Equilibrium among Coalitions

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Majority Rule

• Suppose that three people $A$, $B$, and $C$ may divide $6$ by majority rule.

• The coalition $\{A, B\}$ may achieve the allocation $(3, 3, 0)$, listing dollars allocated to people in alphabetical order.

• To prevent that allocation, $C$ may propose to $B$ forming the coalition $\{B, C\}$ and achieving the allocation $(0, 4, 2)$. 
Endless Incentives

• A core allocation distributes money to players so that each possible coalition receives at least as much as it can obtain by forming and acting alone.

• In the majority-rule game, every allocation leaves some coalition an opportunity to do better. The game lacks a core allocation; its core is empty.
Coalitional Games

• In a coalitional game, players may form coalitions of individuals who act jointly.

• Coalitional games show that rationality does not require pursuit of all incentives. Agents in games with empty cores cannot pursue all incentives, but rationality is still attainable.
Equilibrium

• No coalition has an incentive to act unilaterally to change a core allocation.

• Realizing a core allocation is a type of equilibrium among coalitions. But it is sometimes unattainable.

• This paper introduces a more general type of equilibrium that is always attainable. It draws on *Equilibrium and Rationality* (1998) and *Collective Rationality* (forthcoming).
Outline

• A coalition’s incentives
• A coalition’s pursuit of incentives
• Strategic equilibrium defined in terms of coalitions’ pursuit of incentives.
Elementary Coalitional Games

• This paper treats elementary coalitional games with a single stage.
• In them, utility is transferable, the value function is strictly monotonic, and a coalition’s value is independent of nonmembers’ acts.
• The paper focuses on ideal games in which agents are rational and informed about their game and each other.
A Coalition’s Options

• In a coalitional game, a strategy profile specifies for every coalition (including unit-coalitions) whether the coalition forms and its strategy if it forms.

• A coalition’s contributions to these strategy profiles are its options. For example \{A, B\} may form and achieve \(3, 3, 0\).
Feasible Strategy Profiles

• Selecting one strategy for each coalition may not yield composable strategies.

• If one coalition structure forms, others do not form. They can form, nonetheless, until the game ends without their formation.
A Coalition’s Incentives

• A multimember coalition has an incentive to form when it has a joint strategy that benefits each member.

• It has an incentive not to form when some member does better on her own or in another multimember coalition under some joint strategy of that coalition.
A Path of Incentives

• The nodes of a path of incentives are strategy profiles.

• A step in a path from one profile to another represents a coalition’s incentive to switch from the first profile to the second.
Paths in the Majority-Rule Game

• A multicoalition path: (2, 2, 2) $\rightarrow_{\{A, B\}} (3, 3, 0)$ $\rightarrow_{\{A, C\}} (4, 0, 2) $ $\rightarrow_{\{B, C\}} (0, 2, 4) $ $\rightarrow_{\{A, B\}} (2, 4, 0)$

• A path of incentives for multiple coalitions implies a path of incentives for its initial coalition.

• The derived path for \{A, B\}: (2, 2, 2) $\rightarrow_{\{A, B\}} (3, 3, 0)$ [((0, 2, 4))] $\rightarrow_{\{A, B\}} (2, 4, 0)$. In brackets is the response to \{A, B\}’s pursuing its initial incentive.
Paths of Pursued Incentives

• A path of pursued incentives represents coalitions’ dispositions to pursue incentives.
• Paths of pursued incentives stop.
• In one realization of the majority-rule game, coalitions pursue incentives in the previous slide’s paths.
Selection and Stopping Rules

• Rationality imposes constraints on agents’ pursuit of incentives.
• A coalition that pursues an incentive to switch strategy pursues an optimal incentive.
• A coalition pursues all sufficient incentives.
Insufficient Incentives

• If a path of incentives is endless, then incentives in the path are insufficient.

• The incentives in the following path are insufficient: \((2, 4, 0) \rightarrow \{A, C\} \rightarrow (4, 0, 2) \rightarrow \{B, C\} \rightarrow (0, 2, 4) \rightarrow \{A, B\} \rightarrow (2, 4, 0) \ldots \)
Dynamics

• Paths of pursued incentives constitute a dynamics for deliberations about coalition formation and joint strategies.
• The end of a path of pursued incentives is an equilibrium of that dynamics.
• Deliberations settle at the end of a path as a ball rolling in a basin settles at a low spot in the basin.
Strategic Equilibrium

- A strategic equilibrium is a feasible strategy profile such that no coalition has a path of pursued incentives away from the profile.
- In a strategic equilibrium, coalitions’ strategies are jointly self-supporting in the sense that given the profile no coalition has a sufficient incentive to change strategy.
Generalization and Existence

• Every core allocation is a strategic equilibrium.
• A profile that is not a strategic equilibrium starts a path ending in a strategic equilibrium. Therefore every coalitional game has a strategic equilibrium.
Example with a Nonempty Core

• Take a coalitional game with this value function: $v(A) = v(B) = v(C) = 1$, $v(AB) = v(BC) = v(AC) = 4$, $v(ABC) = 12$.

• Coalitions rationally pursuing incentives continue until the coalition $\{A, B, C\}$ forms and divides its value.

• Only profiles including formation of $\{A, B, C\}$ are strategic equilibria.
The Majority-Rule Game

• The strategic equilibria of a concrete realization of three-person majority-rule division of $6 depend on the agents' pursuit of incentives.

• If A does not pursue incentives and others do, then one strategic equilibrium yields (0, 3, 3), and no strategic equilibrium yields (2, 2, 2).
An Underlying Sequential Game

• In a single-stage coalitional game, a coalitional structure and joint strategies of coalitions that form arise in the same stage. All coalitions realize their strategies in the same stage.

• In the underlying sequential game, individuals propose and accept coalition formation and joint strategies in a sequence of steps.
Coalitions and Individuals

• If no coalition has a sufficient incentive to deviate from a profile, then no individual has a sufficient incentive to deviate either unilaterally or jointly with others.

• Moreover, if some coalition has a sufficient incentive to deviate from a profile, then some individual does.
Compatible Equilibria

• In a coalitional game, a strategy profile has jointly self-supporting strategies for coalitions if and only if its realization in the underlying sequential game has jointly self-supporting strategies for individuals.

• Hence a strategy profile in a coalitional game is a strategic equilibrium if and only if in the underlying sequential game the strategy profile realizing it is a strategic equilibrium.
Identifying Equilibria

In a coalitional game, identifying strategic equilibria for coalitions is a shortcut method of identifying strategic equilibria for individuals in the underlying sequential game.
Solutions

• A solution is a profile of strategies that are jointly rational.
• A solution's realization entails the joint rationality of all agents, and so the rationality of all agents.
• The rationality of all agent entails realization of a strategic equilibrium.
• Hence, a solution's realization entails a strategic equilibrium's realization.
Illustration

Take majority-rule division of $6 among $A$, $B$, and $C$. Suppose that in the game’s realization, the coalition $\{B, C\}$ pursues its incentive from $(2, 2, 2)$ to $(0, 4, 2)$, and its agreement on the division $(0, 4, 2)$ yields the outcome of the underlying sequential game and the coalitional game. The coalition $\{A, C\}$’s halting pursuit of incentives is rational for it and for its members.
Illustration Concluded

• Realization of (0, 4, 2) is a solution and a strategic equilibrium of the coalitional game.

• In the underlying sequential game the strategy profile that realizes (0, 4, 2) is a solution and a strategic equilibrium.
Conclusion

• Strategic equilibrium is an attainable generalization of realization of a core allocation.

• Rational individuals realize a strategic equilibrium in a coalitional game.