Two-Stage Choices from Conditional Choice Functions

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Outline

Choice Functions

- X is a set of *alternatives*.
- $\mathcal{X}$ is the set of all finite, nonempty subsets of X.
- $C : \mathcal{X} \to \mathcal{X}$ is a *choice function* on X just in case $C(Y) \subseteq Y$ for all $Y \in \mathcal{X}$.
- The binary relation R_C is defined as follows: xR_Cy iff $x \in C(Y)$ for some $Y \in \mathcal{X}$.

Conditional Choice Functions

- $\mathcal{X}$ (as before)
- $\mathcal{E} = \langle E, \sqsubseteq \rangle$ is a nonempty poset of *information states*, partially ordered according to strength.
- $\mathcal{C} : \mathcal{E} \times \mathcal{X} \to \mathcal{X}$ is a *conditional choice function* on X just in case the following conditions are satisfied for all $x \in X$, $Y \in \mathcal{X}$ and $e \in E$:
  - $\mathcal{C}(e, Y) \subseteq Y$
  - If $x \in \mathcal{C}(e, Y)$, then there is an $f \in E$ such that $e \sqsubseteq f$ and $x \in \mathcal{C}(g, Y)$ whenever $f \sqsubseteq g$.

Example 1

- $X = \{(x_1, x_2, x_3) \mid x_1, x_2, x_3 \in N\}$
- E is the set of all nonempty subsets of $\{(30, n, 60 - n) \mid 0 \leq n \leq 60\}$.
- $f \sqsubseteq g$ iff $g \subseteq f$.
- $(x_1, x_2, x_3) \in \mathcal{C}(e, Y)$ just in case there is a $(n_1, n_2, n_3) \in e$ such that $\sum_{i=1}^{3} n_i x_i$ is at least as great as $\sum_{i=1}^{3} n_i y_i$ for all $(y_1, y_2, y_3) \in Y$.

Example 2

- $\mathcal{X}$, $\mathcal{E}$, $\mathcal{C}$ (as in Example 1).
- $(x_1, x_2, x_3) \in \mathcal{D}(e, Y)$ iff
  - $(x_1, x_2, x_3) \in \mathcal{C}(e, Y)$,
  - $\min\{\sum_{i=1}^{3} n_i x_i \mid (n_1, n_2, n_3) \in e\} \geq$
    $\min\{\sum_{i=1}^{3} n_i y_i \mid (n_1, n_2, n_3) \in e\}$ for all $(y_1, y_2, y_3) \in \mathcal{C}(e, Y)$.

Outline

Basic Relations

- If $\mathcal{C} : \mathcal{E} \times \mathcal{X} \to \mathcal{X}$ is a conditional choice function and $e \in E$, then let $\mathcal{C}_e$ be the choice function defined by $\mathcal{C}_e(Y) = \mathcal{C}(e, Y)$ for all $Y \in \mathcal{X}$.
- If C is a choice function on $\mathcal{X}$, then let C^* be the conditional choice function defined by $C^*(e, Y) = C(Y)$ for all $e \in E$ and $Y \in \mathcal{X}$.

Extension of Properties

Every property P of choice functions may be extended to a property P^* of conditional choice functions as follows:

> **Property** P^*: *For every $e \in E$ there is an $f \in E$ such that $e \sqsubseteq f$ and $\mathcal{C}_g$ satisfies P for all $g \in E$ such that $f \sqsubseteq g$.*

Moreover, P^* generalizes P in the following sense:

Proposition

Let C be a choice function on X. Let P be a property of choice functions. C satisfies P iff C^* satisfies P^*.

Outline

Preliminaries

Let $\mathcal{C} : \mathcal{E} \times \mathcal{X} \to \mathcal{X}$ be a conditional choice function.

- For each $e \in E$, let $O_e = \{R_{\mathcal{C}_f} \mid e \sqsubseteq f\}$.
- For each $e \in E$, define a binary relation $\succ_e$ on X as follows: $x \succ_e y$ iff there is a $Y \in \mathcal{X}$ and an $f \in E$ such that
  - $e \sqsubseteq f$,
  - $x \in \mathcal{C}(e, Y)$,
  - $y \notin \mathcal{C}(e, Y)$, and
  - $y \in \mathcal{C}(f, Y)$.
- Let $\succ_e^t$ be the transitive closure of $\succ_e$.
- Define $\succsim_e^t$ by $x \succsim_e^t y$ iff not $y \succ_e^t x$.

R1

Property α^*: *For every* $e \in E$ *there is an* $f \in E$ *such that* $e \sqsubseteq f$ *and* $\mathcal{C}_g$ *satisfies* α *for all* $g \in E$ *such that* $f \sqsubseteq g$.

Property β^*: *For every* $e \in E$ *there is an* $f \in E$ *such that* $e \sqsubseteq f$ *and* $\mathcal{C}_g$ *satisfies* β *for all* $g \in E$ *such that* $f \sqsubseteq g$.

Proposition

Let $\mathcal{C}$ *be a conditional choice function that satisfies* α^* *and* β^*. *If* $x \in \mathcal{C}(e, Y)$, *then there is a weak order* $R \in O_e$ *such that* xRy *for all* $y \in Y$.

R2

Property χ: *If $x \succ_e^t y$, then there is no Y such that $x, y \in \mathcal{C}(e, Y)$.*

Proposition

Let $\mathcal{C}$ be a conditional choice function that satisfies α^*, β^*, χ, and such that $\succ_e^t$ is irreflexive for all $e \in E$. $x \in \mathcal{C}(e, Y)$ iff

- $x \in Y$,
- *there is a weak order $R \in O_e$ such that xRy for all $y \in Y$, and*
- *if $y \in Y$ and, for some weak order $R \in O_e$, yRz for all $z \in Y$, then it is not the case that $y \succ_e^t x$.*

Moreover, $\succ_e^t$ asymmetric and transitive.

R3

Proposition

Let $\mathcal{C}$ be a conditional choice function that satisfies α^*, β^*, χ, and such that $\succ_e^t$ is both irreflexive and negatively transitive for all $e \in E$. $x \in \mathcal{C}(e, Y)$ iff

- $x \in Y$,
- *there is a weak order $R \in O_e$ such that xRy for all $y \in Y$, and*
- *if $y \in Y$ and, for some weak order $R \in O_e$, yRz for all $z \in Y$, then $x \succsim_e^t y$.*

Moreover, $\succsim_e^t$ is a weak order.