Rationalizing Two-Tiered Choice Functions through Conditional Choice

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FEW 2010

Outline

Optimize the given index

Select an available alternative that is at least as good as every other available alternative with respect to the given index. Example indices:

- Expected value
- Maximum value
- Minimum value
- Combinations, e.g. linear combinations, of these.

Optimization

Select an available alternative that is at least as good as every other available alternative with respect to the given binary relation.

- Only the ordinal properties of the indices in the previous slide were relevant for optimization.
- Optimization against relation R, often interpreted as weak preference, requires that R is *complete* in the sense that xRy or yRx for all x, y.

Question: Is there any reason to doubt the appropriateness of optimization for rational agents?

Maximization

Select an available alternative that is not strictly worse than any other available alternative with respect to the given binary relation.

- Sen (1997) has argued in favor maximization as an alternative to optimization.
- Maximization makes sense even in the presence of incompleteness.
- Maximization coincides with optimization when in the classical situation.

Maximization is very general, but also very coarse. We now consider alternatives to optimization in more highly structured situations.

Decision Making under Uncertainty

the standard account

Consider the framework of *subjective expected utility* theory:

- Ω is a finite set of *states*.
- K is a finite set of *consequences*.
- The agent's *beliefs* are represented by a probability measure p on Ω.
- The agent's *values* are represented by a cardinal utility function u on K.

Given a set of acts, i.e. functions from Ω to K, the rational agent is supposed to select an available act f that is optimal with respect to the following index:

$$E_p(f) = \sum_{i \in \Omega} p(i)u(f(i))$$

Indeterminate Probabilities

Subjective expected utility theory assumes that the rational agent's credal state should be representable by a probability measure. Not everyone agrees ...

- Epistemic arguments against the requirement of numerically precise probabilities, e.g. Kyburg (1968), Levi (1974).
- Decision theoretic arguments against numerically precise probabilities, e.g. Ellsberg (1961).

Decision Making with Indeterminate Probabilities

Gardenfors and Sahlin

- Ω is a finite set of *states*.
- K is a finite set of *consequences*.
- The agent's *beliefs* are represented by a nonempty *set* P of probability measures on Ω.
- The agent's *values* are represented by a cardinal utility function u on K.

Given a set of acts, i.e. functions from Ω to K, the rational agent is supposed to select an available act f that is optimal with respect to the following index:

$$S(f) = \inf\{\sum_{i \in \Omega} p(i)u(f(i)) \mid p \in P\}$$

Decision Making with Indeterminate Probabilities
Ellsberg

- Ω is a finite set of *states*.
- K is a finite set of *consequences*.
- The agent's *beliefs* are represented by a nonempty *set* P of probability measures on Ω, a distinguished $p_0 \in P$, and parameter value $\lambda \in [0, 1]$.
- The agent's*values* are represented by a cardinal utility function u on K.

Given a set of acts, i.e. functions from Ω to K, the rational agent is supposed to select an available act f that is optimal with respect to the following index:

$$H(f) = \lambda E_{p_0}(f) + (1 - \lambda)S(f)$$

Decision Making with Indeterminate Probabilities
Levi

- Although they allow for indeterminate probabilities, the previous two proposals are compatible with optimization.
- In contrast, the following proposal by Levi *is not*:
  - Ω, K, P, u as before.
  - $f \in Y$ is *E-admissible* in Y iff there is some $p \in P$ such that $E_p(f) \geq E_p(g)$ for all $g \in Y$.
  - $f \in Y$ is *S-admissible* in Y iff it is E-admissible in Y and $S(f) \geq S(g)$ for all g that are E-admissible in Y.

Note: E-admissibility may be regarded as a special case of S-admissibility, one in which the second-tier consideration is vacuous.

Choice Functions

- X is a set of *alternatives*.
- $\mathcal{X}$ is the set of all finite, nonempty subsets of X.
- $C : \mathcal{X} \to \mathcal{X}$ is a *choice function* on X just in case $C(Y) \subseteq Y$ for all $Y \in \mathcal{X}$.

Example

If R is a complete binary relation on X, then R determines a choice function C on $\mathcal{X}$ via optimization.

$$C_R(Y) = \{y \in Y \mid yRz \text{ for all } z \in Y\}$$

Optimization Characterized

It is well known that optimization can be viewed as a fixed point of revealed preference.

- Given $C : \mathcal{X} \to \mathcal{X}$.
- Define R_C by xR_Cy iff $x \in C(\{x, y\})$.
- C is given by optimization just in case $C = C_{R_C}$.

Typically, for rational agents, the generating R is also required to be transitive. It is well known that the class of such C may be characterized in terms of the following properties.

Optimization of Rational Preferences Characterized

C can be represented as optimization of a weak order iff the following conditions hold:

$$\alpha: \text{ If } x \in Y \subseteq Z \text{ and } x \in C(Z), \text{ then } x \in C(Y).$$

$$\beta: \text{ If } Y \subseteq Z, x, y \in C(Y) \text{ and } x \in C(Z), \text{ then } y \in C(Z).$$

Violations of Ordering

E-admissibility

Example (Levi, 1974)

Let P be the set of distributions p on $\{\text{Red}, \text{Yellow}, \text{Blue}\}$ such that $p(\text{Red}) = \frac{1}{3}$, $p(\text{Yellow}) = \frac{n}{90}$, and $p(\text{Blue}) = \frac{60-n}{90}$ for some natural number $n \leq 60$. Consider the following alternatives:

	Red	Yellow	Blue
e	3	0	3
f	3	3	0
g	$\frac{3}{2}$	$\frac{3}{2}$	$\frac{3}{2}$

f and g are E-admissible in $\{f, g\}$. However, f is E-admissible in $\{e, f, g\}$ but g is not. β is violated.

Violations of Ordering

S-admissibility

Example (Levi, 1974)

Let P be the set of distributions p on $\{\text{Red}, \text{Yellow}, \text{Blue}\}$ such that $p(\text{Red}) = \frac{1}{3}$, $p(\text{Yellow}) = \frac{n}{90}$, and $p(\text{Blue}) = \frac{60-n}{90}$ for some natural number $n \leq 60$. Consider the following alternatives:

	Red	Yellow	Blue
e	3	0	3
f	3	3	0
g	$\frac{3}{2}$	$\frac{3}{2}$	$\frac{3}{2}$

e is S-admissible in $\{e, f, g\}$ while g is not. However, g is S-admissible in $\{e, g\}$ while e is not. α is violated.

Other Sources of Indeterminacy

Thus far we have been considering indeterminacy with respect to credal judgments. There are other sources of indeterminacy.

- Levi (1986) presents analogous choice functions in relation to value conflicts.
- Helzner (2009) considers analogous choice functions in the context of an indeterminate weighting of attributes in multiattribute decision making.

Two-Tiered Choice Functions
The General Case

In light of the previous considerations, Helzner (2008) considers the following qualitative formulation of two-tiered choice:

- Let $\mathcal{R}$ be a set of weak orders on X representing first-tier considerations.
- Let S be a weak order on X representing second-tier considerations.
- $y \in C_{\mathcal{R}}(Y)$ iff $y \in Y$ and there is some $R \in \mathcal{R}$ such that yRz for all $y \in Y$.
- $y \in C^S_{\mathcal{R}}(Y)$ iff $y \in C_{\mathcal{R}}(Y)$ and ySz for all $z \in C_{\mathcal{R}}(Y)$.

Attempts at Characterization

It is natural to ask if there is a nice way to characterize those C that are equal to $C^S_{\mathcal{R}}$ for some choice of $\mathcal{R}$ and S.

- Helzner (2008) shows that there is no such characterization in terms of the extensive list of conditions given in Sen (1977).
- There are partial results in more highly structured settings. Seidenfeld, Schervish, and Kadane (2007) characterize E-admissibility in the act-state framework.

However, since indeterminacy may arise with respect to various antecedent judgments, a general analysis should be possible.

Reconsidering the Foundations

- Do choice functions represent enough of the agent to support classification with respect to a given standard of rationality?
- Choice functions simply represent judgments of admissibility across various decision problems.
- Suppose that the agent in credal state P is committed to E-admissibility as a standard of rationality. Shouldn't this commitment extend to its conditional judgment of what it would count as admissible if its credal state were P'?

Conditional Choice Functions

- $\mathcal{X}$ (as before)
- $\mathcal{E} = \langle E, \sqsubseteq \rangle$ is a nonempty poset. Intuitively, an element of E is a potential result of the antecedent judgment(s) on which admissibility depends, and things higher up in the poset are more determinate.
- $\mathcal{C} : \mathcal{E} \times \mathcal{X} \to \mathcal{X}$ is a *conditional choice function* on X just in case the following conditions are satisfied for all $x \in X$, $Y \in \mathcal{X}$ and $e \in E$:
  - $\mathcal{C}(e, Y) \subseteq Y$
  - If $x \in \mathcal{C}(e, Y)$, then there is an $f \in E$ such that $e \sqsubseteq f$ and $x \in \mathcal{C}(g, Y)$ whenever $f \sqsubseteq g$.

Example 1

- $X = \{(x_1, x_2, x_3) \mid x_1, x_2, x_3 \in N\}$
- E is the set of all nonempty subsets of $\{(30, n, 60 - n) \mid 0 \leq n \leq 60\}$.
- $f \sqsubseteq g$ iff $g \subseteq f$.
- $(x_1, x_2, x_3) \in \mathcal{C}(e, Y)$ just in case there is a $(n_1, n_2, n_3) \in e$ such that $\sum_{i=1}^{3} n_i x_i$ is at least as great as $\sum_{i=1}^{3} n_i y_i$ for all $(y_1, y_2, y_3) \in Y$.

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Example 2

- $\mathcal{X}, \mathcal{E}, \mathcal{C}$ (as in Example 1).
- $(x_1, x_2, x_3) \in \mathcal{D}(e, Y)$ iff
  - $(x_1, x_2, x_3) \in \mathcal{C}(e, Y)$,
  - $\min\{\sum_{i=1}^{3} n_i x_i \mid (n_1, n_2, n_3) \in e\} \geq \min\{\sum_{i=1}^{3} n_i y_i \mid (n_1, n_2, n_3) \in e\}$ for all $(y_1, y_2, y_3) \in \mathcal{C}(e, Y)$.

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Basic Relations

- If $\mathcal{C} : \mathcal{E} \times \mathcal{X} \to \mathcal{X}$ is a conditional choice function and $e \in E$, then let $\mathcal{C}_e$ be the choice function defined by $\mathcal{C}_e(Y) = \mathcal{C}(e, Y)$ for all $Y \in \mathcal{X}$.
- If C is a choice function on $\mathcal{X}$, then let C^* be the conditional choice function defined by $C^*(e, Y) = C(Y)$ for all $e \in E$ and $Y \in \mathcal{X}$.

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Extension of Properties

Every property P of choice functions may be extended to a property P^* of conditional choice functions as follows:

P^*: For every $e \in E$ there is an $f \in E$ such that $e \sqsubseteq f$ and $\mathcal{C}_g$ satisfies P for all $g \in E$ such that $f \sqsubseteq g$.

Moreover, P^* generalizes P in the following sense:

Proposition: Let C be a choice function on X. Let P be a property of choice functions. C satisfies P iff C^* satisfies P^*.

Preliminaries

Let $\mathcal{C} : \mathcal{E} \times \mathcal{X} \to \mathcal{X}$ be a conditional choice function.

- For each $e \in E$, let $O_e = \{R_{\mathcal{C}_f} \mid e \sqsubseteq f\}$.
- For each $e \in E$, define a binary relation $\succ_e$ on X as follows: $x \succ_e y$ iff there is a $Y \in \mathcal{X}$ and an $f \in E$ such that
  - $e \sqsubseteq f$,
  - $x \in \mathcal{C}(e, Y)$,
  - $y \notin \mathcal{C}(e, Y)$, and
  - $y \in \mathcal{C}(f, Y)$.
- Let $\succ_e^t$ be the transitive closure of $\succ_e$.
- Define $\succsim_e^t$ by $x \succsim_e^t y$ iff not $y \succ_e^t x$.

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R1

α^*: For every $e \in E$ there is an $f \in E$ such that $e \sqsubseteq f$ and $\mathcal{C}_g$ satisfies α for all $g \in E$ such that $f \sqsubseteq g$.

β^*: For every $e \in E$ there is an $f \in E$ such that $e \sqsubseteq f$ and $\mathcal{C}_g$ satisfies β for all $g \in E$ such that $f \sqsubseteq g$.

Proposition: Let $\mathcal{C}$ be a conditional choice function that satisfies α^* and β^*. If $x \in \mathcal{C}(e, Y)$, then there is a weak order $R \in O_e$ such that xRy for all $y \in Y$.

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R2

χ: If $x \succ_e^t y$, then there is no Y such that $x, y \in \mathcal{C}(e, Y)$.

Proposition: Let $\mathcal{C}$ be a conditional choice function that satisfies α^*, β^*, χ, and such that $\succ_e^t$ is irreflexive for all $e \in E$. $x \in \mathcal{C}(e, Y)$ iff

- $x \in Y$,
- there is a weak order $R \in O_e$ such that xRy for all $y \in Y$, and
- if $y \in Y$ and, for some weak order $R \in O_e$, yRz for all $z \in Y$, then it is not the case that $y \succ_e^t x$.

Moreover, $\succ_e^t$ asymmetric and transitive.

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R3

Proposition: Let $\mathcal{C}$ be a conditional choice function that satisfies α^*, β^*, χ, and such that $\succ_e^t$ is both irreflexive and negatively transitive for all $e \in E$. $x \in \mathcal{C}(e, Y)$ iff

- $x \in Y$,
- there is a weak order $R \in O_e$ such that xRy for all $y \in Y$, and
- if $y \in Y$ and, for some weak order $R \in O_e$, yRz for all $z \in Y$, then $x \succsim_e^t y$.

Moreover, $\succsim_e^t$ is a weak order.

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