $R \subseteq W \times W$, define a map $R^\rightarrow : W \rightarrow \wp(W) :$

for each $w \in W$, let $R^\rightarrow(w) = \{v \mid wRv\}$

Let $\mathcal{N}_w^R$ be the set of sets that are $R$-necessary at $w$:

$$\mathcal{N}_w^R = \{X \mid R^\rightarrow(w) \subseteq X\}$$
Motivating Examples

From Kripke Frames to Neighborhood Frames

Let $R \subseteq W \times W$, define a map $R^\rightarrow : W \to \wp(W)$:

$$\text{for each } w \in W, \text{ let } R^\rightarrow(w) = \{ v \mid wRv \}$$

Let $N^R_w$ be the set of sets that are $R$-necessary at $w$:

$$N^R_w = \{ X \mid R^\rightarrow(w) \subseteq X \}$$

Lemma

Let $R$ be a relation on $W$. Then for each $w \in W$, $N^R_w$ is augmented.
Theorem

- Let $\langle W, R \rangle$ be a relational frame. Then there is an equivalent augmented neighborhood frame.

- Let $\langle W, N \rangle$ be an augmented neighborhood frame. Then there is an equivalent relational frame.
Motivating Examples

From Neighborhood Frames to Kripke Frames

Theorem

Let \( \langle W, R \rangle \) be a relational frame. Then there is an equivalent augmented neighborhood frame.

Let \( \langle W, N \rangle \) be an augmented neighborhood frame. Then there is an equivalent relational frame.

Proof.

For each \( w \in W \), let \( N(w) = N^R_w \).

for all \( X \subseteq W \), \( X \in N(w) \) iff \( X \in N^R_w \).
From Neighborhood Frames to Kripke Frames

Theorem

✓ Let \( \langle W, R \rangle \) be a relational frame. Then there is an equivalent augmented neighborhood frame.

▶ Let \( \langle W, N \rangle \) be an augmented neighborhood frame. Then there is an equivalent relational frame.

Proof.

For each \( w \in W \), let \( N(w) = N^R_w \).
Motivating Examples

From Neighborhood Frames to Kripke Frames

Theorem

▶ Let \( \langle W, R \rangle \) be a relational frame. Then there is an equivalent augmented neighborhood frame.

✓ Let \( \langle W, N \rangle \) be an augmented neighborhood frame. Then there is an equivalent relational frame.

Proof.

For each \( w, v \in W \), \( wR_N v \) iff \( v \in \cap N(w) \).
Definability Results

1. $\mathcal{F} \models \Box (\varphi \land \psi) \rightarrow \Box \varphi \land \Box \psi$ iff $\mathcal{F}$ is closed under supersets (monotonic frames).

2. $\mathcal{F} \models \Box \varphi \land \Box \psi \rightarrow \Box (\varphi \land \Box \psi)$ iff $\mathcal{F}$ is closed under finite intersections.
Motivating Examples

Definability Results

1. $\mathcal{F} \models \Box(\varphi \land \psi) \rightarrow \Box\varphi \land \Box\psi$ iff $\mathcal{F}$ is closed under supersets (monotonic frames).

2. $\mathcal{F} \models \Box\varphi \land \Box\psi \rightarrow \Box(\varphi \land \Box\psi)$ iff $\mathcal{F}$ is closed under finite intersections.

3. $\mathcal{F} \models \Box\top$ iff $\mathcal{F}$ contains the unit

4. $\mathcal{F} \models \text{EMCN}$ iff $\mathcal{F}$ is a filter
Motivating Examples

Definability Results

1. $\mathcal{F} \models \Box(\varphi \land \psi) \rightarrow \Box\varphi \land \Box\psi$ iff $\mathcal{F}$ is closed under supersets (monotonic frames).

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3. $\mathcal{F} \models \Box\top$ iff $\mathcal{F}$ contains the unit

4. $\mathcal{F} \models \text{EMCN}$ iff $\mathcal{F}$ is a filter

5. $\mathcal{F} \models \Box\varphi \rightarrow \varphi$ iff for each $w \in W$, $w \in \cap N(w)$

6. And so on...
Motivating Examples

Completeness Results

- **E** is sound and strongly complete with respect to the class of all neighborhood frames
- **EM** is sound and strongly complete with respect to the class of all monotonic neighborhood frames
- **EC** is sound and strongly complete with respect to the class of all neighborhood frames that are closed under finite intersections
- **EN** is sound and strongly complete with respect to the class of all neighborhood frames that contain the unit
- **K** is sound and strongly complete with respect to the class of all neighborhood frames that are filters
- **K** is sound and strongly complete with respect to the class of all augmented neighborhood frames
Motivating Examples

Completeness Results

- **E**: is sound and strongly complete with respect to the class of all neighborhood frames.
- **EM**: is sound and strongly complete with respect to the class of all **monotonic** neighborhood frames.
- **EC**: is sound and strongly complete with respect to the class of all neighborhood frames that are **closed under finite intersections**.
- **EN**: is sound and strongly complete with respect to the class of all neighborhood frames that **contain the unit**.
- **K**: is sound and strongly complete with respect to the class of all neighborhood frames that are **filters**.
- **K**: is sound and strongly complete with respect to the class of all **augmented** neighborhood frames.
Motivating Examples

Completeness Results

- $E$ is sound and strongly complete with respect to the class of all neighborhood frames.
- $EM$ is sound and strongly complete with respect to the class of all monotonic neighborhood frames.
- $EC$ is sound and strongly complete with respect to the class of all neighborhood frames that are closed under finite intersections.
- $EN$ is sound and strongly complete with respect to the class of all neighborhood frames that contain the unit.
- $K$ is sound and strongly complete with respect to the class of all neighborhood frames that are filters.
- $K$ is sound and strongly complete with respect to the class of all augmented neighborhood frames.
Motivating Examples

Some Results

▶ For each Kripke model $\langle W, R, V \rangle$, there is an pointwise equivalent \textit{augmented} neighborhood model $\langle W, N, V \rangle$.

▶ Bimodal normal modal logics can simulate non-normal modal logics (Kracht and Wolter 1999)

▶ There are logics which are complete with respect to a class of neighborhood frames but not complete with respect to relational frames (D. Gabbay 1975, M. Gerson 1975, M. Gerson 1976).

▶ There are logics incomplete with respect to neighborhood frames (M. Gerson, 1975; T. Litak, 2005).

▶ Model theory of classical modal logics: Bisimulation, van Benthem Characterization Theorem, Interpolation, etc. (H. Hansen, C. Kupke and EP, 2010)
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First-order extensions
First-Order Modal Language: $\mathcal{L}_1$

Extend the propositional modal language $\mathcal{L}$ with the usual first-order machinery (constants, terms, predicate symbols, quantifiers).
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$$A := P(t_1, \ldots, t_n) \mid \neg A \mid A \land A \mid \Box A \mid \forall x A$$

(note that equality is not in the language!)
State-of-the-art


http://lpcs.math.msu.su/~shehtman/n.ps

A **constant domain Kripke frame** is a tuple $\langle W, R, D \rangle$ where $W$ and $D$ are sets, and $R \subseteq W \times W$.

A **constant domain Kripke model** adds a valuation function $V$, where for each $n$-ary relation symbol $P$ and $w \in W$, $V(P, w) \subseteq D^n$.

A **substitution** is any function $\sigma : \mathcal{V} \rightarrow D$ ($\mathcal{V}$ the set of variables).

A substitution $\sigma'$ is said to be an $x$-**variant** of $\sigma$ if $\sigma(y) = \sigma'(y)$ for all variable $y$ except possibly $x$, this will be denoted by $\sigma \sim_x \sigma'$. 
A constant domain Kripke frame is a tuple \( \langle W, R, D \rangle \) where \( W \) and \( D \) are sets, and \( R \subseteq W \times W \).

A constant domain Kripke model adds a valuation function \( V \), where for each \( n \)-ary relation symbol \( P \) and \( w \in W \), \( V(P, w) \subseteq D^n \).

Suppose that \( \sigma \) is a substitution.

1. \( \mathcal{M}, w \models_\sigma P(x_1, \ldots, x_n) \) iff \( \langle \sigma(x_1), \ldots, \sigma(x_n) \rangle \in V(P, w) \)
2. \( \mathcal{M}, w \models_\sigma \Box A \) iff \( R(w) \subseteq (A)^{\mathcal{M}, \sigma} \)
3. \( \mathcal{M}, w \models_\sigma \forall x A \) iff for each \( x \)-variant \( \sigma' \), \( \mathcal{M}, w \models_{\sigma'} A \)
A **constant domain Neighborhood frame** is a tuple $\langle W, N, D \rangle$ where $W$ and $D$ are sets, and $N : W \rightarrow \wp(\wp(W))$.

A **constant domain Neighborhood model** adds a valuation function $V$, where for each $n$-ary relation symbol $P$ and $w \in W$, $V(P, w) \subseteq D^n$.

Suppose that $\sigma$ is a substitution.

1. $M, w \models_\sigma P(x_1, \ldots, x_n)$ iff $\langle \sigma(x_1), \ldots, \sigma(x_n) \rangle \in V(P, w)$
2. $M, w \models_\sigma \Box A$ iff $(A)^M,\sigma \in N(w)$
3. $M, w \models_\sigma \forall x A$ iff for each $x$-variant $\sigma'$, $M, w \models_{\sigma'} A$
Let $S$ be any (classical) propositional modal logic, by $\text{FOL} + S$ we mean the set of formulas closed under the following rules and axioms:

(S) All instances of axioms and rules from $S$.

(∀) $\forall x A \rightarrow A^x_t$ (where $t$ is free for $x$ in $A$)

(Gen) $\frac{A \rightarrow B}{A \rightarrow \forall x B}$, where $x$ is not free in $A$. 
Barcan Schemas

- **Barcan formula (BF):** $\forall x \Box A(x) \rightarrow \Box \forall x A(x)$
- **converse Barcan formula (CBF):** $\Box \forall x A(x) \rightarrow \forall x \Box A(x)$
Barcan Schemas

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- **converse Barcan formula** (*CBF*): \( \Box \forall x A(x) \rightarrow \forall x \Box A(x) \)

**Observation 1:** *CBF* is provable in **FOL + EM**

**Observation 2:** *BF* and *CBF* both valid on relational frames with constant domains

**Observation 3:** *BF* is valid in a *varying* domain relational frame iff the frame is anti-monotonic; *CBF* is valid in a *varying* domain relational frame iff the frame is monotonic.

*See* (Fitting and Mendelsohn, 1998) *for an extended discussion*
The system **EMN** and seems to play a central role in characterizing monadic operators of high probability (See Kyburg and Teng 2002, Arló-Costa 2004).
Constant Domains without the Barcan Formula

The system \textbf{EMN} and seems to play a central role in characterizing monadic operators of high probability (See Kyburg and Teng 2002, Arló-Costa 2004).

Of course, \textbf{BF} should fail in this case, given that it instantiates cases of what is usually known as the ‘\textit{lottery paradox}’:

For each individual $x$, it is \textit{highly probably} that $x$ will loose the lottery; however it is not necessarily highly probably that each individual will loose the lottery.
Converse Barcan Formulas and Neighborhood Frames

A frame $\mathcal{F}$ is **consistent** iff for each $w \in W$, $N(w) \neq \emptyset$

A first-order neighborhood frame $\mathcal{F} = \langle W, N, D \rangle$ is **nontrivial** iff $|D| > 1$

**Lemma** Let $\mathcal{F}$ be a consistent constant domain neighborhood frame. The converse Barcan formula is valid on $\mathcal{F}$ iff either $\mathcal{F}$ is trivial or $\mathcal{F}$ is supplemented.
$X \in N(w)$
First-Order Classical Modal Logic

$Y \not\in N(w)$
First-Order Classical Modal Logic

\[ F = \emptyset \]

\[ \forall v \not\in Y, \quad I(F, v) = \emptyset \]
∀v ∈ X, I(F, v) = D = \{a, b\}
First-Order Classical Modal Logic

\[ F = \emptyset \]

\[ F = \{ a \} \]

\[ F = D = \{ a \} \]

\[ \forall v \in Y - X, \quad I(F, v) = D = \{ a \} \]
\[ (F[a])^M = Y \not\in N(w) \quad \text{hence} \quad w \not\models \forall x \Box F(x) \]
$F = \emptyset$

$F = \{a\}$

$F = D$

$X \in N(w)$

$(\forall x F(x))^M = (F[a])^M \cap (F[b])^M = X \in N(w)$

hence $w \models \square \forall x F(x)$
Barcan Formulas and Neighborhood Frames

We say that a frame closed under $\leq \kappa$ intersections if for each state $w$ and each collection of sets $\{X_i \mid i \in I\}$ where $|I| \leq \kappa$, $\cap_{i \in I} X_i \in N(w)$.

**Lemma** Let $\mathcal{F}$ be a consistent constant domain neighborhood frame. The Barcan formula is valid on $\mathcal{F}$ iff either

1. $\mathcal{F}$ is trivial or
2. if $D$ is finite, then $\mathcal{F}$ is closed under finite intersections and if $D$ is infinite and of cardinality $\kappa$, then $\mathcal{F}$ is closed under $\leq \kappa$ intersections.
Completeness Theorems

Theorem FOL + E is sound and strongly complete with respect to the class of all frames.
Completeness Theorems

**Theorem FOL + E** is sound and strongly complete with respect to the class of all frames.

**Theorem FOL + EC** is sound and strongly complete with respect to the class of frames that are closed under intersections.
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**Theorem FOL + E + CBF** is sound and strongly complete with respect to the class of frames that are either non-trivial and supplemented or trivial and not supplemented.
**FOL + K** and **FOL + K + BF**

**Theorem**  
**FOL + K** is sound and strongly complete with respect to the class of filters.
**FOL + K and FOL + K + BF**

**Theorem** FOL + K is sound and strongly complete with respect to the class of filters.

**Observation** The augmentation of the smallest canonical model for FOL + K is not a canonical model for FOL + K. In fact, the closure under infinite intersection of the minimal canonical model for FOL + K is not a canonical model for FOL + K.
**FOL + K and FOL + K + BF**

**Theorem**  FOL + K is sound and strongly complete with respect to the class of filters.

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**Lemma**  The augmentation of the smallest canonical model for FOL + K + BF is a canonical for FOL + K + BF.

**Theorem**  FOL + K + BF is sound and strongly complete with respect to the class of augmented first-order neighborhood frames.
Is the addition of quantifiers straightforward?

1. S4M is complete for the class of all frames that are reflexive, transitive and final (every world can see an ‘end-point’). However, FOL + S4M is incomplete for Kripke models based on S4M-frames. (see Hughes and Cresswell, pg. 283).

2. S4.2 is S4 with $\Diamond\Box\phi \rightarrow \Box\Diamond\phi$. This logic is complete for the class of frames that are reflexive, transitive and convergent. However, FOL + S4M + BF is incomplete for the class of constant domain models based on reflexive, transitive and convergent frames. (see Hughes and Cresswell, pg. 271).

3. The quantified extension of GL is not recursively axiomatizable (Cresswell, 1997).
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3. The quantified extension of **GL** is not recursively axiomatizable (Cresswell, 1997).
What is going on?
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There are (consistent) modal logics that are incomplete.

A general model is a structure $\langle W, R, V, A \rangle$ where $A$ is a suitable boolean algebra with an operator of propositions.

All modal logics are sound and strongly complete with respect to general frames.
**Theorem** (Goldblatt and Mares) For any canonical propositional modal logic $S$, its quantified extension $QS$ is complete over a class of general frames for which the underlying propositional frame are just the $S$-frames.
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- New perspective on the Barcan formula: it corresponds to Tarskian models
- There is a trade-off between having the underlying Kripke frame validate the propositional logic in question and having a Tarskian-reading of the quantifier.
Central Idea

**Algebraic reading of the universal quantifier:** $\forall x \varphi$ is true at a world $w$ iff there is some proposition $X$ such that $X$ entails every instantiation of $\varphi$ and $X$ obtains at $w$. 
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$M, w \models_{\sigma} \forall x A$ iff there is a proposition $X$ such that $w \in X$ and $X \subseteq (A)^{M}_{\sigma(x|d)}$ for all $d \in D$.

vs.

$M, w \models_{\sigma} \forall x A$ iff for all $d \in D$, $M, w \models_{\sigma(x|d)} A$
General Frames

Let $\langle W, R \rangle$ be a frame.

$[R] : \wp W \to \wp W$ where

$[R](X) = \{ w \in W \mid \text{for all } v \in W, wRv \text{ implies } v \in X \}$

So $(\square \alpha)^{\mathcal{M}} = [R](\alpha)^{\mathcal{M}}$

$X \Rightarrow Y = (W - X) \cup Y$

So $(\alpha \rightarrow \beta)^{\mathcal{M}} = (\alpha)^{\mathcal{M}} \Rightarrow (\beta)^{\mathcal{M}}$. 
Halmos Functions

\[ \varphi : D^V \to \wp W \]

Let \( \varphi \) and \( \psi \) be two such functions, we can lift \([R]\) and \(\Rightarrow\) to operations of functions: Eg., if \( \varphi : D^V \to \wp W \) and \( f \in D^V \).

\[
([R]\varphi)(f) = [R](\varphi(f))
\]
Halmos Functions

Let $\varphi$ and $\psi$ be two such functions, we can lift $[R]$ and $\Rightarrow$ to operations of functions: Eg., if $\varphi : D^\gamma \to \wp W$ and $f \in D^\gamma$. 

$$([R] \varphi)(f) = [R](\varphi(f))$$

Fix a set $Prop \subseteq \wp W$. This defines for each $S \subseteq \wp W$,

$$\cap S = \bigcup\{X \in Prop \mid X \subseteq \bigcap S\}$$
Suppose $\text{Prop} \subseteq \wp W$ and let $\varphi : D^V \to \text{Prop}$,

$$(\forall x \varphi) f = \bigcap_{d \in D} \varphi(f[x|d])$$
Suppose $\text{Prop} \subseteq \wp W$ and let $\varphi : D^\gamma \to \text{Prop}$,

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$\langle W, R, V, \text{Prop}, \text{PropFun} \rangle$ where

- $\text{Prop}$ contains $\emptyset$ and is closed under $\Rightarrow$ and $[R]$
- Contains the function $\varphi_\emptyset(f) = \emptyset$ for all $f \in D^\gamma$
- $\text{PropFun}$ is closed under $\Rightarrow$, $[R]$ and $\forall x$.
- Assume $(P)^M : D^\gamma \to \wp W$ is an element of $\text{PropFun}$ for each atomic predicate $P$. 
General Completeness

**Theorem** For any propositional modal logic $S$, the quantified logic $QS$ is complete for the class of (all validating) quantified general frames.

*Note that the canonical model construction has as worlds maximally consistent sets that need not be $\forall$-complete.*
Key Results

**Theorem** (Goldblatt and Mares) If $S$ is a canonical propositional logic, then $QS$ is characterized by the class of all $QS$-frames whose underlying propositional frames validate $S$. 

Logics containing the Barcan formula have two characterizing canonical general frames: one that is Tarskian and one that is not.

1. If $S$ is canonical, then the second canonical model will have an underlying propositional frame that validates $S$ (e.g., $S4.2$), but may not be Tarskian.
2. On the other hand, the Tarskian canonical model may not have an underlying propositional frame that is a frame for $S$ (again $S4.2$ is an example). 

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Conclusions

- Characterize other “modality-quantifier interactions” in terms of properties on neighborhoods (eg. $\exists x \Box \varphi(x) \rightarrow \Box \exists x \varphi(x)$)
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▶ Equality, existence predicate, free systems, etc.
General Frames for First-Order Modal Logics

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▶ Extensions with higher-order quantification

Conclusions

- Characterize other “modality-quantifier interactions” in terms of properties on neighborhoods (e.g. $\exists x \Box \varphi(x) \rightarrow \Box \exists x \varphi(x)$)

- Equality, existence predicate, free systems, etc.

- Extensions with higher-order quantification


- Different perspectives on the first-order modal language.

Thank you.
General Frames for First-Order Modal Logics

Extensions: Higher-Order Coalition Logic

Strategy Logics

- **Coalitional Logic**: Reasoning about (local) group power.

  \([C]\varphi\): coalition \(C\) has a **joint action** to bring about \(\varphi\).

Strategy Logics

- **Coalitional Logic**: Reasoning about (local) group power.

  \[ [C] \varphi \]: coalition \( C \) has a **joint action** to bring about \( \varphi \).


- **Alternating-time Temporal Logic**: Reasoning about (local and global) group power:

  \( \langle \langle A \rangle \rangle \Box \varphi \): The coalition \( A \) has a **joint action** to ensure that \( \varphi \) will remain true.

Multi-agent Transition Systems

\[
\langle *, \text{deny} \rangle \\
q_0 \quad x = 1 \\
\langle \text{set2, grant} \rangle \\
q_1 \quad x = 2 \\
\langle *, \text{deny} \rangle
\]

\[
(P_{x=1} \rightarrow [s]P_{x=1}) \land (P_{x=2} \rightarrow [s]P_{x=2})
\]
Multi-agent Transition Systems

\[ P_{x=1} \rightarrow \neg[s] P_{x=2} \]
Multi-agent Transition Systems

\[
P_{x=1} \rightarrow [s, c]P_{x=2}
\]
Coalition Logic: $\varphi ::= p \mid \neg \varphi \mid \varphi \land \varphi \mid [C]\varphi$
Coalition Logic: $\varphi ::= p \mid \neg \varphi \mid \varphi \land \varphi \mid [C]\varphi$

$\mathcal{M}, w \models [C]\varphi$ iff $(\varphi)^{\mathcal{M}} \in N(w, C)$: “Coalition $C$ has a joint strategy to force the outcome to satisfy $\varphi$.”
Coalition Logic: $\varphi := p \mid \neg \varphi \mid \varphi \land \varphi \mid [C]\varphi$

$\mathcal{M}, w \models [C]\varphi$ iff $(\varphi)^\mathcal{M} \in N(w, C)$: “Coalition $C$ has a joint strategy to force the outcome to satisfy $\varphi$”.

Higher-Order Coalition Logic: $\varphi := F(x_1, \ldots, x_n) \mid Xx \mid \neg \varphi \mid \varphi \land \varphi \mid \forall X \varphi \mid \forall x \varphi \mid \{x\}\varphi \mid \langle \{x\}\varphi \rangle \varphi$
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Higher-Order Coalition Logic: $\varphi :=
F(x_1, \ldots, x_n) \mid Xx \mid \neg \varphi \mid \varphi \land \varphi \mid \forall X \varphi \mid \forall x \varphi \mid [{\{x}\}]\varphi \mid \langle {\{x}\} \rangle \varphi$

- $F(x_1, \ldots, x_n)$ is a first-order atomic formula
- $x$ is a first-order variable
- $X$ is a set variable
- $\{x\} \psi$ is a group operator representing the set of all $d$ such that $\psi[d/x]$ holds
HCL: Expressivity

What does the added expressive power give you?
HCL: Expressivity

What does the added expressive power give you?

- Relationships between coalitions:
  \[ \forall x (super\_user(x) \rightarrow user(x)) \]
HCL: Expressivity

What does the added expressive power give you?

▶ Relationships between coalitions:
$$\forall x (\text{super } user(x) \rightarrow user(x))$$

▶ General quantification over coalitions:
$$\forall X (\forall x (Xx \rightarrow user(x)) \rightarrow [{y}Xy]\varphi)$$

*Every coalition such that all of its members are users can achieve \( \varphi \).*
HCL: Expressivity

What does the added expressive power give you?

- Relationships between coalitions:
  \[ \forall x (super\_user(x) \rightarrow user(x)) \]

- General quantification over coalitions:
  \[ \forall X (\forall x (Xx \rightarrow user(x)) \rightarrow [{y} Xy] \varphi) \]

  Every coalition such that all of its members are users can achieve \( \varphi \).

- Complex relationships between coalitions and agents:
  \[ [{x} \varphi(x)] \psi \rightarrow [{y} \exists x (\varphi(x) \land collaborates(y, x))] \psi \]

  If the coalition represented by \( \varphi \) can achieve \( \psi \) then so can any group that collaborates with at least one member of \( \varphi(x) \).
HCL: Barcan/Converse Barcan Formulas

Converse Barcan: $[\{x\}\varphi(x)]\forall y \psi(y) \rightarrow \forall y[\{x\}\varphi(x)]\varphi(y)$

Barcan: $\forall y[\{x\}\varphi(x)]\varphi(y) \rightarrow [\{x\}\varphi(x)]\forall y \psi(y)$
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Converse Barcan: $\{x\} \varphi(x) \forall y \psi(y) \rightarrow \forall y \{x\} \varphi(x) \varphi(y)$

Barcan: $\forall y \{x\} \varphi(x) \varphi(y) \rightarrow \{x\} \varphi(x) \forall y \psi(y)$

$\{x\} x = Eric \forall y (CMU(y) \rightarrow happy(y)) \rightarrow$
$\forall y \{x\} x = Eric (CMU(y) \rightarrow happy(y))$
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Converse Barcan: \([\{x\}\varphi(x)]\forall y\psi(y) \rightarrow \forall y[\{x\}\varphi(x)]\varphi(y)\)

Barcan: \(\forall y[\{x\}\varphi(x)]\varphi(y) \rightarrow [\{x\}\varphi(x)]\forall y\psi(y)\)

\([\{x\}x = Eric]\forall y(CMU(y) \rightarrow happy(y)) \rightarrow \forall y[\{x\}x = Eric](CMU(y) \rightarrow happy(y))\)

*If I can do something to make everyone happy at CMU implies for each person at CMU, I can do something to make them happy.*
HCL: Barcan/Converse Barcan Formulas

Converse Barcan: $\{x\} \varphi(x) \forall y \psi(y) \rightarrow \forall y [\{x\} \varphi(x)] \varphi(y)$

Barcan: $\forall y [\{x\} \varphi(x)] \varphi(y) \rightarrow [\{x\} \varphi(x)] \forall y \psi(y)$

$[\{x\} x = \text{Eric}] \forall y (CMU(y) \rightarrow \text{happy}(y)) \rightarrow \\
\forall y [\{x\} x = \text{Eric}] (CMU(y) \rightarrow \text{happy}(y))$

*If I can do something to make everyone happy at CMU implies for each person at CMU, I can do something to make them happy.*

$\forall y [\{x\} x = \text{Eric}] (CMU(y) \rightarrow \text{happy}(y)) \not\rightarrow \\
[\{x\} x = \text{Eric}] \forall y (CMU(y) \rightarrow \text{happy}(y))$
HCL: Barcan/Converse Barcan Formulas

Converse Barcan: \([\{x\}\varphi(x)] \forall y \psi(y) \rightarrow \forall y[\{x\}\varphi(x)]\varphi(y)\]

Barcan: \(\forall y[\{x\}\varphi(x)]\varphi(y) \rightarrow [\{x\}\varphi(x)]\forall y \psi(y)\)

\([\{x\}x = Eric] \forall y (CMU(y) \rightarrow happy(y)) \rightarrow \forall y[\{x\}x = Eric](CMU(y) \rightarrow happy(y))\)

*If I can do something to make everyone happy at CMU implies for each person at CMU, I can do something to make them happy.*

\(\forall y[\{x\}x = Eric](CMU(y) \rightarrow happy(y)) \not\rightarrow [\{x\}x = Eric] \forall y (CMU(y) \rightarrow happy(y))\)

*For each person at CMU, I can make them happy does not imply that I can do something to make everyone at CMU happy.*
Higher-Order Coalition Logic

Sound and complete axiomatization combines ideas from coalition logic, first-order extensions of non-normal modal logics and Henkin-style completeness for second-order logic.