Conglomerability and Disintegrability for Unbounded Random Variables

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This presentation engages two challenges for an *Expected Utility* theory of *coherent* preferences over *random quantities* when:

1. **Utilities for (outcomes of) random variables are *unbounded*.**

2. ***Coherence* (that is, avoidance of uniform dominance in the partition by *states*) is the liberal standard for rational preference afforded by de Finetti's theory.**

That standard allows merely finitely additive probability; so, conditional probability is *not* conglomerable in every partition.

***Probability conglomerable in a partition*: An unconditional probability lies inside the closed interval of conditional probabilities in a partition: $\pi = \{h_1, h_2, \ldots\}$.**

$$inf_{h\in\pi}\, P(E \mid h) \;\leq\; P(E) \leq\; sup_{h\in\pi}\, P(E \mid h)$$

- **One central goal in our paper is to develop a theory of finitely additive *expectations* that accommodates these two challenges.**

***Aside*: Savage's theory accommodates the second challenge, but not the first.**

- **As a second central goal, we seek to extend Lester Dubins' (1975) work on the theory of finitely additive expectations. Hereafter, think of the outcome of a variable as its utility.**

Dubins relies on a finitely additive expectation for bounded random variables that can be written as an integral.

***X* is a bounded, real-valued variable defined on a set of states $X: \Omega \rightarrow \Re$**

$$EU(X) = \int_{\Omega} X(\omega) d\mathrm{P}(\omega)$$

Consider:

(i) **Probability is an expectation for events (treated as indicator functions)**

$$\mathrm{P}(F) = EU(F) = \int_{\Omega} F(\omega) d\mathrm{P}(\omega).$$

(ii) **Conditional probabilities are random quantities, $\{\mathrm{P}(E \mid h): h \in \pi\}$.**

One of Dubins' main (1975) results is that with respect to the class χ of all bounded variables, a finitely additive expectation is conglomerable over χ in a partition π $\quad \forall X \in \chi \;\; inf_{h\in\pi}\, \mathrm{EU}(X \mid h) \leq EU(X) \leq sup_{h\in\pi}\, \mathrm{P}(X \mid h)$

just in case each expectation is an integral of its conditional expectations in π.

$$\forall X \in \chi \quad EU(X) = \int_{h\in\pi} EU(X \mid h)\, d\mathrm{P}(h).$$

- **In this paper we develop an account of finitely additive expectations for unbounded variables that extends this result.**

Challenge 1 (SSK 2009) – With unbounded utilities, *coherent* preferences,

i.e. preferences that respect simple dominance, state-by-state,

do _not_ also respect indifferences between *equivalent variables*.

***Definition*: Two variables are *equivalent* if they have the same Probability distribution over outcomes.**

***Example:* Consider a fair coin toss with P(*H*) = P(*T*) = ½**
Let *X* be the variable *X*(*H*) = 1 and *X*(*T*) = 0
Let *Y* be the variable *Y*(*H*) = 0 and *Y*(*T*) = 1.
***X* and *Y* are equivalent as P(*X*=1) = P(*Y*=1) = ½, etc.**

- **In canonical *EU*-theories utility is over the outcomes of variables: the decision maker is *indifferent* between equivalent variables.**

  See: von Neumann-Morgenstern (1947); Savage (1954); Anscombe-Aumann (1963).

- **In these theories, preference is defined over *lotteries* (aka *gambles*), which are the equivalence classes of equivalent variables.**

Two Heuristic Examples illustrating *Challenge #*1

Each of the following two examples provides a collection of unbounded but equivalent variables that cannot all be indifferent to each other.

Common structure for both heuristic examples

- **Let events E_n (n = 1, …) form a partition $\pi_E = \{E_n\}$ with a Geometric(½) probability distribution: P(E_n) = 2^{-n} (n = 1, 2, …).**

  Flip a fair coin until the first head. E_n = first head on flip #n.

- **Let $\pi_A = \{A_{\mathrm{H}}, A_{\mathrm{T}}\}$ be the outcome of another fair-coin flip, independent of the events E_n. P($A_{\mathrm{H}}|E_n$) = P(A_H) = ½.**

- **Consider the countable state-space $\pi_E \times \pi_A$.**

Heuristic Example 1: St. Petersburg variables

Define three (equivalent) St. Petersburg random variables, *X*, *Y*, and *Z*, as follows.

	E_1	E_2	....	E_n	....
	$Z = 2$	$Z = 4$		$Z = 2^n$	
A_H	$X = 4$	$X = 8$		$X = 2^{n+1}$	
	$Y = 2$	$Y = 2$		$Y = 2$	
	$Z = 2$	$Z = 4$		$Z = 2^n$	
A_T	$X = 2$	$X = 2$		$X = 2$	
	$Y = 4$	$Y = 8$		$Y = 2^{n+1}$	

For each state in $\pi_E \times \pi_A$,

$$X + Y - 2Z = 2, \text{ a constant quantity.}$$

This situation contradicts indifference between all 3 pairs of these equivalent variables. Such indifference requires that the expected utility [*EU*(·)] for the difference between two equivalent variables is 0. In this example, that entails,

$$EU(X - Z) + EU(Y - Z) = EU(X + Y - 2Z) = 0.$$

But the utility of a constant is that constant.

So, $EU(X + Y - 2Z) = 2$ **a contradiction.**

Thus, coherent preferences, here, are <u>not</u> defined merely by the probability distribution of utility outcomes.

Aside: ***Heuristic Example*** **1 uses non-Archimedean preference.**
The St. Petersburg variables do not have finite utility.
Heuristic Example 2 **uses Archimedean preferences.**

***Heuristic Example* 2 – Coherent *boost* for unbounded variables.**

As before, consider the countable state-space $\pi_E \times \pi_A$,
with the Geometric(½) probability distribution on π_E,
and with an independent "fair coin" distribution on π_A.

Define the three equivalent (Geometric) random variables *X*, *Y*, and *Z*.

	E_1	E_2	....	E_n	....
	$X = 1$	$X = 2$		$X = n$	
A_H	$Y = 2$	$Y = 3$		$Y = n+1$	
	$Z = 1$	$Z = 1$		$Z = 1$	
	$X = 1$	$X = 2$		$X = n$	
A_T	$Y = 1$	$Y = 1$		$Y = 1$	
	$Z = 2$	$Z = 3$		$Z = n+1$	

- ***X*, *Y*, and *Z* are equivalent Geometric(½) variables.**

But for each state in $\pi_E \times \pi_A$, $\quad Y + Z - X = 2.$

Thus for equivalent variables to have equal Expected Utility

$$EU(Y - X) + EU(Z - X) = 0 \qquad \textit{if and only if}$$

$$EU(Y) = EU(Z) = EU(X) = 2.$$

Then Expected Utility for a Geometric(½) variable *X* is its *countably additive* expectation, 2, and Expected Utility is continuous from below.

Specifically, **if a sequence of variables $<X_n> \rightarrow X$ (pointwise convergence)**
and for each state ω, $X_n(\omega) \leq X(\omega)$,
then $lim_{n\to\infty}\, EU(X_n) = EU(X)$.

That is, in order to have indifference over equivalent Geometric(½) random variables, preferences must be continuous from below.

However, de Finetti's theory of *coherence* requires only that preference respects (uniform) dominance in the partition by *states*. This entails respecting *bounds* from sequences of bounded random variables without requiring continuity from below.

Consider, the unbounded Geometric(½) variable X from the example, where

$$X(\{A_T, E_n\}) = X(\{A_T, E_n\}) = n; \text{ with } P(E_n) = 2^{-n}.$$

Let X_n be the bounded, truncated variable:

$$X_n(\{A_T, E_m\}) = X(\{A_T, E_m\}) = m \text{ for } m \leq n$$

and $$X_n(\{A_H, E_m\}) = X(\{A_H, E_m\}) = 0 \text{ for } m > n.$$

So, for each n = 1, 2, …, and for each state ω,

$$X_n(\omega) \leq X(\omega).$$

Also, $$\langle X_n \rangle \rightarrow X.$$

Respect for (uniform) simple dominance in the partition by states entails merely that $$\lim_{n\to\infty} EU(X_n) \leq EU(X).$$

Thus, if we start with the class of bounded variables and extend to included X, Y and Z, there is no sure-loss that results from the values $EU(X) = 10$, $EU(Y) = 4$, and $EU(Z) = 8$; when, X has *boost* 8, Y has *boost* 2, and Z has *boost* 6.

Unless preferences are continuous from below (entailing probability is countably additive) *Utility for unbounded variables will not be a function merely of the probability distribution of outcomes*!

***Aside*: The notion of *state* carries no metaphysical significance here. *States* are the elements of a state-space partition used to fix the values of variables.**

***Related Definitions*:**

Let <Ω, *B*, P> be a (finitely additive) measure space, where

$\Omega = \{\omega_1, \omega_2, \ldots.\}$ a set of *states* – a countable Ω is enough for our needs here.

***B* is a σ-field of sets – used for the domain of the probability P and for measurability conditions on random variables.**

***B* may be the powerset of Ω when state space is countable.**

P is a (finitely additive) probability with domain *B*.

Each variable *X* is real-valued, $X: \Omega \rightarrow \Re$, a *B*-measurable function.

***Challenge #2*: Non-conglomerable conditional probabilities.**

Recall when probability is conglomerable in a partition π.

***Conglomerability in a partition*: Probability is *conglomerable in a partition* $\pi = \{h_1, h_2, \ldots\}$ provided that, for each event E in the algebra, the unconditional probability $P(E)$ lies inside the closed interval of conditional probabilities $\{P(E \mid h)\}$.**

$$inf_{h\in\pi}\, P(E \mid h) \;\leq\; P(E) \leq\; sup_{h\in\pi}\, P(E \mid h)$$

- ***Theorem* (*SSK*, 1984): Each finitely but not countably additive probability fails to be conglomerable in some countable partition.**

***Example* (Dubins, 1975):**

Let <Ω, ℬ, P> be a finitely additive measure space with

A countable $\Omega = \pi_E \times \pi_N$, where $\pi_E = \{E_C, E_F\}$ and $\pi_N = \{1, 2, \ldots\}$.
ℬ is the powerset of Ω.
$P(E_C) = P(E_F) = ½$.
$P(N \mid E_C)$ is Geometric(½)
$P(N \mid E_F)$ is *purely finitely additive* – pick a “random” number.

	N=1	N=2	....	N=m	....
E_C	$1/2^2$	$1/2^3$	....	$1/2^{(m+1)}$	....
EF	0	0	....	0	....

Table of unconditional probabilities for states in Dubins’ example.

$P(N=m) = 2^{-(m+1)} > 0$. So conditional probability given N is determined.
$P(E_C) = ½ < 1 = P(E_C \mid N=m)$.
and P fails to be conglomerable in the partition π_N.

In the light of non-conglomerable probability, the probability of an event is not always an "average" of its conditional probabilities.

When probability is not conglomerable for event E in partition π, then P is not *disintegrable* in π either: $\quad P(E) \neq \int_{h\in\pi} P(E|h)\, dP(h)$.

But Probability is merely the special case of Expected Utility restricted to indicator functions: $P(E) = EU(E(\omega))$

So, the concepts of *conglomerability* and *disintegration* apply also to Expected Utility and Conditional Expected Utility.

***Aside*: de Finetti calls these values *Previsions*, not Expected Utilities.**

<u>SO</u>, for random variables in a class $\chi = \{X\}$

an Expected Utility function is *disintegrable* over χ in partition π if

$$\forall X\in\chi \quad EU(X) = \int_{h\in\pi} EU(X|h)\, dP(h).$$

and it is *conglomerable* over χ in π

if $\quad \forall X\in\chi \quad inf_{h\in\pi}\, EU(X \mid h) \leq EU(X) \leq sup_{h\in\pi}\, EU(X \mid h).$

Dubins (1975) established that if an expected utility function *EU*(·) is defined for a class χ of bounded variables that includes all linear combinations of these variables then:

$EU(\cdot)$ is conglomerable in partition π for each $X \in \chi$

iff **$EU(\cdot)$ is disintegrable in π for each $X \in \chi$.**

The two challenges stand in the way of our central goals:

Goal 1) Define f.a. expectations for unbounded variables so that expectations have an integral representation.

Goal 2) Have this integral extend Dubins' result that, for a sufficiently rich class of unbounded variables *conglomerability* and *disintegrability* are coextensive.

- **Challenge 1: Finitely additive expectations for unbounded variables are not a function of distributions over outcomes.**
- **Challenge 2: Of course, a theory of finitely additive conditional expectations for unbounded variables will display both non-conglomerable and non-disintegrable conditional expectations, because it does for probability and conditional probability, i.e. because it does so already for bounded variables.**

Summary of our progress

Goal 1) We have adapted an existing theory of integrals – the *Daniell* integral (see Royden, 1968) – so that it matches de Finetti's coherence criterion for a class of functions forming a linear space and including all constants.

This class includes the unbounded variables from the 2nd heuristic example.

Thus, we are able to incorporate finite *boost* into our integral theory of expectations. The finitely additive *Daniell* integral is not required to be a function of the distribution of outcomes.

***Aside*: The first heuristic example with St. Petersburg variables involves infinite expectations. Since $(\infty - \infty)$ is not well defined, those variables are not included in our analysis.**

- **There is work yet to be done on an integral representation for non-Archimedean, finitely additive expected utility!**

Goal 2) Under the following finiteness conditions on unbounded variables, we extend Dubins' result that conglomerable and disintegrable expectations are coextensive, and show somewhat more.

Conditions on (unbounded) random variables:

- **The variables are real-valued – no St. Petersburg variables.**
- **The variables have finite absolute expectations:** $EU(|X|) < \infty$.
- **Each conditional expectation is finite:** $EU(X \mid h) < \infty$.
- **Expectation of conditional expectation is finite:** $EU(\, EU(X|h)\,) < \infty$.

***Note*: The set of all variables that satisfy these conditions form a linear space.**

Let $EU(\cdot)$ be a (de Finetti) coherent expectation, and π be a partition.

Let W be a class of variables that meet the finiteness conditions.

- ***Definitions***

Say that W is of *Class-0* relative to an $EU(\cdot)$ and a partition π if
$EU(\cdot)$ is not conglomerable (hence, also not disintegrable) in π over W.

***Aside*: Let $W \subseteq Z$. Non-conglomerability is inherited by the larger class Z.**
So, if W is of *Class-0* and then Z also is of *Class 0*.

Say that W is of *Class-1* relative to an $EU(\cdot)$ and a partition π if
$EU(\cdot)$ is conglomerable but not disintegrable in π over W.

Say that W is of *Class-2* relative to an $EU(\cdot)$ and a partition π if
$EU(\cdot)$ is both conglomerable and disintegrable in π over W.

- **Dubins' (1975) result, applied to the class of all bounded random variables is that, either it is of Class-0 or of Class 2 relative to an $EU(\cdot)$ and a π.**
- **We show the same for classes of unbounded random variables that satisfy the finiteness conditions mentioned before, and which form a linear space.**
- **However, also we display a partition π, and a subclass (not a linear space) that includes all the bounded random variables that is of Class-1.**

References

Anscombe, F.J. and Aumann, R. (1963) A definition of subjective probability. *Annals of Math. Stat*. 34: 199-205.

de Finetti, B. (1974) *Theory of Probability* (2 volumes). New York, John Wiley.

Dubins, L. (1975) Finitely additive conditional probabilities, conglomerability, and disintegrations. *Ann. Prob*. 3: 89-99.

Royden, H. L. (1968) *Real Analysis*. London, Macmillan.

Savage, L.J. (1954) *Foundations of Statistics*. New York, John Wiley.

Scherivsh, M.J., Seidenfeld, T. and Kadane, J.B. (1984) The Extent of non-conglomerability of finitely additive probabilities. *Z. f. Wahrscheinlichkeitstheorie* 66: 205-226.

Seidenfeld, T., Schervish, M.J., and Kadane, J.B. (2009) Preference for equivalent random variables: a price for unbounded utilities. *J. Math. Econ.* 45: 329-340.

Von Neumann, J. and Morgenster, O. (1947) Theory of Games and Economic Behavior (2nd ed.) Princeton, Princeton University Press.