THREE PROOFS AND THE KNOWER IN THE QUANTIFIED LOGIC OF PROOFS

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Abstract. The Knower Paradox demonstrates that any theory T which 1) extends Robinson arithmetic Q, 2) includes a predicate K(x) intended to formalize the formula with Godel number x is known by agent i, and 3) contains certain elementary epistemic principles involving K(x) is inconsistent. In [3] Dean and Kurokawa show that the Knower can be proved in the Quantified Logic of Proofs (QLP) formulated by Mel Fitting in [6] – extending work presented by Artemov in [1]. Dean and Kurokawa argue that QLP is more expressive than most modal logics and that this permits to identify more clearly the epistemic principles used in the proof of the Knower. They also argue that the proof seems to require the use of a suspect principle – the Uniform Barcan Formula (UBF). So, they propose to resolve the paradox by abandoning UBF.

In this note we offer three independent proofs of the Knower in QLP that do not require the use of UBF. So, it seems that the resolution of the paradox proposed by Dean and Kurokawa is not a viable option. We conclude with some observations about possible resolutions of the paradox compatible with our results.

1. Introduction

Montague and Kaplan present the paradox of the Knower at the end of their joint paper [8]. The central idea is that any formal system containing the apparatus of elementary syntax (Robinson's arithmetic Q) and including among its theorems all instances of three basic principles regulating knowledge is inconsistent. The principles establish that knowledge entails truth, that this fact is known and a closure principle for knowledge. Here is a formal presentation of the basic principles about knowledge (where $\overline{\phi}$ is the *standard name* of ϕ):

(T): $\Box(\overline{\phi}) \to \phi$
(U): $\Box(\overline{\Box(\phi) \to \phi})$
(I): $[I(\overline{\phi}, \overline{\psi}) \wedge \Box(\overline{\phi})] \to \Box(\overline{\psi})$

T and U are self-explanatory. I uses the operator I. The idea of this operator is to express in elementary syntax the fact that ϕ entails ψ. This is done by the sentence $I(\overline{\phi}, \overline{\psi})$.

The derivation of a contradiction passes through the fact that the existence of self-referential statements about the modality $\Box$ such as:

(i): $\delta \leftrightarrow \Box(\overline{\neg\delta})$
(ii): $\delta \leftrightarrow \neg\Box(\overline{\delta})$

The proof in [8] uses (i). The proof is rather direct. Here is a sketch of it:

1. $\vdash \delta \leftrightarrow \Box(\overline{\neg\delta})$ (i)
2. $\vdash \delta \to \Box(\overline{\neg\delta})$ Classical logic (1).
3. $\vdash \Box(\overline{\neg\delta}) \to \neg\delta$ An instance of axiom T. Call it E_1

4. $E_1 \vdash \delta \to \neg\delta$ — From (2), classical logic.
5. $E_1 \vdash \neg\delta$ — From (4), classical logic.
6. $I(\overline{E_1}, \overline{\neg\delta})$ — From (5) and the definition of the I-operator.
7. $\vdash [I(\overline{E_1}, \overline{\neg\delta}) \wedge \Box(\overline{E_1})] \to \Box(\overline{\neg\delta})$ — Instance of axiom I. Call it E_3.
8. $\vdash \Box(\overline{E_1})$ — Instance of axiom U. Call it E_2.
9. $E_2, E_3 \vdash \Box(\overline{\neg\delta})$ — Classical Logic
10. $\vdash \delta$ — (1), (9), (7) and (8), Classical Logic.
11. $\vdash \neg\delta$ — (5) and (3), MP.

In [7] Montague proved a different theorem that appeals to (ii) and slightly different epistemic principles. Here it is the central theorem proved in [7]:

Theorem 1.1. *Suppose that T is any theory such that:*

(t): *T is an extension of Q.*
(T): $\Box(\overline{\delta}) \to \delta$
(Int): *If $T \vdash \delta$, then $T \vdash \Box(\overline{\delta})$*

Then T is inconsistent.

Proof. The proof proceeds along similar lines than the previous proof.

1. $T \vdash \delta \leftrightarrow \neg\Box(\overline{\delta})$ — An instance of (ii).
2. $T \vdash \neg\Box(\overline{\delta}) \to \delta$ — Logic (1).
3. $T \vdash \Box(\overline{\delta}) \to \neg\delta$ — (1) Logic.
4. $T \vdash \Box(\overline{\delta}) \to \delta$ — An instance of axiom T.
5. $T \vdash \neg\Box(\overline{\delta})$ — (3), (4), classical logic.
6. $T \vdash \delta$ — (2), (5), classical logic.
7. $T \vdash \Box(\overline{\delta})$ — Int, (6).
8. $T \vdash \bot$ — (5), (7), Logic.

□

Dean and Kurokawa suggest that a faithful interpretation of the meaning of the modality in the previous proof is given by the following:

(□): $\Box(\overline{\phi}) \rightleftharpoons$ there exists a proof of ϕ in T.

They propose in addition to formalize this informal interpretation of the modality in the context of an extension to the first order case of the Logic of Proofs proposed by Artemov in [1]. This logic implements statements of the form $[t]\phi$ known as *explicit modalities* with the following intended interpretation:

(Explicit Modalities): $[t]\phi \rightleftharpoons$ t is a proof of ϕ.

Herein t is a potentially complex expression known as a proof term intended to denote an informal proof. In addition to explicit modalities, QLP also possess first-order quantifiers intended to range over informal proofs. In this system, a statement of the form $\exists x[x]\phi$ thus receives the intended interpretation "there exists a proof of ϕ," which thereby provide with a means of formalizing the right hand side of (□).

The main goal of [3] is to show that if the second aforementioned proof of the Knower is reconstructed in QLP a suspect epistemic principle is needed in order to justify step (7) in the derivation of the paradox. Our goal in this note is to show that both proofs of the Knower are reconstructible in QLP without appealing to this principle.

2. Some background about QLP

The class of QLP proof terms is specified by the following grammar:

$$t := x_i|a_i(x)|!t|t_1.t_2|t_1 + t_2|(t\forall x)$$

$x_1, x_2, \ldots$ are known as proof variables, $a_1(x), a_2(x), \ldots$ as primitive proof terms. !, ., +, (.∀.) denote proof operations respectively called proof checker (unary), application (binary), sum (binary) and uniform verifier (binary). The intuition is as follows: $t.u$ is meant to be the result of joining together the two reasons t and u; typically if t justifies $X \to Y$ and u justifies X then $t.u$ justifies Y. $t + u$ is a kind of union or choice operation; it justifies what either t or u justifies. And ! is a verification operator; $!t$ verifies the correctness of an application of t. The axioms below articulate these ideas formally.

The class of formulas and sentences of QLP is defined as follows (where P_i are propositional letters) :

$$\phi :=\perp |P_i|\phi_1 \wedge \phi_2|\phi_1 \vee \phi_2|\phi_1 \to \phi_2|\neg\phi|[t]\phi|\forall x\phi_x|\exists x\phi_x$$

Definition 2.1. A primitive proof term is a proof term of the form $f(x_1, ..., x_n)$, where f is a primitive function symbol and $x_1, ..., x_n$ are proof variables. A primitive term specification is a mapping $\mathcal{F}$, assigning to each primitive proof term p some set (possibly empty) of formulas. Think of $\mathcal{F}$ as mapping a primitive proof term to the set of formulas it potentially justifies. A formula X has a primitive proof term with respect to $\mathcal{F}$ if $X \in \mathcal{F}(p)$ for some primitive proof term p. Likewise a primitive proof term p is for a formula X if $X \in \mathcal{F}(p)$.

2.1. An Axiom System.

- **1:** A finite set of classical axiom schemas, sufficient for tautologies.
- **2:** $[t](X \to Y) \to ([s]X \to [t.s]Y)$
- **3:** $[t]X \to X$
- **4:** $[t]X \to [!t]([t]X)$
- **5:** $[s]X \to (s + t)X$ and $[t]X \to (s + t)X$

These axioms are the usual axioms of LP, taken from [1]. To these axioms Fitting adds the following axioms for quantification:

- **6:** $\forall x\phi(x) \to \phi(t)$, for any proof term t that is free for x in $\phi(x)$.
- **7:** $\forall x(\psi \to \phi(x)) \to (\psi \to \forall x\phi(x))$, where x does not occur free in ψ.
- **8:** $\exists y[y]\forall x[t]\phi \to [t\forall x]\forall x\phi$, y does not occur free in t or ϕ.

The last axiom is called a *uniformity formula* by Fitting. The justification of the axiom is as follows. Suppose that we can produce a proof in a uniform way for each instance of $\phi(x)$ and that we can verify that. From this we can conclude that we have a proof of $\forall x\phi(x)$, which we can calculate from the uniform proof of instances of ϕx, a calculation represented by $(t\forall x)$.

Axiom 8 is Fitting's weaker version of the Uniform Barcan Formula used in [3]:

UBF: $\forall x[t(x)]\phi \to [t\forall x]\forall x\phi$, y does not occur free in t or ϕ.

Dean and Kurokawa find UBF objectionable. Perhaps Axiom 8 is less transparent than other axioms or rules of inference. But it does not be a necessary ingredient in the proof of the Knower.

The rules of inference are Modus Ponens and two additional rules that correspond to necessitation and the standard universal generalization rule.

A primitive term specification $\mathcal{F}$ is axiomatically appropriate if $\mathcal{F}$ provides primitive proof terms for exactly the axioms listed above. This amounts to a requirement that primitive proof terms serve to justify the obvious, which in this case are elementary logical truths. The next rule depends on the choice of $\mathcal{F}$, which is assumed to be axiomatically appropriate.

$\mathcal{F}$ **Necessitation Rule:** If X is an axiom and $X \in \mathcal{F}$(p), one may conclude $[p]X$.

Justified Universal Generalization Rule: If $[t]\phi(x)$, infer $[t\forall x]\forall x\phi(x)$.

We have an important derived result:

Proposition 2.2. *If X is a theorem of QLP, then for some proof term p the formula $[p]X$ is also a theorem, where all free variables of p are also free variables of X.*

For a proof see [6]. We have now an axiom system that characterizes QLP. We will now consider various proofs of the Knower in this system.

3. The First Proof of the Knower

Consider the self referential axiom schema S: $\Box(\varphi \leftrightarrow \Box\neg\varphi)$. For any system of modal logic X we can call X^* its extension with the axiom schema S.

Theorem 3.1. $(KT)^*$ *is inconsistent.*

Proof. The idea is to use Kaplan and Montague's strategy of proof while simplifying the presentation and eliminating the operator I. Instead we use the axiom K. We have:

1. $\vdash \Box(\varphi \leftrightarrow \Box\neg\varphi)$	Axiom S
2. $\vdash \varphi \leftrightarrow \Box\neg\varphi$	By T in (1)
3. $\vdash \Box\neg\varphi \to \neg\varphi$	An instance of T. Call it E_1.
4. $\vdash \varphi \to \Box\neg\varphi$	From (2), classical logic.
5. $E_1 \vdash \varphi \to \neg\varphi$	From (4), classical logic.
6. $\vdash E_1 \to \neg\varphi$	From (5), classical logic.
7. $\vdash \Box(E_1 \to \neg\varphi)$	Necessitation in (6)
8. $\vdash [\Box(E_1 \to \neg\varphi) \wedge \Box(E_1)] \to \Box(\neg\varphi)$	Instance of K
9. $\vdash \Box(E_1)$	Necessitation in (3)
10. $\vdash \Box(\neg\varphi)$	(7), (8), (9) and MP.
11. $\vdash \varphi$	(2) and (10), Classical logic.
12. $\vdash \neg\varphi$	(3), (6) and MP.

□

Now consider the axiom schema S: $[p](\varphi \leftrightarrow \exists x[x]\neg\varphi)$. $(QLP)^*$ is the extension of QLP with this axiom schema. We will show that this extension is inconsistent.

Theorem 3.2. $(QLP)^*$ *is inconsistent.*

Proof. The proof proceeds in a similar manner than the modal proof.

1. $\vdash \varphi \leftrightarrow \exists x[x]\neg\varphi$	Axiom S and Axiom 3.
2. $\vdash [x]\neg\varphi \to \neg\varphi$	An instance of Axiom 3.
3. $\vdash [q_0]([x]\neg\varphi \to \neg\varphi)$	2, $\mathcal{F}$-Necessitation
4. $\vdash [q_1](([x]\neg\varphi \to \neg\varphi) \to (\varphi \to \neg[x]\neg\varphi))$	$\mathcal{F}$-Necessitation for CPL
5. $\vdash [q_1 \cdot q_0](\varphi \to \neg[x]\neg\varphi)$	3, 4, Axiom-2-rule
6. $\vdash [(q_1 \cdot q_0)\forall x]\forall x(\varphi \to \neg[x]\neg\varphi)$	5, JUG
7. $\vdash [q_2](\forall x(\varphi \to \neg[x]\neg\varphi) \to (\varphi \to \forall x\neg[x]\neg\varphi))$	Axiom 7, $\mathcal{F}$-Necessitation

8. $\vdash [q_2 \cdot ((q_1 \cdot q_0)\forall x)](\varphi \to \forall x \neg [x] \neg \varphi)$ 6, 7, Axiom-2-rule
9. $\vdash [q_3]((\varphi \to \forall x \neg [x] \neg \varphi) \to (\exists x [x] \neg \varphi \to \neg \varphi))$ $\mathcal{F}$-Necessitation for CPL
10. $\vdash [q_3 \cdot (q_2 \cdot ((q_1 \cdot q_0)\forall x))](\exists x [x] \neg \varphi \to \neg \varphi)$ 8, 9, Axiom-2-rule
11. $\vdash \exists x [x] \neg \varphi \to \neg \varphi$ 10, Axiom 3. Call this formula E_1.
12. $\vdash \varphi \to \exists x [x] \neg \varphi$ From (1), classical logic.
13. $E_1 \vdash \varphi \to \neg \varphi$ From (12), classical logic.
14. $\vdash E_1 \to \neg \varphi$ From (13), classical logic.
15. $\vdash [q](E_1 \to \neg \varphi)$ Proposition 2.2 in (14)
16. $\vdash ([q](E_1 \to \neg \varphi) \wedge [q_3 \cdot (q_2 \cdot ((q_1 \cdot q_0)\forall x))](E_1)) \to [q \cdot (q_3 \cdot (q_2 \cdot ((q_1 \cdot q_0)\forall x)))](\neg \varphi)$ Axiom 2.
17. $\vdash [q \cdot (q_3 \cdot (q_2 \cdot ((q_1 \cdot q_0)\forall x)))](\neg \varphi)$ 10, 16 and 15, MP.
18. $\vdash \exists x [x](\neg \varphi)$ Axiom 6 applied to (17)
19. $\vdash \varphi$ (1) and (18), Classical logic.
20. $\vdash \neg \varphi$ 14 and 11, MP.

□

Remark 3.3. As the reader can see there is no appeal to Axiom 8 or UBF. An important ingredient of the proof is the use of Justified Universal Generalization. The alternative proofs that follow also appeal to this rule.

Remark 3.4. We justified step 15 in terms of Proposition 2.2. Notice, again, that we do not need the full force of Proposition 2.2 in the proof. We only need to verify the axioms and rules of inference previously used in the proof. None of them involves the use of Axiom 8.

4. The Second Proof of the Knower

This proof proceeds similarly than the proof offered in [3]. The central idea is to produce a proof of $\neg \Box (A \leftrightarrow \neg \Box A)$ in S4 and then translate the proof to QLP.

The main difference is that we make explicit the intermediate steps needed to go from step 5 to step 6. Dean and Kurokawa appeal to a derived rule that they call Necessitation (although the rule is only remotely connected to the rule of Necessitation used in modal logic). We here make explicit various intermediate steps that appeal to axiom 4 and K. The resulting argument does not seem to appeal to Axiom 8 or UBF. As in the previous case, a central ingredient of the proof is the use of the Justified Universal Generalization Rule.

4.1. The Modal Argument.

0: $\Box(A \leftrightarrow \neg \Box A) \vdash A \leftrightarrow \neg \Box A$
1: $\Box(A \leftrightarrow \neg \Box A) \vdash \neg \Box A \to A$
2: $\Box(A \leftrightarrow \neg \Box A) \vdash \Box A \to \neg A$
3: $\Box(A \leftrightarrow \neg \Box A) \vdash \Box A \to A$
4: $\Box(A \leftrightarrow \neg \Box A) \vdash \neg \Box A$
5: $\Box(A \leftrightarrow \neg \Box A) \vdash A$
5': $\vdash \Box(A \leftrightarrow \neg \Box A) \to A$
5": $\vdash \Box(\Box(A \leftrightarrow \neg \Box A) \to A)$
5"': $\vdash \Box\Box(A \leftrightarrow \neg \Box A) \to \Box A$
5"": $\vdash \Box(A \leftrightarrow \neg \Box A) \to \Box\Box(A \leftrightarrow \neg \Box A)$
6: $\Box(A \leftrightarrow \neg \Box A) \vdash \Box A$
7: $\Box(A \leftrightarrow \neg \Box A) \vdash \bot$

4.2. **The argument translated to QLP.**

0: $\exists x[x](\varphi \leftrightarrow \neg\exists x[x]\varphi) \vdash \varphi \leftrightarrow \neg\exists x[x]\varphi$
1: $\exists x[x](\varphi \leftrightarrow \neg\exists x[x]\varphi) \vdash \neg\exists x[x]\varphi \rightarrow \varphi$
2: $\exists x[x](\varphi \leftrightarrow \neg\exists x[x]\varphi) \vdash \exists x[x]\varphi \rightarrow \neg\varphi$
3: $\exists x[x](\varphi \leftrightarrow \neg\exists x[x]\varphi) \vdash \exists x[x]\varphi \rightarrow \varphi$ by derivation in QLP
4: $\exists x[x](\varphi \leftrightarrow \neg\exists x[x]\varphi) \vdash \neg\exists x[x]\varphi$
5: $\exists x[x](\varphi \leftrightarrow \neg\exists x[x]\varphi) \vdash \varphi$
5': $\vdash \exists x[x](\varphi \leftrightarrow \neg\exists x[x]\varphi) \rightarrow \varphi$
5": $\vdash \exists x[x](\exists x[x](\varphi \leftrightarrow \neg\exists x[x]\varphi) \rightarrow \varphi)$ by QLP derivation
5"': $\vdash \exists x[x]\exists x[x](\varphi \leftrightarrow \neg\exists x[x]\varphi) \rightarrow \exists x[x]\varphi$
5"": $\vdash \exists x[x](\varphi \leftrightarrow \neg\exists x[x]\varphi) \rightarrow \exists x[x]\exists x[x](\varphi \leftrightarrow \neg\exists x[x]\varphi)$ QLP derivation
6: $\exists x[x](\varphi \leftrightarrow \neg\exists x[x]\varphi) \vdash \exists x[x]\varphi$
7: $\exists x[x](\varphi \leftrightarrow \neg\exists x[x]\varphi) \vdash \perp$

4.2.1. *The first QLP argument: Step (3).*

1.	$[q_0]([x]\varphi \rightarrow \varphi)$	Axiom 3, $\mathcal{F}$-Necessitation
2.	$[q_1](([x]\varphi \rightarrow \varphi) \rightarrow (\neg\varphi \rightarrow \neg[x]\varphi))$	$\mathcal{F}$-Necessitation for CPL
3.	$[q_1 \cdot q_0](\neg\varphi \rightarrow \neg[x]\varphi)$	1, 2, Axiom-2-rule
4.	$[(q_1 \cdot q_0)\forall x]\forall x(\neg\varphi \rightarrow \neg[x]\varphi)$	3, JUG
5.	$[q_2](\forall x(\neg\varphi \rightarrow \neg[x]\varphi) \rightarrow (\neg\varphi \rightarrow \forall x\neg[x]\varphi))$	Axiom 7, $\mathcal{F}$-Necessitation
6.	$[q_2 \cdot ((q_1 \cdot q_0)\forall x)](\neg\varphi \rightarrow \forall x\neg[x]\varphi)$	4, 5, Axiom-2-rule
7.	$[q_3]((\neg\varphi \rightarrow \forall x\neg[x]\varphi) \rightarrow (\exists x[x]\varphi \rightarrow \varphi))$	$\mathcal{F}$-Necessitation for CPL
8.	$[q_3 \cdot (q_2 \cdot ((q_1 \cdot q_0)\forall x))](\exists x[x]\varphi \rightarrow \varphi)$	6, 7, Axiom-2-rule
9.	$\exists x[x]\varphi \rightarrow \varphi$	8, Axiom-3-rule

4.2.2. *The second QLP argument: Step (5").*

1.	$\vdash \exists y[y](\varphi \leftrightarrow \neg\exists x[x]\varphi) \rightarrow \varphi$	Proof in QLP established above
2.	$\vdash [p](\exists y[y](\varphi \leftrightarrow \neg\exists x[x]\varphi) \rightarrow \varphi)$	Proposition 2.2 (Fitting)
3.	$\vdash \exists z[z](\exists y[y](\varphi \leftrightarrow \neg\exists x[x]\varphi) \rightarrow \varphi)$	Axiom 6

Notice that we do not need the full force of Proposition 2.2. We only need to appeal to a proof of a part of this proposition involving the axioms and rules of inference used so far: Axiom 3, 6 and 7 and the rule JUG. Nothing of this requires to appeal to UBF.

4.2.3. *The third QLP argument: Step (5"").* We will prove: $\exists x[x]X \rightarrow \exists x[x]\exists x[x]X$

1.	$[x]X \rightarrow [!x]([x]X)$	Axiom 4
2.	$[x]X \rightarrow \exists x[x]X$	Instance of axiom 6
3.	$[p]([x]X \rightarrow \exists x[x]X)$	2 and $\mathcal{F}$ Necessitation.
4.	$[p]([x]X \rightarrow \exists x[x]X) \rightarrow ([!x]([x]X) \rightarrow [p.!x]\exists x[x]X)$	Axiom 2
5.	$([!x]([x]X) \rightarrow [p.!x]\exists x[x]X)$	3,4 and MP
6.	$([!x]([x]X) \rightarrow \exists x[x]\exists x[x]X)$	from 5
7.	$[x]X \rightarrow \exists x[x]\exists x[x]X$	from 1 and 6
8.	$\exists x[x]X \rightarrow \exists x[x]\exists x[x]X$	By a modification of the proof given in 4.2.1.

4.3. **An additional argument for the step from 5"' from 5".** We define $\mathbf{QLP}^-$ to be the logic consisting of Axioms 1–7 (note that we omit Axiom 8, i.e., UF), Modus Ponens, and $\mathcal{F}$-Necessitation of Fitting 2005, plus a rule of universal generalization. It is of some interest to show that there is a derivation of 5"' from 5" in $\mathbf{QLP}^-$. The justification of the steps uses both the notation of [3] and ours.

(5‴) $\vdash \exists x[x]\exists x[x](\varphi \leftrightarrow \neg\exists x[x]\varphi) \rightarrow \exists x[x]\varphi$

from

(5″) $\vdash \exists x[x](\exists x[x](\varphi \leftrightarrow \neg\exists x[x]\varphi) \rightarrow \varphi)$

when $x \notin \mathrm{FV}(\varphi)$. We may assume $y \notin \mathrm{FV}(\varphi)$.

1. $[x](\exists x[x](\varphi \leftrightarrow \neg\exists x[x]\varphi) \rightarrow \varphi) \rightarrow ([y]\exists x[x](\varphi \leftrightarrow \neg\exists x[x]\varphi) \rightarrow [y \cdot x]\varphi)$ LP2 (Axiom 2)
2. $[y \cdot x]\varphi \rightarrow \exists x[x]\varphi$ QLP2 (Axiom 6)
3. $[x](\exists x[x](\varphi \leftrightarrow \neg\exists x[x]\varphi) \rightarrow \varphi) \rightarrow ([y]\exists x[x](\varphi \leftrightarrow \neg\exists x[x]\varphi) \rightarrow \exists x[x]\varphi)$ 1, 2, classical propositional logic
4. $\forall x([x](\exists x[x](\varphi \leftrightarrow \neg\exists x[x]\varphi) \rightarrow \varphi) \rightarrow ([y]\exists x[x](\varphi \leftrightarrow \neg\exists x[x]\varphi) \rightarrow \exists x[x]\varphi))$ 3, RQLP3 (universal generalization)
5. $\forall x([x](\exists x[x](\varphi \leftrightarrow \neg\exists x[x]\varphi) \rightarrow \varphi) \rightarrow ([y]\exists x[x](\varphi \leftrightarrow \neg\exists x[x]\varphi) \rightarrow \exists x[x]\varphi))$ $\rightarrow (\exists x[x](\exists x[x](\varphi \leftrightarrow \neg\exists x[x]\varphi) \rightarrow \varphi) \rightarrow ([y]\exists x[x](\varphi \leftrightarrow \neg\exists x[x]\varphi) \rightarrow \exists x[x]\varphi))$ QLP4 (Axiom 7)
6. $\exists x[x](\exists x[x](\varphi \leftrightarrow \neg\exists x[x]\varphi) \rightarrow \varphi) \rightarrow ([y]\exists x[x](\varphi \leftrightarrow \neg\exists x[x]\varphi) \rightarrow \exists x[x]\varphi)$ 4, 5, RQLP2 (Modus Ponens)
7. $\exists x[x](\exists x[x](\varphi \leftrightarrow \neg\exists x[x]\varphi) \rightarrow \varphi)$ Assumption (5″)
8. $[y]\exists x[x](\varphi \leftrightarrow \neg\exists x[x]\varphi) \rightarrow \exists x[x]\varphi$ 6, 7, RQLP2 (Modus Ponens)
9. $\forall y([y]\exists x[x](\varphi \leftrightarrow \neg\exists x[x]\varphi) \rightarrow \exists x[x]\varphi)$ 8, RQLP3 (universal generalization)
10. $\forall y([y]\exists x[x](\varphi \leftrightarrow \neg\exists x[x]\varphi) \rightarrow \exists x[x]\varphi)$ $\rightarrow ([x]\exists x[x](\varphi \leftrightarrow \neg\exists x[x]\varphi) \rightarrow \exists x[x]\varphi)$ QLP1 (Axiom 6)
11. $[x]\exists x[x](\varphi \leftrightarrow \neg\exists x[x]\varphi) \rightarrow \exists x[x]\varphi$ 9, 10, RQLP2 (Modus Ponens)
12. $\forall x([x]\exists x[x](\varphi \leftrightarrow \neg\exists x[x]\varphi) \rightarrow \exists x[x]\varphi)$ 11, RQLP3 (universal generalization)
13. $\forall x([x]\exists x[x](\varphi \leftrightarrow \neg\exists x[x]\varphi) \rightarrow \exists x[x]\varphi)$ $\rightarrow (\exists x[x]\exists x[x](\varphi \leftrightarrow \neg\exists x[x]\varphi) \rightarrow \exists x[x]\varphi)$ QLP4 (Axiom 7)
14. $\exists x[x]\exists x[x](\varphi \leftrightarrow \neg\exists x[x]\varphi) \rightarrow \exists x[x]\varphi$ 12, 13, RQLP2 (Modus Ponens)

5. The Third and Last Proof of the Knower

This proof does not use a modal argument as a heuristic device. It is a pure proof of the Knower in QLP.

1. $[p](\varphi \leftrightarrow \neg\exists y[y]\varphi) \rightarrow (\varphi \leftrightarrow \neg\exists y[y]\varphi)$ Axiom 3
2. $[y]\varphi \rightarrow \varphi$ Axiom 3
3. $\neg\varphi \rightarrow \neg[y]\varphi$ 2, CPL
4. $\forall y(\neg\varphi \rightarrow \neg[y]\varphi)$ 3, SUG
5. $\forall y(\neg\varphi \rightarrow \neg[y]\varphi) \rightarrow (\neg\varphi \rightarrow \forall y\neg[y]\varphi)$ Axiom 7
6. $\neg\varphi \rightarrow \forall y\neg[y]\varphi$[1] 4, 5, MP
7. $[p](\varphi \leftrightarrow \neg\exists y[y]\varphi) \rightarrow \neg\exists y[y]\varphi$ 1, 6, CPL
8. $[q_0]([y]\varphi \rightarrow \varphi)$ Axiom 3, $\mathcal{F}$-Necessitation
9. $[q_1](([y]\varphi \rightarrow \varphi) \rightarrow (\neg\varphi \rightarrow \neg[y]\varphi))$ $\mathcal{F}$-Necessitation for CPL
10. $[q_1 \cdot q_0](\neg\varphi \rightarrow \neg[y]\varphi)$ 8, 9, Axiom-2-rule
11. $[(q_1 \cdot q_0)\forall y]\forall y(\neg\varphi \rightarrow \neg[y]\varphi)$ 10, JUG
12. $[q_2]((\forall y(\neg\varphi \rightarrow \neg[y]\varphi) \rightarrow (\neg\varphi \rightarrow \forall y\neg[y]\varphi))$ Axiom 7, $\mathcal{F}$-Necessitation

[1]The derivation from 2 to 6 can be done by using the proof in 4.2.1. This proof depends directly on JUG rather than the rule of universal generalization used in step 4 here.

13.	$[q_2 \cdot ((q_1 \cdot q_0)\forall y)](\neg\varphi \to \forall y \neg [y]\varphi)$	11, 12, Axiom-2-rule
14.	$[q_3]((\neg\varphi \to \forall y \neg [y]\varphi) \to ((\varphi \leftrightarrow \neg\exists y[y]\varphi) \to \varphi))$	$\mathcal{F}$-Necessitation for CPL
15.	$[q_3 \cdot (q_2 \cdot ((q_1 \cdot q_0)\forall y))]((\varphi \leftrightarrow \neg\exists y[y]\varphi) \to \varphi)$	13, 14, Axiom-2-rule
16.	$(15) \to ([p](\varphi \leftrightarrow \neg\exists y[y]\varphi) \to [(q_3 \cdot (q_2 \cdot ((q_1 \cdot q_0)\forall y))) \cdot p]\varphi)$	Axiom 2
17.	$[p](\varphi \leftrightarrow \neg\exists y[y]\varphi) \to [(q_3 \cdot (q_2 \cdot ((q_1 \cdot q_0)\forall y))) \cdot p]\varphi$	15, 16, MP
18.	$\forall y \neg [y]\varphi \to \neg [(q_3 \cdot (q_2 \cdot ((q_1 \cdot q_0)\forall y))) \cdot p]\varphi$	Axiom 6
19.	$[(q_3 \cdot (q_2 \cdot ((q_1 \cdot q_0)\forall y))) \cdot p]\varphi \to \exists y[y]\varphi$	18, CPL
20.	$[p](\varphi \leftrightarrow \neg\exists y[y]\varphi) \to \exists y[y]\varphi$	17, 19, CPL
21.	$[p](\varphi \leftrightarrow \neg\exists y[y]\varphi) \to \bot$	7, 20, CPL

6. Discussion

When it comes to assess the paradox of the Knower most authors seem to blame the axiom U used in the original proof. The closure condition I is quite strong but it can be substituted by more reasonable closure conditions like K and Charles Cross has argued in [2] that one can prove the Knower without a closure condition at all (see the arguments in [4] which put the last strong statement in context). The proofs in terms if Int continue to use closure conditions although they are milder than the ones used in the original proof.

The proofs of the Knower offered above use analogues of condition U although the direct translation of U is not used. A direct translation of U in the logic of proofs would be:

UP: $\exists y[y](\exists x[x]\phi \to \phi)$

If we focus on the last proof (which is the most direct of the previously offered proofs) what we use is $[q_0]([y]\varphi \to \varphi)$, which is justified by an application of the rule of $\mathcal{F}$-Necessitation to Axiom 3. Perhaps $\mathcal{F}$-Necessitation can be weakened, but it is not clear how such a weakening should be implemented.

Other axioms used in the proof are Axiom 3, 6, 7 and 2. Axiom 3 seems a solid axiom one should want to keep.[2] Axioms 6 and 7 are classical axioms of quantification, which seem solid also. Axiom 2 is a closure condition in the logic of proofs. Perhaps it is possible to adopt a weaker version of this axiom but it is not clear either how to implement this.

Finally we did appeal to the rule of Justified Universal Generalization (JUG) in all the proofs offered in this note. Other versions of QLP, like the one used in [3] and [5] , do not appeal explicitly to this rule. But, as we show below, JUG is a theorem in the system used in [3] and [5]. So, our proofs appeal in all cases to theses that are provable in the axiom system adopted by Dean and Kurokawa. It is clear that one has to have a generalization rule of some type in QLP. And JUG seems a reasonable generalization rule that one would like to preserve. The state of the art in this area is still in flux. Fitting offered an axiom system in [5] that appealed directly to UBF. In more recent work [6] UBF is abandoned as an explicit axiom

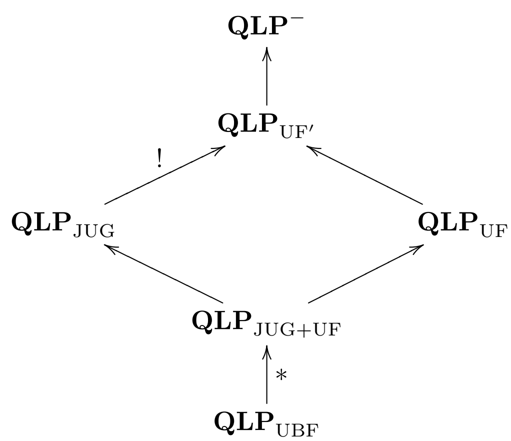
[2]At least there is no proof of the Knower revealing a direct conflict between the Axiom T and self reference. In the area of truth some people have claimed that any language strong enough to have self reference should not have a predicate satisfying the axiom of truth. In a similar vein one may claim that a language having self reference should not have a predicate or operator satisfying the T axiom. But, as we explained above, there is no evidence in the current literature of a direct conflict between T and self reference. It seems that one needs some form of the axiom U and or some form of closure. Following Kaplan and Montague contemporary students of the paradox have preferred to preserve T and blame U or closure.

(and probably as a theorem) but JUG is adopted as an explicit rule of inference. Our proofs show that the Knower is provable as well in these weaker systems which seem to encode Fitting's most recent ideas about QLP.

6.1. **JUG, UBF and UF.** Consider the following axioms and rules of inference previously used:

SUG:	$\dfrac{\varphi}{\forall x\varphi}$
JUG:	$\dfrac{[t]\varphi}{[t\forall x]\forall x\varphi}$
UBF:	$\forall x[t]\varphi \to [t\forall x]\forall x\varphi$
UF:	$\exists y[y]\forall x[t]\varphi \to [t\forall x]\forall x\varphi$
UF′:	$\exists y[y]\forall x[t]\varphi \to \exists y[y]\forall x\varphi$

We define $\mathbf{QLP}^-$ to be the logic consisting of Axioms 1–7 (note that we omit Axiom 8, i.e., UF), Modus Ponens, and $\mathcal{F}$-Necessitation of Fitting 2005, plus SUG. We also write $\mathbf{QLP}_X$, for X from the latter three rule / axioms above, to denote the logic gained by adding X to $\mathbf{QLP}^-$; e.g., $\mathbf{QLP}_{\text{JUG+UF}}$ is $\mathbf{QLP}^-$ plus JUG and UF. Then we have the following diagram of strength among different logics.



The arrow marked with $*$ holds because JUG and UF are derivable in $\mathbf{QLP}_{\text{UBF}}$, as shown below. The one marked with ! holds because UF′ is derivable in $\mathbf{QLP}_{\text{JUG}}$ as below. All the other arrows are trivial.

The first proof is to show JUG to be derivable in $\mathbf{QLP}_{\text{UBF}}$:

1. $[t]\varphi$ — hypothesis
2. $\forall x[t]\varphi$ — SUG
3. $\forall x[t]\varphi \to [t\forall x]\forall x\varphi$ — UBF
4. $[t\forall x]\forall x\varphi$ — 2, 3, MP

Then here is a proof of $\mathbf{QLP}_{\text{UBF}} \vdash$ UF:

1. $[y]\forall x[t]\varphi \to \forall x[t]\varphi$ — Axiom 3
2. $\neg\forall x[t]\varphi \to \neg[y]\forall x[t]\varphi$ — 1, classical prop. logic
3. $\forall y(\neg\forall x[t]\varphi \to \neg[y]\forall x[t]\varphi)$ — 2, SUG
4. $\forall y(\neg\forall x[t]\varphi \to \neg[y]\forall x[t]\varphi) \to (\neg\forall x[t]\varphi \to \forall y\neg[y]\forall x[t]\varphi)$ — Axiom 7
5. $\neg\forall x[t]\varphi \to \forall y\neg[y]\forall x[t]\varphi$ — 3, 4, MP
6. $\exists y[y]\forall x[t]\varphi \to \forall x[t]\varphi$ — 5, classical prop. logic

7. $\forall x[t]\varphi \to [t\forall x]\forall x\varphi$ — UBF
8. $\exists y[y]\forall x[t]\varphi \to [t\forall x]\forall x\varphi$ — 6, 7, classical prop. logic

Lastly, here is a proof of $\mathbf{QLP}_{\mathrm{JUG}} \vdash \mathrm{UF}'$:

1. $[a_0](\forall x[t]\varphi \to [t]\varphi)$ — Axiom 6, $\mathcal{F}$-Necessitation
2. $[a_1]([t]\varphi \to \varphi)$ — Axiom 3, $\mathcal{F}$-Necessitation
3. $[p]((\forall x[t]\varphi \to [t]\varphi) \to (([t]\varphi \to \varphi) \to (\forall x[t]\varphi \to \varphi)))$ — $\mathcal{F}$-Necessitation for CPL
4. $[p \cdot a_0](([t]\varphi \to \varphi) \to (\forall x[t]\varphi \to \varphi))$ — 1, 3, Axiom-2-rule
5. $[(p \cdot a_0) \cdot a_1](\forall x[t]\varphi \to \varphi)$ — 2, 4, Axiom-2-rule
6. $[((p \cdot a_0) \cdot a_1)\forall x]\forall x(\forall x[t]\varphi \to \varphi)$ — 5, JUG
7. $[a_2](\forall x(\forall x[t]\varphi \to \varphi) \to (\forall x[t]\varphi \to \forall x\varphi))$ — Axiom 7, $\mathcal{F}$-Necessitation
8. $[a_2 \cdot (((p \cdot a_0) \cdot a_1)\forall x)](\forall x[t]\varphi \to \forall x\varphi)$ — 6, 7, Axiom-2-rule
9. $[y]\forall x[t]\varphi \to [(a_2 \cdot (((p \cdot a_0) \cdot a_1)\forall x)) \cdot y]\forall x\varphi$ — 8, Axiom 2
10. $[y]\forall x[t]\varphi \to \exists y[y]\forall x\varphi$ — 9, Axiom 6
11. $\forall y([y]\forall x[t]\varphi \to \exists y[y]\forall x\varphi)$ — 10, SUG
12. $\exists y[y]\forall x[t]\varphi \to \exists y[y]\forall x\varphi$ — 11, Axiom 7

Dear and Kurokawa present some negative arguments against UBF. Do they transfer to JUG as well? Dean and Kurokawa offer formal arguments and conceptual arguments against UBF. The formal arguments show the arithmetic unsoundness of $\mathbf{QLP}_{\mathrm{JUG}}$. But this is does not really offer a proof of the arithmetic unsoundness of JUG per se. It is far from obvious that JUG is the culprit there.

Then there are conceptual arguments against UBF. These arguments seem problematic even in the case of UBF and they seem difficult to generalize in order to affect JUG.

A quantified logic of evidence should contain some proof-term expression of universal justification. JUG seems to offer a reasonable alternative. Criticism against UBF might have motivated Fitting to abandon the principle. But still he did not retreated to $\mathbf{QLP}^-$. His proposal is to use $\mathbf{QLP}_{\mathrm{JUG+UF}}$, a system that does appeal to JUG. We agree with Fitting that $\mathbf{QLP}_{\mathrm{JUG+UF}}$ seems to offer a reasonable representation of the quantified logic of proofs. Unfortunately, as we showed, the Knower is also derivable in this new version of the quantified logic of evidence.

The use of QLP to represent the Knower has some obvious representational advantages. But it seems that what is puzzling about the Knower remains puzzling in this new setting. One faces in QLP similar choices to block the paradox than in other logics. And none of the options seems entirely satisfying. So, in a way the use of QLP permits new insights but not a full solution of the paradox.

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