Formal Epistemology Workshop 2010

Two Ways of Measuring Degrees of Incoherence

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Standard subjective Bayesianism only allows us to distinguish between agents whose credences are perfectly probabilistically coherent and agents whose credences are incoherent. It is desirable to develop a measure of degrees of incoherence. One way of measuring degrees of incoherence is by ascertaining how much money an incoherent person can be made to lose in a Dutch book. I will compare two Dutch book measures, one proposed by Schervish, Seidenfeld & Kadane, and my own, and argue that only my method works as a plausible measure of incoherence.

I. SS&K's method for measuring degrees of incoherence in a nutshell

1. Start with a partition of propositions on which the agent is incoherent.
2. Set up a Dutch book against the agent in the normal way, and calculate the guaranteed loss for the agent.
3. Calculate the degree of incoherence for the agent for that partition in one of three ways:

a) Rate of loss for the agent:
Calculate the sum of all the amounts the agent needs to cover her part of the individual bets in the Dutch book (called the agent's "escrow"). Divide the guaranteed loss the agent faces by that sum. So: guaranteed loss divided by agent's escrow = rate of loss for the agent.

b) Rate of gain for the bookie:
Calculate the sum of all the amounts the bookie needs to cover her part of the individual bets in the Dutch book (called the bookie's "escrow"). Divide the guaranteed loss the agent faces by that sum. So: guaranteed loss divided by bookie's escrow = rate of gain for the bookie.

c) Neutral/sum normalization:
Divide the guaranteed loss by the sum of the stakes for all the bets in the Dutch book.

Example:
P(X) = 0.6, P(~X) = 0.6, Stakes: $1 per bet
Guaranteed loss: 0.2
Rate of loss: 0.2/1.20 = 1/6
Rate of gain: 0.2/0.8 = 1/4
Neutral/sum: 0.2/2 = 1/10

II. Problems with this method

1. Special problems for the rate of gain:
- can't handle bets on contradictions and tautologies, because the bookie does not need any money to cover the bets. In this case, the rate of gain is x/0, which is undefined.
- can't handle credences above 1 or below 0, because the bookie's escrow in those cases is either 0 or negative.

Problems that affect all three measures:

2. The method is designed to calculate the degree of incoherence for partitions, but not for an entire credence function.

3. Depending on which proposition the agent is incoherent on, the number of partitions on which she can be Dutch-booked varies. This leads to problems, because the differences in guaranteed loss generated by different numbers of partitions don't get evened out correctly by any of the three normalizations. As a result, different ways of being incoherent that should lead to the same degree of incoherence don't get the same score given any of the three measures.

Example:
1. P (X) = 0.5
2. P (~X) = 0.5
3. P (Y) = 0.5
4. P (~Y) = 0.5
5. P (X & Y) = 0.25
6. P (X & ~Y) = 0.25
7. P (~X & Y) = 0.25
8. P (~X & ~Y) = 0.25
9. P (X ∨ Y) = 0.75
10. P (X ∨ ~Y) = 0.75
11. P (~X ∨ Y) =0.75
12. P (~X ∨ ~Y) = 0.75
13. P ((X & ~Y) ∨ (~X & Y)) = 0.5
14. P ((X & Y) ∨ (~X & ~Y)) = 0.5
15. P (X ∨ ~X) = 1
16. P (X & ~X) = 0

The following four ways of being incoherent should plausibly be measured as the same degree of incoherence:
A. P(X&Y) = 0.3 (being 0.05 overconfident)
B. P(X&Y) = 0.2 (being 0.05 underconfident)
C. P(~X∨~Y) = 0.8 (being 0.05 overconfident)
D. P(~X∨~Y) = 0.7 (being 0.05 underconfident)

However, it turns out that none of the three measures delivers this result if we Dutch book the agent on all partitions that she is incoherent on. This is because (X&Y) is contained in 5 different partitions, but (~X∨~Y) is only contained in one.
Results ($1 Stakes for all bets):

Case	Guaranteed loss	Rate of loss	Rate of Gain	Neutral/Sum
A	0.25	1/21	1/39	1/60
B	0.25	1/41	1/19	1/60
C	0.05	1/21	1/19	1/40
D	0.05	1/21	1/19	1/40

3. Not all incoherencies in a credence function can be captured if we only allow Dutch books on partitions. Consider the following case: assume someone's credence function is just like in the previous example, except for the following credences:
5. P (X & Y) = 0.3
12. P (~X ∨ ~Y) = 0.7

If we allow only Dutch books on partitions, we can capture the incoherence in 5. in a Dutch book, but there is no partition that allows us to set up a Dutch book that captures the underconfidence in 12. However, we could do so if we also allowed Dutch books on anti-partitions.

Definition (Anti-Partition): Given a partition of propositions PT, the anti-partition PT′ is the set that contains all and only the propositions that are the negations of the propositions in PT. Given that every partition contains exactly one true proposition, every anti-partition correspondingly contains exactly one false proposition.

If an agent has coherent credences in the propositions in an anti-partition containing n propositions, the credences must add up to n-1. Otherwise, the agent can be Dutch-booked. One anti-partition the agent is incoherent on due to her credence in 12. is {X, ~X∨~Y, ~X∨Y}, which is the anti-partition corresponding to the partition {~X, (X & Y), (X & ~Y)}.

III. The maximum Dutch book measure

The Dutch book measure I propose is called the "maximum Dutch book measure", because it involves making the incoherent agent bet on or against each proposition in her credence function (which I assume to be a Boolean algebra).

1. Take a Boolean algebra B of n atomic propositions in which the agent S has degrees of belief.
2. List all the propositions that can be obtained in B, which will be 2^2^n (= m), not listing propositions that are equivalent to propositions already on the list.
3. Make a list of the degrees of belief S assigns to each proposition in B. Call this credence function of S "b".
4. For each proposition X_i in B, S can either bet on or against X_i.

If S bets on X_i, then S will win $1 - b(X_i)$ if X_i is true and lose $b(X_i)$ if X_i is false.
If S bets against X_i, then S will receive $b(X_i)$ if X_i is false, and lose $1 - b(X_i)$ if X_i is true.

5. Then figure out which combination of betting on and against the propositions in B will result in the maximum loss across the 2^n state descriptions.

For a given combination of bets (of which there are 2^m, because every proposition in B can be bet on or against), consider all state descriptions (of which there are 2^n) and see how much the agent wins or loses given each state description. If there is a state description in which the agent gains or breaks even, then the score of this combination of bets is 0, and otherwise the score is the minimum amount lost in any state description. The score for the credence function is the highest score for any of the 2^m combinations of bets. Thus, the combination of bets that will result in the highest guaranteed loss across all 2^n state descriptions is the Dutch book we are looking for (sometimes there is more than one combination of bets that results in the highest guaranteed loss across all possible worlds).

In order to show that this method works, need to show that an agent is maximum-Dutch-bookable iff the agent is incoherent. Clearly, if an agent is maximum Dutch-bookable, then the agent is Dutch-bookable, and thus incoherent. I need to show that if an agent is incoherent, then she is maximum Dutch-bookable.

Lemma 1: If an agent is incoherent then there is some partition on which she is incoherent.

Lemma 2: If there exists a partition with respect to which an agent is incoherent, then the Boolean algebra is divisible into partitions and anti-partitions, such that she is incoherent with respect to at least one of them.

Proof: Assume that there is some partition on which the agent is incoherent. We can now identify the anti-partition that corresponds to the partition on which the agent is incoherent. Since this anti-partition contains exactly the propositions that are the negations of the propositions in the partition on which the agent is incoherent, the part of the Boolean algebra that is encompassed by the resulting partition/anti-partition duo does not contain any proposition without also containing its negation. The now remaining part of the credence function can be divided into partitions by simply pairing each remaining proposition with its negation.

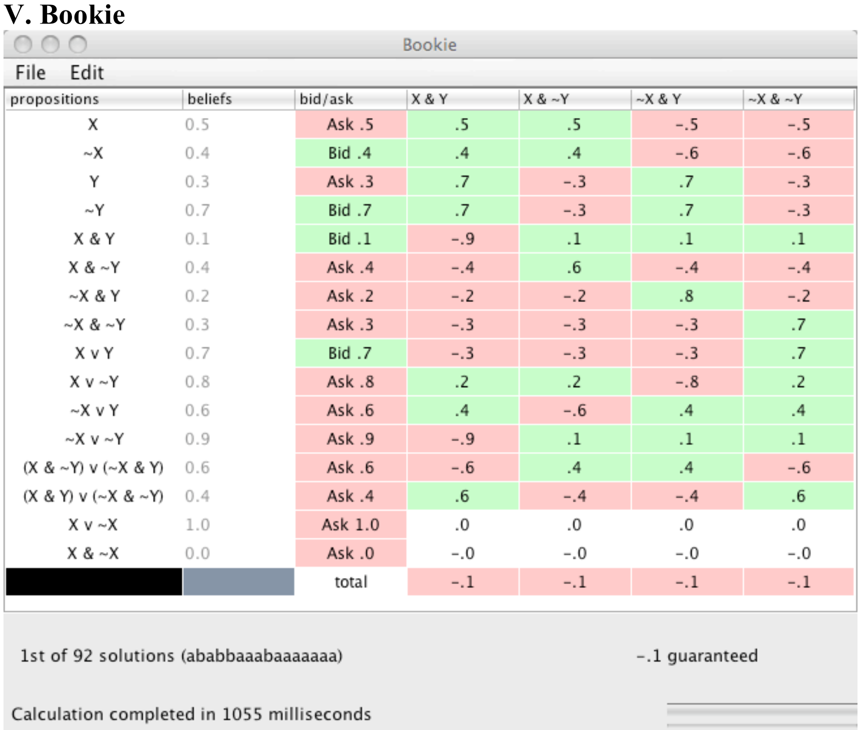
Lemma 3: If the Boolean algebra is divisible into partitions and anti-partitions such that the agent is incoherent with respect to at least one of them, then she is maximum Dutch-bookable.

Proof: Any partition or anti-partition on which the agent is incoherent will generate a Dutch book loss. For all of the remaining partitions or anti-partitions on which the agent is coherent, the bookie can set up the bets in such a way that the agent neither gains nor loses money. This is important, because it guarantees that there won't be any accidental gains from these bets that decrease the guaranteed loss from the bets on the incoherent parts of the credence function.

IV. Comparing S,S&K's Dutch book measure to the maximum Dutch book measure

1. The maximum Dutch book measure can easily handle bets on contradictions and tautologies, as well as credences below 0 and above 1.
2. The maximum Dutch book measure does not have the limitation of applying only to a part of the agent's credence function.
3. In the example under II.2, the maximum Dutch book measure gives the same result for all four cases of over- and underconfidence: 0.05 guaranteed loss for each.
4. The maximum Dutch book measure, because it is sensitive to incoherence on anti-partitions, captures both the incoherence on 5. and 12, and delivers 0.10 as the result.

V. Bookie


Bookie

File Edit

propositions	beliefs	bid/ask	X & Y	X & ~Y	~X & Y	~X & ~Y
X	0.5	Ask .5	.5	.5	-.5	-.5
~X	0.4	Bid .4	.4	.4	-.6	-.6
Y	0.3	Ask .3	.7	-.3	.7	-.3
~Y	0.7	Bid .7	.7	-.3	.7	-.3
X & Y	0.1	Bid .1	-.9	.1	.1	.1
X & ~Y	0.4	Ask .4	-.4	.6	-.4	-.4
~X & Y	0.2	Ask .2	-.2	-.2	.8	-.2
~X & ~Y	0.3	Ask .3	-.3	-.3	-.3	.7
X v Y	0.7	Bid .7	-.3	-.3	-.3	.7
X v ~Y	0.8	Ask .8	.2	.2	-.8	.2
~X v Y	0.6	Ask .6	.4	-.6	.4	.4
~X v ~Y	0.9	Ask .9	-.9	.1	.1	.1
(X & ~Y) v (~X & Y)	0.6	Ask .6	-.6	.4	.4	-.6
(X & Y) v (~X & ~Y)	0.4	Ask .4	.6	-.4	-.4	.6
X v ~X	1.0	Ask 1.0	.0	.0	.0	.0
X & ~X	0.0	Ask .0	-.0	-.0	-.0	-.0
		total	-.1	-.1	-.1	-.1

1st of 92 solutions (ababbaaabaaaaaaa) -.1 guaranteed

Calculation completed in 1055 milliseconds

VI. Selected References
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