

# Equilibrium among Coalitions

Paul Weirich  
University of Missouri

# Majority Rule

- Suppose that three people  $A$ ,  $B$ , and  $C$  may divide \$6 by majority rule.
- The coalition  $\{A, B\}$  may achieve the allocation  $(3, 3, 0)$ , listing dollars allocated to people in alphabetical order.
- To prevent that allocation,  $C$  may propose to  $B$  forming the coalition  $\{B, C\}$  and achieving the allocation  $(0, 4, 2)$ .

# Endless Incentives

- A core allocation distributes money to players so that each possible coalition receives at least as much as it can obtain by forming and acting alone.
- In the majority-rule game, every allocation leaves some coalition an opportunity to do better. The game lacks a core allocation; its core is empty.

# Coalitional Games

- In a coalitional game, players may form coalitions of individuals who act jointly.
- Coalitional games show that rationality does not require pursuit of all incentives. Agents in games with empty cores cannot pursue all incentives, but rationality is still attainable.

# Equilibrium

- No coalition has an incentive to act unilaterally to change a core allocation.
- Realizing a core allocation is a type of equilibrium among coalitions. But it is sometimes unattainable.
- This paper introduces a more general type of equilibrium that is always attainable. It draws on *Equilibrium and Rationality* (1998) and *Collective Rationality* (forthcoming).

# Outline

- A coalition's incentives
- A coalition's pursuit of incentives
- Strategic equilibrium defined in terms of coalitions' pursuit of incentives.

# Elementary Coalitional Games

- This paper treats elementary coalitional games with a single stage.
- In them, utility is transferable, the value function is strictly monotonic, and a coalition's value is independent of nonmembers' acts.
- The paper focuses on ideal games in which agents are rational and informed about their game and each other.

# A Coalition's Options

- In a coalitional game, a strategy profile specifies for every coalition (including unit-coalitions) whether the coalition forms and its strategy if it forms.
- A coalition's contributions to these strategy profiles are its options. For example  $\{A, B\}$  may form and achieve  $(3, 3, 0)$ .



# Feasible Strategy Profiles

- Selecting one strategy for each coalition may not yield compossible strategies.
- If one coalition structure forms, others do not form. They can form, nonetheless, until the game ends without their formation.

# A Coalition's Incentives

- A multimember coalition has an incentive to form when it has a joint strategy that benefits each member.
- It has an incentive not to form when some member does better on her own or in another multimember coalition under some joint strategy of that coalition.

# A Path of Incentives

- The nodes of a path of incentives are strategy profiles.
- A step in a path from one profile to another represents a coalition's incentive to switch from the first profile to the second.

# Paths in the Majority-Rule Game

- A multicoalition path:  $(2, 2, 2) \xrightarrow{\{A, B\}} (3, 3, 0) \xrightarrow{\{A, C\}} (4, 0, 2) \xrightarrow{\{B, C\}} (0, 2, 4) \xrightarrow{\{A, B\}} (2, 4, 0)$
- A path of incentives for multiple coalitions implies a path of incentives for its initial coalition.
- The derived path for  $\{A, B\}$ :  $(2, 2, 2) \xrightarrow{\{A, B\}} (3, 3, 0) [(0, 2, 4)] \xrightarrow{\{A, B\}} (2, 4, 0)$ . In brackets is the response to  $\{A, B\}$ 's pursuing its initial incentive.

# Paths of Pursued Incentives

- A path of pursued incentives represents coalitions' dispositions to pursue incentives.
- Paths of pursued incentives stop.
- In one realization of the majority-rule game, coalitions pursue incentives in the previous slide's paths.

# Selection and Stopping Rules

- Rationality imposes constraints on agents' pursuit of incentives.
- A coalition that pursues an incentive to switch strategy pursues an optimal incentive.
- A coalition pursues all sufficient incentives.

# Insufficient Incentives

- If a path of incentives is endless, then incentives in the path are insufficient.
- The incentives in the following path are insufficient:  $(2, 4, 0) \xrightarrow{\{A, C\}} (4, 0, 2) \xrightarrow{\{B, C\}} (0, 2, 4) \xrightarrow{\{A, B\}} (2, 4, 0) \dots$

# Dynamics

- Paths of pursued incentives constitute a dynamics for deliberations about coalition formation and joint strategies.
- The end of a path of pursued incentives is an equilibrium of that dynamics.
- Deliberations settle at the end of a path as a ball rolling in a basin settles at a low spot in the basin.



# Strategic Equilibrium

- A strategic equilibrium is a feasible strategy profile such that no coalition has a path of pursued incentives away from the profile.
- In a strategic equilibrium, coalitions' strategies are jointly self-supporting in the sense that given the profile no coalition has a sufficient incentive to change strategy.

# Generalization and Existence

- Every core allocation is a strategic equilibrium.
- A profile that is not a strategic equilibrium starts a path ending in a strategic equilibrium. Therefore every coalitional game has a strategic equilibrium.

# Example with a Nonempty Core

- Take a coalitional game with this value function:  $v(A) = v(B) = v(C) = 1$ ,  $v(AB) = v(BC) = v(AC) = 4$ ,  $v(ABC) = 12$ .
- Coalitions rationally pursuing incentives continue until the coalition  $\{A, B, C\}$  forms and divides its value.
- Only profiles including formation of  $\{A, B, C\}$  are strategic equilibria.

# The Majority-Rule Game

- The strategic equilibria of a concrete realization of three-person majority-rule division of \$6 depend on the agents' pursuit of incentives.
- If  $A$  does not pursue incentives and others do, then one strategic equilibrium yields  $(0, 3, 3)$ , and no strategic equilibrium yields  $(2, 2, 2)$ .

# An Underlying Sequential Game

- In a single-stage coalitional game, a coalitional structure and joint strategies of coalitions that form arise in the same stage. All coalitions realize their strategies in the same stage.
- In the underlying sequential game, individuals propose and accept coalition formation and joint strategies in a sequence of steps.

# Coalitions and Individuals

- If no coalition has a sufficient incentive to deviate from a profile, then no individual has a sufficient incentive to deviate either unilaterally or jointly with others.
- Moreover, if some coalition has a sufficient incentive to deviate from a profile, then some individual does.

# Compatible Equilibria

- In a coalitional game, a strategy profile has jointly self-supporting strategies for coalitions if and only if its realization in the underlying sequential game has jointly self-supporting strategies for individuals.
- Hence a strategy profile in a coalitional game is a strategic equilibrium if and only if in the underlying sequential game the strategy profile realizing it is a strategic equilibrium.

# Identifying Equilibria

In a coalitional game, identifying strategic equilibria for coalitions is a shortcut method of identifying strategic equilibria for individuals in the underlying sequential game.



# Solutions

- A solution is a profile of strategies that are jointly rational.
- A solution's realization entails the joint rationality of all agents, and so the rationality of all agents.
- The rationality of all agent entails realization of a strategic equilibrium.
- Hence, a solution's realization entails a strategic equilibrium's realization.

# Illustration

Take majority-rule division of \$6 among  $A$ ,  $B$ , and  $C$ . Suppose that in the game's realization, the coalition  $\{B, C\}$  pursues its incentive from  $(2, 2, 2)$  to  $(0, 4, 2)$ , and its agreement on the division  $(0, 4, 2)$  yields the outcome of the underlying sequential game and the coalitional game. The coalition  $\{A, C\}$ 's halting pursuit of incentives is rational for it and for its members.

# Illustration Concluded

- Realization of  $(0, 4, 2)$  is a solution and a strategic equilibrium of the coalitional game.
- In the underlying sequential game the strategy profile that realizes  $(0, 4, 2)$  is a solution and a strategic equilibrium.

# Conclusion

- Strategic equilibrium is an attainable generalization of realization of a core allocation.
- Rational individuals realize a strategic equilibrium in a coalitional game.