Frege on Conceptual Analysis and Logical Entailment

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I. Analysis and Proof

1884: "… the fundamental propositions of arithmetic should be proved, if in any way possible, with the utmost rigor; for only if every gap in the chain of deductions is eliminated with the greatest care can we say with certainty upon what primitive truths the proof depends …

If we now try to meet this demand, we very soon come to propositions which cannot be proved so long as we do not succeed in analyzing concepts which occur in them into simpler concepts or in reducing them to something of greater generality. Now here it is above all Number which has to be either defined or recognized as indefinable. This is the point which the present work is meant to settle. On the outcome of this task will depend the decision as to the nature of the laws of arithmetic." [*Grundlagen* p. 5; emphasis added]

1914: "In the development of science it can indeed happen that one has used a word, a sign, an expression, over a long period under the impression that its sense is simple until one succeeds in analysing it into simpler logical constituents. By means of such an analysis, we may hope to reduce the number of axioms; for it may not be possible to prove a truth containing a complex constituent so long as that constituent remains unanalysed; but it may be possible, given an analysis, to prove it from truths in which the elements of the analysis occur." [*Logic in Mathematics* manuscript. PW 209]

The Epistemological Role of Proof and the Logicist Thesis

1884 "The aim of proof is, in fact, not merely to place the truth of a proposition beyond all doubt, but also to afford us insight into the dependence of truths upon one another. After we have convinced ourselves that a boulder is immovable, by trying unsuccessfully to move it, there remains the further question, what is it that supports it so securely? The further we pursue these enquiries, the fewer become the primitive truths to which we reduce everything; and this simplification is in itself a goal worth pursuing." [*Grundlagen* 2]

1884 "I hope I may claim in the present work to have made it probable that the laws of arithmetic are analytic judgements and consequently a priori. Arithmetic thus becomes simply a development of logic, and every proposition of arithmetic a law of logic, albeit a derivative one." [*Grundlagen* §87]

1893 "In my *Grundlagen der Arithmetik*, I sought to make it plausible that arithmetic is a branch of logic and need not borrow any ground of proof whatever from either experience or intuition. In the present book this shall now be confirmed, by the derivation of the simplest laws of numbers by logical means alone" [*Grundgesetze* I, §1]

Analysis precedes proof

1914: "The effect of the logical analysis of which we spoke will then be precisely this – to articulate the sense clearly. Work of this kind is very useful; it does not, however, form part of the construction of the system, but must take place beforehand." [1914] PW 210-211

Examples:

Example 1: *Begriffsschrift* treatment of Series

1879 "Through the present example .. we see how pure thought, irrespective of any content given by the senses or even by an intuition a priori, can, solely from the content that results from its own constitution, bring forth judgments that at first sight appear to be possible only on the basis of some intuition" [*Bs §23*]

Analysis of the ancestral of a binary relation *f* (as Frege puts it, of "y follows x in the *f*-sequence"):

Def. 76: $||\!-\ (P)(Her(P,f) \rightarrow ((z)(fxz \rightarrow Pz) \rightarrow Py)) \equiv Fol_f\,(y,x)$

where Her(F,*f*) ("Property P is hereditary in the *f*-sequence") is:

Def. 69: $||\!-\ (x)(Px \rightarrow (y)(fxy \rightarrow Py)) \equiv Her(P,f)$

"Proposition (69) could be expressed in words as follows:

If from the proposition that x has property *P it can be inferred generally, whatever x may be, that every result of an application of the procedure f to x has property P, then I say: "Property P is hereditary in the f-sequence.*" [*Bs §24*]

Derivation of:

Proposition 98: $|\!-\ Fol_f\,(y,x) \rightarrow (Fol_f\,(z, y) \rightarrow Fol_f\,(z, x))$

"If y follows x in the f-sequence and if z follows y in the f-sequence, then z follows x in the f-sequence"

Note that this derivation only shows what Frege takes it to show if "y follows x in the *f*-sequence" is adequately analyzed as the left-hand side of Def. 76.

Example 2: *Grundlagen* on Number

Goal: an analysis of

(N=) The number that belongs to F = the number that belongs to G

First try:

(map) $\exists f$(*f* maps F 1-1 onto G) [I.e.: $F \approx G$]

Final version:

(Gl=) $ext(\xi \approx F) = ext(\xi \approx G)$ [where $\approx$ cashed out in terms of 1-1 mapping]

Use this to give the accounts of 0, successor, then finite number. Then sketch proofs.

1884 "I proceed to give here a list of several propositions to be proved by means of our definitions. The reader will easily see for himself in outline how this can be done.

(1) If *a* follows in the natural series of numbers directly after 0, then *a* = 1.

(2) If 1 is the Number which belongs to a concept, then there exists an object which falls under that concept.

…

(6) Every Number except 0 follows in the natural series of numbers directly after a Number." (Gl §78)

Note: Though Frege regularly characterizes the claim proven in ordinary German, using ordinary terminology, the claim actually proven is its highly-analyzed counterpart.

II. Fregean Thoughts

1892 "…[D]ifferent expressions quite often have something in common, which I call the sense, or, in the special case of sentences, the thought. In other words, we must not fail to recognize that the same sense, the same thought, maybe be variously expressed;… It is possible for one sentence to give no more and no less information than another; and, for all the multiplicity of languages, mankind has a *common stock of thoughts*. If all transformation of the expression were forbidden on the plea that this would alter the content as well, logic would simply be crippled; for the task of logic can hardly be performed without trying to recognize the thought in its manifold guises." [*On Concept & Object* 196 note; emphasis added.]

1906 "When one uses the phrase 'prove a proposition'' in mathematics, then by the word 'proposition'' we clearly mean not a sequence of words or a group of signs, but a thought; something of which one can say that it is true. Frege [1906] 401 ([1984] 332)

1914 "What we prove is not a sentence, but a thought. And it is neither here nor there which language is used in giving the proof." PW206

- Thoughts
  - As material of common science;
  - As relata of logical relations, and things proven.
- General Picture:
  - Start with arithmetical thoughts; subject them to analysis;
  - Prove the thus-analyzed versions of the thoughts from pure logic;
  - Conclude that the truths of arithmetic are truths of logic.

III The Crucial Inference

- Question #1: What justifies the inference from:
  - The highly-analyzed τ^* is provable via purely-logical principles (and hence is grounded solely in logic) *to*
  - The original thought τ is grounded in pure logic?

  - One answer (Frege's?): $\tau = \tau^*$, which is to say that analysis is thought-preserving.
    - Not very helpful, for two reasons:
      - On the most natural (if vague) Fregean criterion of thought-identity, his analysans- and analysandum-sentences don't express the same thoughts, since those sentences are very far from being synonymous.
      - The lack of a clear Fregean criterion of identity means that this isn't a very clear idea anyway.
  - Another answer (Benacerraf, Weiner): it's a bad question; Frege makes no such inference, since he is not interested in the content of ordinary arithmetic (Benacerraf). He's giving a replacement for, not an analysis of, arithmetic (Weiner).
    - Motivations for this view:
      - Defects in ordinary language
      - Failure of Frege's definitions to preserve reference, esp of the numerals. Note that (Gl=) is no better than other possibilities, e.g. (Gg=)
    - Problems with this view:
      - It doesn't make sense of much of what Frege does.
      - The adequacy-criteria favored on this view are too loose to make sense of Frege's rejection of competing reductions

      As Frege puts it:

    1893: "Mathematicians generally are indeed only concerned with the content of a proposition and with the fact that it is to be proved. What is new in this book is not the content of the proposition, but the way in which the proof is carried out and the foundations on which it rests." (*Gg* viii)

- Question #2 Can one make sense of the project of analysis as content-preserving in the relevant sense, given the highly non-trivial nature of Frege's analyses and the failure of his definitions to preserve reference?
    - Yes, I think so. Note the following two features of Frege's analyses:

First: The, or a, crucial thing is the analysis of the canonical identity-statements.

1884 "In our present case, we have to define the sense of the proposition
'the number which belongs to the concept F is the same as that which belongs to the concept G';
that is to say, we must reproduce the content of this proposition in other terms, avoiding the use of the expression
'the number which belongs to the concept F'.
In doing this, we shall be giving a general criterion for the identity of numbers." [*Gl* §62]

1884 ""In the same way with the definitions of fractions, complex numbers and the rest, everything will in the end come down to the search for a judgment-content which can be transformed into an identity whose sides precisely are the new numbers."

Secondly: The following are all (given Frege's logic) easily and obviously logically equivalent:

(map) $\exists f$(f maps F 1-1 onto G) [I.e.: $F \approx G$]

(Gl=) $\text{ext}(\xi \approx F) = \text{ext}(\xi \approx G)$

(Gg=) $\text{ext}(\exists H(x = \text{ext}(H)\ \&\ H \approx F) = \text{ext}(\exists H(x = \text{ext}(H)\ \&\ H \approx G)$

And there are infinitely many others.

For Fregean purposes, if τ^* is an adequate analysis of τ, then so too are any thoughts τ^{**} which are obviously logically equivalent with τ^*. So if (map) is OK, then so too are (Gl=), (Gg=), and so on. The preservation of numeral-reference is not a criterion of adequate analysis.

- The Better Answer to Question #1:
    - There is no general, one-size-fits-all answer to the question of what constitutes an adequate analysis.
    - But this is no reason to reject the general idea that some analyses are successful, and that the proof of the analyzed version τ^* can inform us about the grounds of the original τ.
    - Note that if we recognize (however we do this) τ^* as an adequate analysis of τ, then any τ^{**} that's obviously logically equivalent with τ^* will do as well, for Frege's purposes.

Frege's Historical Picture

1884 "Proof is now demanded of many things that formerly passed as self-evident. Again and again the limits to the validity of a proposition have been in this way established for the first time. The concepts of function, of continuity, of limit and of infinity have been shown to stand in need of sharper definition. Negative and irrational numbers, which had long since been admitted into science, have had to submit to a closer scrutiny of their credentials.

In all directions these same ideals can be seen at work – rigor of proof, precise delimitation of the extent of validity, and as a means to this, sharp definition of concepts.

Proceeding along these lines, we are bound eventually to come to the concept of Number and to the simplest propositions holding of positive whole numbers…" [*Grundlagen* §§1-2]

General picture:

- To show that τ is logically entailed by premises P, it suffices to give analyses, yielding τ^* and P^*, and then prove τ^* from P^*.

IV – IMPLICATIONS

Consider:

Σ a set of sentences;

φ a sentence;

$\tau(\Sigma)$ and $\tau(\varphi)$ the thoughts expressed;

|- an ordinary, reliable, topic-neutral relation of sentential derivability:

1. If $\Sigma \vdash \varphi$, then $\tau(\varphi)$ is logically implied by $\tau(\Sigma)$;
2. Not the converse. Failure occurs when the logical implication is due to latent semantic features which can in principle be brought out by conceptual analysis. Such an analysis will yield more-complex Σ' and φ' such that $\Sigma' \vdash \varphi'$.

Question: Can we tell by inspection of our interpreted language whether its terms contain such latent relations of content?

Frege's answer is: "Typically, no."

1884 "That a concept contains a contradiction is not always obvious without investigation" [*Grundlagen* §74]

1892 "Now something logically simple is no more given us at the outset than most of the chemical elements are; it is reached only by means of scientific work." (C&O 193)

1903 "How do we tell that properties are not mutually inconsistent? There seems to be no criterion for this except the occurrence of the properties in question in one and the same object. … [N]ot every contradiction lies quite open to view." [Gg II §143]

Modern picture (Hilbert, Tarski):

- Take a set Σ of sentences as partially interpreted. Σ defines a general condition, satisfied (if at all) by various structures; if satisfied by A, then satisfied (at least) by every structure isomorphic with A.
- What's primarily interesting about Σ is the class of structures it characterizes.
  - If Σ has a model, then it's consistent. (The condition it defines is satisfiable, and no contradiction is derivable from Σ.)
  - If a sentence φ is satisfied by each of Σ's models ($\Sigma \models \varphi$), then there's a clear sense in which φ is entailed by Σ.
  - What matters is the syntax of the sentence, not the contents of the non-logical terms.

Compare with…

Fregean picture

- A set Σ of sentences is of interest only in the sense that it expresses $\tau(\Sigma)$.
- Because the logical properties and relations (consistency, entailment) are due in part to what's expressed by syntactically simple terms, these properties and relations don't supervene on the syntax of the sentences.
- For the same reason, models are not of central importance. In particular:
    - Two theories can be indistinguishable except for choice of non-logical terms, and hence have all the same models, but have very different logical properties. E.g.:
        - T1's axiom-sentences express thoughts that are all independent of one another; T2's are redundant.
        - Though both theories have finite models (in Hilbert's sense of model), the set of thoughts expressed by T1 is true in finite domains, while the set of thoughts expressed by T2 requires an infinite domain.
        - T1 expresses a consistent (in Frege's sense) set of thoughts; T2 an inconsistent one.
    - The "interpretation" (in Hilbert's sense) of a theory can bring in contradictions not there in the axiom-sentences as schematically understood.

1885 "A proof of non-contradictoriness, then, cannot be given by saying that these rules have been proved as laws for the positive whole numbers and therefore must be without contradiction; for after all, they might conflict with the peculiar properties of the higher numbers, e.g. that of yielding -1 when squared. And in fact, not all rules can be retained... It is therefore evident that in virtue of the peculiar nature of the complex higher numbers there may arise a contradiction where so far as the positive whole numbers are concerned, no contradiction obtains." ["On Formal Theories of Arithmetic" 102]

- Model-theoretic entailment can miss logical entailment. I.e. $\Sigma \not\models \varphi$ although $\tau(\Sigma)$ logically implies $\tau(\varphi)$. This is obvious; happens whenever the entailment has to do with the contents of the terms occurring in those sentences.
- And the converse, depending on the system. Consider some nice rich language, e.g. second-order. General truths about the size and structure of the sets used as models will turn up as model-theoretic truths of such a language; but from the Fregean point of view this is no reason to take them to be truths of logic.
    - Hence a completeness theorem is not a requirement of a good formal system, from a Fregean point of view.

- What if we have a language all of whose terms are conceptually simple, consistent, and independent? (i.e., no further analysis will reveal latent contradictions or entailments.)
    - In principle, in this case, the Fregean and the modern approaches converge.
    - But, crucial for Frege: you generally can't know that this is the case.