Dilemmas and Models

A Nominalist's Response to Burgess's Arguments for Realism

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On October 22, 1987, at a meeting of the Berkeley Philosophy Department's Colloquium, the Princeton logician John Burgess delivered a paper directed at refuting nominalism in the philosophy of mathematics.

One of the chief points of contention concerned some such theorem of arithmetic as:

[1] There are numbers greater than 10^{10} that are prime.

Either [1] is true or [1] is not true.

Hence, the nominalist must choose between the following horns of a dilemma:

(1) He could maintain that [1] is true, and then hold that [1] does not imply that numbers exist;

(2) Or he could maintain that [1] is not true, and then hold that the mathematician's justification (or proof) does not in fact justify the belief in the truth of [1].

The first option, which Burgess calls the "hermeneutic alternative", requires maintaining that [1] does not actually mean what it seems to mean (since it seems to imply that there are numbers).

This option, Burgess believes, runs into the difficulty of explaining the meaning of [1] in a way that is compatible with denying the implication in question.

The second option—the "revolutionary alternative"-- also faces some difficult questions.
It denies that the expert's proof of [1] does in fact justify its acceptance as true. Then, he writes:

> *Is the aim of the mathematician, in deciding what results to accept, that of arriving at justified beliefs, or is it something else, perhaps that of devising useful fictions? If mathematicians are aiming to arrive at justified beliefs and are failing to do so, should philosophers attempt to get them to recognize their failure and take corrective measures?*

Since neither of the two described options seem to be at all attractive, Burgess concluded that the nominalist faces quite a troublesome dilemma.

Burgess, a mathematical realist, on the other hand, is free to conclude that mathematical objects truly e:

Burgess discussed the above dilemma in connection with his attempt to answer a question he calls the "Benacerraf Problem":

How could we come justifiably to believe anything implying that there are numbers, given that it does not make sense to ascribe location or causal powers to numbers?

Burgess's response to the problem is summed up in the motto: "Don't think, look!" In particular, look at how mathematicians come to accept such theorems as [1].

Burgess believes that one need only look at how mathematicians have come to accept that there are infinitely many prime numbers to see how we can come justifiably to believe something that implies that there are numbers.

Thus, one can answer Benacerraf's Problem.

In a work Burgess co-authored with Gideon Rosen, it is asserted that, having studied Euclid's Theorem that there are infinitely many prime numbers: "we are prepared to say that there exist infinitely many prime numbers".

This prompts Burgess and Rosen to "acquiesce in the existence of numbers", even though "numbers are not supposed to be like ordinary concrete things like rocks or trees or people".

These authors are clearly taking Euclid's Theorem to be an *"ontologically weighty assertion"*—that is, an assertion about what "actually exists" (or to put it in terms used by modal logicians, "what exists in the actual universe").

The theorem is taken to assert that prime numbers a exist, even though they are not like ordinary concret like rocks or trees.

It can be seen that, instead of the bifurcation of the possibilities that Burgess makes, we can express what is possible, more illuminatingly, in the following trifurcation:

[1] is an ontologically weighty statement, and [1] is true, but [1]does not imply that numbers exist;

or

[1] is an ontologically weighty statement, but [1] is not true;

or

[1] is not an ontologically weighty statement.

The Second Option

How might [1] not be true, given that there is a mathematically acceptable proof of [1]?

Consider the following theorem of group theory:

There is one and only one right identity.

Suppose we take this theorem as an ontologically weighty assertion. In that case, the theorem would be asserting that there actually exists one and only one right identity.

Would that statement be true?

Does that statement even make sense?

How many mathematicians would understand the theorem in the above ontologically weighty way?

Few, if any.

Practically all mathematicians would regard the theorem as an assertion **not** about what exists in the real world, but rather about what holds *in any group structure*:

It tells us that in any group there is one and only one right identity.

The above structural way of understanding theorems of group theory is a big part of a central idea of my book *A Structural Account of Mathematics.*

The rough idea is this: whenever one proves a mathematical theorem, the sentence proved has not been proved to be true in the actual world, but rather to hold in structures of a certain sort.

According to my *Structural Account of Mathematics*, every mathematical theorem has what I call a "structural content", which may or may not coincide with the literal meaning of theorem itself.

To see what "structural content" is, let us examine an example in which a mathematical theorem ϕ is proved (derived) in **PA**. It follows that:

Any model of **PA** would have to be a model of ϕ.

The above displayed sentence gives what I call "the structural content of ϕ".

What the proof of ϕ proves is that the structural content of ϕ is true—it does not prove that ϕ is true.

Why do I say that the derivation of ϕ does not prove that ϕ is true?

Clearly, for the derivation of ϕ to constitute a proof that ϕ is true, the axioms of PA must also be interpreted to be ontologically weighty propositions and we must have good grounds for believing that these axioms, so interpreted, are true.

But who developed these good grounds?

And what are these good grounds?

The nominalist can reasonably question that there are such good grour

The point being made here can be seen more clearly by considering a derivation in ZF.

In some versions of ZF, one can derive as a theorem 'There is an empty set'.

On the assumption that all the sentences of ZF are to be interpreted as ontologically weighty assertions, should we conclude:

No. For what reason do we have for thinking that the axioms of ZF, taken to be ontologically weighty, are themselves true? Has anyone proved the axioms to be true? Are they self-evident truths about what exists in the actual world?

No: a derivation of a sentence from the axioms does not constitute a proof that the sentence is true

Up to now, I have reasoned under the assumption that [1] is an ontologically weighty assertion. Let us now consider the possibility that [1] is not ontologically weighty. In other words, consider the THIRD OPTION to see if it might be defended as acceptable to the nominalist?

In his influential logic text book *Mathematical Logic*, Joseph Shoenfield sets out to describe *"the methods of the mathematician"*, writing that a "conspicuous feature of mathematics, as opposed to other sciences, is the use of proofs".

"However, since it is impossible to prove all mathematical laws, there are
in mathematics certain first laws (called "axioms") which mathematicians accept without proof."
He asks: "For what reason do we accept the axioms?"

His answer is that we "select as axioms certain laws which we feel are evident from *the nature of the concepts involved*".

Underlying the axioms, there are "basic concepts" which are not defined, while the remaining concepts that the mathematicians use are defined in terms of the basic concepts. The basic concepts are supposed to be "so simple and clear that we can understand them without a precise definition".

Shoenfield then gives the following account of what an axiom system is:

The mathematician presents us with certain basic concepts and certain axioms about these concepts. He then explains these concepts to us until we understand them sufficiently well to see that the axioms are truths about the concept. He then proceeds to define derived concepts and to prove theorems about both basic and derived concepts. The entire edifice which he constructs, consisting of basic concepts, derived concepts, axioms, and theorems, is called an *axiom system*.

Shoenfield then describes a specific axiom system ZF. He describes the basic concept of set by describing what is sometimes called the "iterative concept of set": he does this by describing in some detail the way in which sets are formed in successive stages to form a hierarchy of sets. In effect, he describes the well-known type of structure sometimes called the cumulative hierarchy that characterizes Zermelo-Fraenkel set theory.
This characterization is supposed to convince the reader that the basic concept of set is "so simple and clear that we can understand [it] without a precise definition".

The next step is to convince the reader that the axioms of ZF are evident from the nature of the basic concept of set we had been given. What follows are the usual "proofs" of the axioms

What is striking is the way Shoenfield describes what his proofs show. For example, he tells us that one proof he gives enables us to "see that the regularity axiom is true".

What is clear from such examples is that, in asserting the truth of a theorem he is proving, Shoenfield is not taking the theorem to be an ontologically weighty assertion but is only regarding the theorem to be true of some structure (such as a group or a cumulative hierarchy) which is determined by some basic concept).

Furthermore, it is clear that Shoenfield regards any theorem of a system that a mathematician may construct to be true in the way that the theorems of ZF are true—that is, true to the basic concepts of the system or true in the structure that is conveyed by the basic concept.

And he is not alone in this regard. Thus, Hirst and Rhodes write:

> It is not the aim of mathematics to declare what is true and what is false in the world, but rather to produce conceptual models [structures] of certain aspects of the world and to study within the models [structures] which statements are logical consequences of other statements.

This discussion makes it clear that there are at least two distinct ways of interpreting mathematical theorems:

There is the traditional ontologically weighty way that Burgess favors;
and there is the conceptual/structural way that Shoenfield advocates.
Burgess seems to think that any interpretation that deviates significantly from the ontologically weighty one he favors must undertake the burden of producing evidence supporting the interpretation—evidence of a linguistic nature.

But I see the situation quite differently: this is not a question of interpreting words in sentences, but rather one of interpreting a practice—a mathematical practice.

In "Platonism and Aristotelianism in Mathematics" , Richard Pettigrew argues that "the platonist interpretation [the one Burgess espouses] is not the default interpretation that we should favour and only abandon if it leads to serious problems" (p. 2).

To support this view, he goes on to show how the language of arithmetic can be understood in a way that treats the constant terms of arithmetic, such as '0', '1', '+' and 'x', not as names, but rather as variables or parameters.

As this interpretation fits nicely with Shoenfield's understanding of mathematics, we have significant support for Shoenfield's view and against the idea that Burgess's ontological weighty interpretation should be taken to be the obviously correct one.

There is another reason for preferring Shoenfield's interpretation of mathematical theorems to Burgess's.
Consider Euclid's proof of [1]:

Suppose that there are only finitely many primes.

Then the product of all the primes, say N, exists.
Then the number N + 1 also exists.
N + 1 is not prime, since it is larger than all primes.

Then, there is a prime number P which divides N + 1.

Since (by hypothesis) N is a product of all the primes,
P must be a divisor of N and hence must also be a prime divisor of 1.

But that is impossible, since P must be greater than 1.

So we can conclude that there are infinitely many primes.

If the proof is supposed to justify the belief in numbers, then the existence of numbers cannot be assumed or presupposed at any point in the above proof.

However, it is clear that the above proof does presuppose that numbers exist.

It is assumed that there are numbers right from the start.

In general, if one assumes that all mathematical theorems are ontologically weighty, the standard proofs of theorems can be seen to be not valid.

This gives us additional reasons for thinking that Shoenfield's interpretation, and not Burgess's ontologically weighty interpretation, is correct.

There is a third reason for thinking Shoenfield's interpretation is correct: When set theorists justify the axioms ZF, what they all do is to justify asserting that the axioms are true in the intended iterative structure. That is just what one would expect if Shoenfield's understanding of mathematical theorems was correct.

In summary, we can conclude:

1. Burgess's answer to Benacerraf's problem fails.
2. Burgess's dilemma argument for realism does not present insuperable difficulties for the nominalist.

   a] The second option can be maintained, even if one accepted the assumption that all mathematical theorems are ontologically weighty

   b] The third option can also be maintained, since there are several significant grounds for accepting the Shoenfield's interpretation.