The Bounded Strength of Weak Expectations

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Formal Epistemology Workshop

Konstanz, 2-5 September 2010

The Pasadena and the St. Petersburg Game

- The Pasadena Game – the topic of this talk – is a variation on the St. Petersburg Game familiar from decision theory.
- The main question concerns the price which a rational agent should assign to the game. This has been the subject of several papers in the journal *Mind*.
- We analyze the scope of the weak expectations approach, a solution suggested by Easwaran (2008).

The St. Petersburg Game

The St. Petersburg Game:

- A fair coin is tossed repeatedly until it first comes up heads, say, at toss n.
- The agent receives €2^n.
- What is the rational price of this game?

Expected Utility Theory

Expected Utility Theory: Take a game with countably many outcomes. If s_j denotes the game's outcomes (e.g. "heads comes up first at toss j"), $X(s_j)$ the associated payoff and $P(s_j)$ the associated probability, then the rational price of the game is its expected utility

$$\sum_{j\in\mathbb{Z}} P(s_j)X(s_j). \tag{1}$$

Games where $\sum_{j\in\mathbb{Z}} |P(s_j)X(s_j)| < \infty$ are called (strongly) integrable, and Expected Utility Theory applies.

The St. Petersburg Game: Failure of (Strong) Integrability

- Problem: The St. Petersburg Game is not strongly integrable.
- More precisely, its "expected utility" is

$$\sum_{n\in\mathbb{N}} \frac{1}{2^n} \cdot 2^n = \sum_{n\in\mathbb{N}} 1 = \infty. \quad (2)$$

- So it seems that the game is infinitely desirable.
- But intuitively, it is only moderately desirable.

The Pasadena Game

An even trickier variation of the St. Petersburg Game is the Pasadena Game (Nover and Hájek 2004).

- A fair coin is tossed repeatedly until it first comes up heads, say, at toss n.
- If n is an odd number, the agent receives €$2^n/n$.
- If n is an even number, the agent has to pay €$2^n/n$.
- Is this game desirable or not, and what is its rational price?

Expected Utility Theory and the Pasadena Game

State s_i	...	s_{-2}	s_{-1}	s_0	s_1	s_2	...
Probability $P(s_i)$	...	1/16	1/4	1/2	1/8	1/32	...
Payoff (in €) $X(s_i)$	...	-4	-2	2	8/3	32/5	...

- Note: $\sum_{j\in\mathbb{Z}} |P(s_j)X(s_j)| = \infty$ – the game is not (strongly) integrable
- Thus: if $(s_j)_{j\in\mathbb{Z}}$ is the collection of outcomes, the sum $\sum_{j\in\mathbb{Z}} P(s_j)X(s_j)$ has no definite value.
- There is a problem of arbitrariness: the value of the sum depends on the order of summation. But which order is "the right one"?

The Pasadena Game: Some Examples

The order of summation makes a crucial difference:

$$\sum_{j=0}^{\infty} \left(P(s_j)X(s_j) + P(s_{-j-1})X(s_{-j-1})\right) = \log 2$$

$$\sum_{j=0}^{\infty} \left(P(s_j)X(s_j) + \sum_{k=1}^{5} P(s_{-5j-k})X(s_{-5j-k})\right) = \log 2 + \frac{1}{2}\log\frac{1}{5}$$

$$\sum_{j=0}^{\infty} \left(P(s_{-j})X(s_{-j}) + \sum_{k=2^j}^{2^{j+1}-1} P(s_k)X(s_k)\right) = \infty$$

Dominance Heuristics

State s_i	...	s_{-2}	s_{-1}	s_0	s_1	s_2	...
Probability $P(s_i)$	...	1/16	1/4	1/2	1/8	1/32	...
Payoff (in €) $X(s_i)$	...	-3	-1	3	11/3	37/5	...

- What shall we do with Expected Utility Theory? Apparent failure? Not applicable?
- Still, we can say something about the Pasadena Game: it is worse than the Altadena Game where all payoffs are increased by 1 Euro (see table above).

There has been some debate in *Mind* about the implications of this result for decision theory as a whole.

Weak Expectations: The Definition

Easwaran (2008): The rational price of the Pasadena Game is its weak expectation.

> **Weak Expectation Rule (WER):** *A probabilistic game X with i.i.d. realizations $(X_n)_{n\in\mathbb{N}}$ and with $S_n := \sum_{i=1}^n X_i$ should be valued at its weak expectation μ. This value μ satisfies for any tolerance margin ε:*

$$\forall \varepsilon, \delta > 0 \; \exists N_0 \; \forall n \geq N_0 : P\left(\left|\frac{1}{n}S_n - \mu\right| \geq \delta\right) \leq \varepsilon. \quad (3)$$

In other words, we will, with probability $1-\varepsilon$, in the long run end up with an average payoff that is close to μ.

Games that satisfy (3) are called weakly integrable.

Weak Expectations: Benefits

WER has several attractive features:

- It resolves the arbitrariness inherent in the Pasadena Game. $\log 2$ is the only rational price of the game.
- It respects the dominance heuristics for the Pasadena and the Altadena Game.
- It has a clear and natural anchoring in probability theory.
- It is a conservative extension of Expected Utility Theory that successfully deals with problematic cases.

Weak Expectations: Objections

The crucial equation for WER was

$$\forall \varepsilon, \delta > 0 \; \exists N_0 \; \forall n \geq N_0 : P\left(\left|\frac{1}{n}S_n - \mu\right| \geq \delta\right) \leq \varepsilon. \quad (4)$$

- The crucial rationale is this: if we neglect events of total probability smaller than ε, then the repeated, averaged game S_n/n is almost equal to a sure-thing game with payoff μ.
- **Question:** Are we entitled to neglect these outcomes – even when we can make their probability arbitrarily small?

Weak Expectations: Objections (cont'd)

- A first answer could be: "why not?" In daily life, we often ignore dangers that occur with very small chances.
- Games where payoffs increase without bounds are, however, different from daily life: it is completely unclear which outcomes we should ignore.
- Should we ignore the extremely positive, the extremely negative outcomes or some in between?

$\Rightarrow$ It is arbitrary to neglect only those outcomes with extreme payoffs, as WER does.

Weak Expectations: Reasons for Failure

- WER moves the problem from a single game to the repeated game S_n/n, but that one inherits the structure of the original game. Why should the problem vanish then?
- $\rightarrow$ The rational price of the game is again in the eye of the beholder; weak expectations fail to develop normative force for the valuation of the game.

A New Research Program

Question Can we find a theoretical framework where weak expectations do have normative force?

Proposal A psychologically realistic bounded utility framework.

Bounded Utility: Assumptions

- All agents i have utility functions u_i that map monetary payoffs to utility units.
- These functions are bounded, i.e., there is a maximal amount of utility that money can confer, even if we have infinite amounts of money.

There are two evident dangers:

Trivialization All games become integrable, i.e. EUT applies. The paradoxes vanish trivially.

Subjectivism Agents have different utility functions and assign different (subjectively rational) prices.
$\rightarrow$ price of a game in the eye of the beholder, no interesting results.

The Agreement Theorem

We prove a strong theoretical result in the following

Setup: Take a group of M agents $G = \{1, 2, 3, \ldots, M\}$ with monotonously increasing, bounded and continuous utility functions $u_i : \mathbb{R} \to \mathbb{R}$, $i \in G$. Let $\|f\|_\infty := \sup_{x \in \mathbb{R}} |f(x)|$ denote the supremum norm. Then there is a common bound for the u_i:

$$C := \sup_{i \in G} \|u_i\|_\infty < \infty. \quad (5)$$

The Agreement Theorem (cont'd)

Theorem: In the above setup, let $\Delta > 0$ and let S_n denote the payoff sum of n i.i.d. realizations of a weakly integrable game with weak expectation μ. Then, there is an $N_0 \in \mathbb{N}$ such that for all $n \geq N_0$ and all $i, j \in G$,

$$\left| u_i^{-1}\left(\mathbb{E}\left[u_i\left(\frac{1}{n}S_n\right)\right]\right) - \mu \right| \leq \Delta \quad (6)$$

i.e. each agent regards μ as the rational price of the game, and the differences between the individual valuations of the game vanish.

The Agreement Theorem (cont'd)

The theorem has a number of remarkable implications:

- The theorem applies to the Pasadena and Altadena Game and leads, as the number of games increases, to a rational price of $\log 2$ for the Pasadena and $1 + \log 2$ for the Altadena Game.
- The theorem shows that agents agree on the rational price of the repeated game, regardless of the nature of an individual utility function.
- The theorem saves the dominance heuristics.
- Trivialization and subjectivism are avoided.
- The single case and the long run are not isomorphic (confirming one of Easwaran's worries).

Conclusions

- The normative force of weak expectations is undercut by the arbitrariness inherent in the Weak Expectation Rule.
- There is no unique rational price for a single Pasadena Game.
- In a bounded utility framework (with different utility functions), the weak expectation determines the rational price for a repeated, averaged game. Easwaran's conjecture is vindicated when choosing a psychologically realistic framework.
- Marrying bounded utility to weak expectations preserves the best of both worlds.

Thanks a lot for your attention!

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