

# PrSAT Tutorial (*Mathematica* Notebook)

June 16, 2011

## ■ First, load in the PrSAT package

See my **PrSAT website** for instructions on downloading and installing **PrSAT** (assuming you have *Mathematica* installed).

```
<< PrSAT`
```

## ■ Example #1

The first example of a probability model that we saw was the following:

$$\text{MODEL1} = \text{PrSAT} \left[ \left\{ \text{Pr}[X \wedge Y] == \frac{1}{6}, \text{Pr}[X \wedge \neg Y] == \frac{1}{4}, \text{Pr}[\neg X \wedge Y] == \frac{1}{8}, \text{Pr}[\neg X \wedge \neg Y] == \frac{11}{24} \right\} \right]$$
$$\left\{ \{X \rightarrow \{a_2, a_4\}, Y \rightarrow \{a_3, a_4\}, \Omega \rightarrow \{a_1, a_2, a_3, a_4\}\}, \left\{ a_1 \rightarrow \frac{11}{24}, a_2 \rightarrow \frac{1}{4}, a_3 \rightarrow \frac{1}{8}, a_4 \rightarrow \frac{1}{6} \right\} \right\}$$

**PrSAT** will show us an STT representation of **MODEL1**:

```
TruthTable[MODEL1]
```

X	Y	var	Pr
T	T	a <sub>4</sub>	$\frac{1}{6}$
T	F	a <sub>2</sub>	$\frac{1}{4}$
F	T	a <sub>3</sub>	$\frac{1}{8}$
F	F	a <sub>1</sub>	$\frac{11}{24}$

We can use **PrSAT** to calculate probability, using **MODEL1**:

```
EvaluateProbability[{Pr[X ∨ Y], Pr[X], Pr[Y]}, MODEL1]
```

$$\left\{ \frac{13}{24}, \frac{5}{12}, \frac{7}{24} \right\}$$

We can also check arbitrary claims to see if they are *true on MODEL1*:

```
EvaluateProbability[Pr[X | Y] > Pr[X], MODEL1]
```

True

## ■ Example #2

The second example we saw was an algebraic proof of the following theorem:

$$\text{Pr}(X \vee Y) = \text{Pr}(X) + \text{Pr}(Y) - \text{Pr}(X \wedge Y)$$

$$\text{Pr}[X \mid \mid Y] == \text{Pr}[X] + \text{Pr}[Y] - \text{Pr}[X \&\& Y]$$

**PrSAT** easily verifies this theorem (note: it does not present a readable proof).

```
PrSAT[ {Pr[X ∨ Y] ≠ Pr[X] + Pr[Y] - Pr[X ∧ Y] } ]
```

```
PrSAT::srchfail : Search phase failed; attempting FindInstance
```

```
{ }
```

This output means there are no probability models on which  $\Pr[X \vee Y] \neq \Pr[X] + \Pr[Y] - \Pr[X \wedge Y]$ . That "proves" that the above statement is a theorem of probability calculus.

### ■ Example #3

The second example we saw was an algebraic proof of the following theorem:

$$\Pr(X) = \Pr(X \wedge Y) + \Pr(X \wedge \neg Y)$$

$$\Pr[X] = \Pr[X \ \&\& \ Y] + \Pr[X \ \&\& \ \neg \ Y]$$

**PrSAT** easily verifies this theorem (note: it does not present a readable proof).

```
PrSAT[ {Pr[X] ≠ Pr[X ∧ Y] + Pr[X ∧ ¬ Y] } ]
```

```
PrSAT::srchfail : Search phase failed; attempting FindInstance
```

```
{ }
```

This output means there are no probability models on which  $\Pr[X] \neq \Pr[X \wedge Y] + \Pr[X \wedge \neg Y]$ . That "proves" that the above statement is a theorem of probability calculus.

### ■ Example #4

The next example involves the following theorem:

$$\Pr(X \rightarrow Y) \geq \Pr(Y | X)$$

$$\Pr[X \rightarrow Y] \geq \Pr[Y | X]$$

**PrSAT** easily verifies this theorem (note: it does not present a readable proof). First, we need to define the conditional operator.

$$X \rightarrow Y := \neg X \vee Y;$$

```
PrSAT[ {Pr[X → Y] < Pr[Y | X] } ]
```

```
PrSAT::srchfail : Search phase failed; attempting FindInstance
```

```
{ }
```

This output means there are no probability models on which  $\Pr[X \rightarrow Y] \geq \Pr[Y | X]$ . That "proves" that the above statement is a theorem of probability calculus.

### ■ Example #5

The next example involves the following theorem:

$$d(X, Y) = d(X \vee Y, Y) + d(X \vee \neg Y, Y), \text{ where } d(X, Y) = \Pr(X | Y) - \Pr(X). \quad \text{+}$$

**PrSAT** easily verifies this theorem (note: it does not present a readable proof). First, we need to define  $\mathbf{d}(X, Y)$ .

$$\mathbf{d}[X_, Y_] := \Pr[X | Y] - \Pr[X];$$

```
PrSAT[ {d[X, Y] ≠ d[X ∨ Y, Y] + d[X ∨ ¬ Y, Y] } ]
```

```
PrSAT::srchfail : Search phase failed; attempting FindInstance
```

```
{ }
```

This output means there are no probability models on which  $\mathbf{d}[X, Y] \neq \mathbf{d}[X \vee Y, Y] + \mathbf{d}[X \vee \neg Y, Y]$ . That "proves" that the above statement is a theorem of probability calculus.

## ■ Example #6

The next example involves the fact that  $\Pr(X | Y \vee Z) = \Pr(X | Y \wedge Z)$  is NOT a theorem:

**PrSAT** easily finds a counter-model to this claim.

**PrSAT** [ { **Pr** [ **X** | **Y**  $\vee$  **Z** ]  $\neq$  **Pr** [ **X** | **Y**  $\wedge$  **Z** ] } ]

$$\left\{ \begin{array}{l} \{X \rightarrow \{a_2, a_5, a_6, a_8\}, Y \rightarrow \{a_3, a_5, a_7, a_8\}, \\ Z \rightarrow \{a_4, a_6, a_7, a_8\}, \Omega \rightarrow \{a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8\}\}, \\ \left\{ a_1 \rightarrow \frac{63}{125528}, a_2 \rightarrow 0, a_3 \rightarrow \frac{1}{568}, a_4 \rightarrow \frac{110}{221}, a_5 \rightarrow 0, a_6 \rightarrow 0, a_7 \rightarrow 0, a_8 \rightarrow \frac{1}{2} \right\} \end{array} \right\}$$

The model **PrSAT** finds by default is *non-regular*. We can force it to find a *regular* counter-model, as follows:

**MODEL2** = **PrSAT** [ { **Pr** [ **X** | **Y**  $\vee$  **Z** ]  $\neq$  **Pr** [ **X** | **Y**  $\wedge$  **Z** ] }, **Probabilities**  $\rightarrow$  **Regular** ]

$$\left\{ \begin{array}{l} \{X \rightarrow \{a_2, a_5, a_6, a_8\}, Y \rightarrow \{a_3, a_5, a_7, a_8\}, \\ Z \rightarrow \{a_4, a_6, a_7, a_8\}, \Omega \rightarrow \{a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8\}\}, \\ \left\{ a_1 \rightarrow \frac{6053}{6111882}, a_2 \rightarrow \frac{1}{999}, a_3 \rightarrow \frac{1}{999}, a_4 \rightarrow \frac{1}{999}, a_5 \rightarrow \frac{1}{46}, a_6 \rightarrow \frac{10}{21}, a_7 \rightarrow \frac{85}{171}, a_8 \rightarrow \frac{1}{999} \right\} \end{array} \right\}$$

Here is an STT representation of **MODEL2**:

**TruthTable** [ **MODEL2** ]

X	Y	Z	var	Pr
T	T	T	a <sub>8</sub>	$\frac{1}{999}$
T	T	F	a <sub>5</sub>	$\frac{1}{46}$
T	F	T	a <sub>6</sub>	$\frac{10}{21}$
T	F	F	a <sub>2</sub>	$\frac{1}{999}$
F	T	T	a <sub>7</sub>	$\frac{85}{171}$
F	T	F	a <sub>3</sub>	$\frac{1}{999}$
F	F	T	a <sub>4</sub>	$\frac{1}{999}$
F	F	F	a <sub>1</sub>	$\frac{6053}{6111882}$

We can calculate the values of **Pr** [ **X** | **Y**  $\wedge$  **Z** ], **Pr** [ **X** | **Y**  $\vee$  **Z** ] on this model as follows:

**EvaluateProbability** [ { **Pr** [ **X** | **Y**  $\wedge$  **Z** ], **Pr** [ **X** | **Y**  $\vee$  **Z** ] }, **MODEL2** ]

$$\left\{ \frac{19}{9454}, \frac{3049405}{6099711} \right\}$$

We can look at decimal representations of these exact real numbers, as follows:

**% // N**

$$\{0.00200973, 0.499926\}$$

We gave a different model in the lecture notes. We can enter that model in by hand, and then verify it has the desired properties, as follows:

**MODEL3 =**

$$\text{PrSAT} \left[ \left\{ \begin{aligned} \Pr[X \wedge Y \wedge Z] &= \frac{1}{6}, \Pr[X \wedge Y \wedge \neg Z] = \frac{1}{6}, \Pr[X \wedge \neg Y \wedge Z] = \frac{1}{4}, \Pr[X \wedge \neg Y \wedge \neg Z] = \frac{1}{16}, \\ \Pr[\neg X \wedge Y \wedge Z] &= \frac{1}{6}, \Pr[\neg X \wedge Y \wedge \neg Z] = \frac{1}{12}, \Pr[\neg X \wedge \neg Y \wedge Z] = \frac{1}{24}, \Pr[\neg X \wedge \neg Y \wedge \neg Z] = \frac{1}{16} \end{aligned} \right\} \right]$$

$$\left\{ \begin{aligned} X &\rightarrow \{a_2, a_5, a_6, a_8\}, Y \rightarrow \{a_3, a_5, a_7, a_8\}, \\ Z &\rightarrow \{a_4, a_6, a_7, a_8\}, \Omega \rightarrow \{a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8\}, \\ \left\{ a_1 &\rightarrow \frac{1}{16}, a_2 \rightarrow \frac{1}{16}, a_3 \rightarrow \frac{1}{12}, a_4 \rightarrow \frac{1}{24}, a_5 \rightarrow \frac{1}{6}, a_6 \rightarrow \frac{1}{4}, a_7 \rightarrow \frac{1}{6}, a_8 \rightarrow \frac{1}{6} \right\} \end{aligned} \right\}$$

**TruthTable[MODEL3]**

X	Y	Z	var	Pr
T	T	T	a <sub>8</sub>	$\frac{1}{6}$
T	T	F	a <sub>5</sub>	$\frac{1}{6}$
T	F	T	a <sub>6</sub>	$\frac{1}{4}$
T	F	F	a <sub>2</sub>	$\frac{1}{16}$
F	T	T	a <sub>7</sub>	$\frac{1}{6}$
F	T	F	a <sub>3</sub>	$\frac{1}{12}$
F	F	T	a <sub>4</sub>	$\frac{1}{24}$
F	F	F	a <sub>1</sub>	$\frac{1}{16}$

**EvaluateProbability[{Pr[X | Y ∧ Z], Pr[X | Y ∨ Z]}, MODEL3]**

$$\left\{ \frac{1}{2}, \frac{2}{3} \right\}$$

We can also see the algebraic form of an expression, as follows:

**AlgebraicForm[Pr[X | Y ∧ Z] == Pr[X | Y ∨ Z], {X, Y, Z}]**

$$\frac{a_8}{a_7 + a_8} = \frac{a_5 + a_6 + a_8}{a_3 + a_4 + a_5 + a_6 + a_7 + a_8}$$

Note that **PrSAT** uses different conventions (*i.e.*, a different ordering in the truth-table) for the  $a_i$  than I use in the lecture notes.

Here is a verification of some of our *independence* theorems:

```
PrSAT[{
  Pr[P ∧ Q] == Pr[P] Pr[Q],
  Pr[P | Q] ≠ Pr[P],
}]
```

PrSAT::srchfail : Search phase failed; attempting FindInstance

{}

```
PrSAT[{
  Pr[P ∧ ¬ Q] == Pr[P] Pr[¬ Q],
  Pr[P ∧ Q] ≠ Pr[P] Pr[Q]
}]
```

PrSAT::srchfail : Search phase failed; attempting FindInstance

{}

```
PrSAT[{
  Pr[P ∧ Q] == Pr[P] Pr[Q],
  Pr[P ∧ ¬ Q] ≠ Pr[P] Pr[¬ Q]
}]
```

PrSAT::srchfail : Search phase failed; attempting FindInstance

```
{}
```

Here is a (regular) model showing that pairwise independence does *not* entail mutual independence (for a set of 3 events)

```
PrSAT[{
  Pr[X ∧ Y] == Pr[X] Pr[Y],
  Pr[X ∧ Z] == Pr[X] Pr[Z],
  Pr[Y ∧ Z] == Pr[Y] Pr[Z],
  Pr[(X ∧ Y) ∧ Z] ≠ Pr[X ∧ Y] Pr[Z],
  Pr[X] == Pr[Y] == Pr[Z] == 1 / 2
}, Probabilities → Regular]
```

```
{X → {a2, a5, a6, a8}, Y → {a3, a5, a7, a8},
 Z → {a4, a6, a7, a8}, Ω → {a1, a2, a3, a4, a5, a6, a7, a8}},
 {a1 → 1/676, a2 → 42/169, a3 → 42/169, a4 → 42/169, a5 → 1/676, a6 → 1/676, a7 → 1/676, a8 → 42/169}}
```

TruthTable[%]

X	Y	Z	var	Pr
T	T	T	a <sub>8</sub>	$\frac{42}{169}$
T	T	F	a <sub>5</sub>	$\frac{1}{676}$
T	F	T	a <sub>6</sub>	$\frac{1}{676}$
T	F	F	a <sub>2</sub>	$\frac{42}{169}$
F	T	T	a <sub>7</sub>	$\frac{1}{676}$
F	T	F	a <sub>3</sub>	$\frac{42}{169}$
F	F	T	a <sub>4</sub>	$\frac{42}{169}$
F	F	F	a <sub>1</sub>	$\frac{1}{676}$

Here is a (regular) model showing that independence is *not* transitive:

```
PrSAT[{
  Pr[X ∧ Y] == Pr[X] Pr[Y],
  Pr[Y ∧ Z] == Pr[Y] Pr[Z],
  Pr[X ∧ Z] ≠ Pr[X] Pr[Z],
  Pr[X] == Pr[Y] == Pr[Z] == 1 / 2
}, Probabilities → Regular]
```

```
{X → {a2, a5, a6, a8}, Y → {a3, a5, a7, a8},
 Z → {a4, a6, a7, a8}, Ω → {a1, a2, a3, a4, a5, a6, a7, a8}},
 {a1 → 1/999, a2 → 995/3996, a3 → 1/999, a4 → 995/3996, a5 → 995/3996, a6 → 1/999, a7 → 995/3996, a8 → 1/999}}
```

**TruthTable[%]**

X	Y	Z	var	Pr
T	T	T	a <sub>8</sub>	$\frac{1}{999}$
T	T	F	a <sub>5</sub>	$\frac{995}{3996}$
T	F	T	a <sub>6</sub>	$\frac{1}{999}$
T	F	F	a <sub>2</sub>	$\frac{995}{3996}$
F	T	T	a <sub>7</sub>	$\frac{995}{3996}$
F	T	F	a <sub>3</sub>	$\frac{1}{999}$
F	F	T	a <sub>4</sub>	$\frac{995}{3996}$
F	F	F	a <sub>1</sub>	$\frac{1}{999}$

Extra Example (not from my lecture #1 notes): Here's an example of a set of constraints that are satisfiable but only on probability models containing *irrational* numbers

$$\text{PrSAT}\left[\left\{\text{Pr}[Y | X] = \text{Pr}[Y \vee X], \text{Pr}[X \wedge Y] = \frac{1}{4}, \text{Pr}[\neg X \wedge Y] = \frac{1}{4}\right\}\right]$$

$$\left\{\{X \rightarrow \{a_2, a_4\}, Y \rightarrow \{a_3, a_4\}, \Omega \rightarrow \{a_1, a_2, a_3, a_4\}\}, \left\{a_1 \rightarrow \frac{1}{8} (7 - \sqrt{17}), a_2 \rightarrow \frac{1}{8} (-3 + \sqrt{17}), a_3 \rightarrow \frac{1}{4}, a_4 \rightarrow \frac{1}{4}\right\}\right\}$$

**TruthTable[%]**

X	Y	var	Pr
T	T	a <sub>4</sub>	$\frac{1}{4}$
T	F	a <sub>2</sub>	$\frac{1}{8} (-3 + \sqrt{17})$
F	T	a <sub>3</sub>	$\frac{1}{4}$
F	F	a <sub>1</sub>	$\frac{1}{8} (7 - \sqrt{17})$

Can you prove that this is the *only* model for this set of constraints?

### ■ The first numerical probability model appearing in the lecture notes

$$\text{MODEL1} = \text{PrSAT}\left[\left\{\begin{array}{l} \text{Pr}[X \wedge Y] = \frac{1}{6}, \\ \text{Pr}[X \wedge \neg Y] = \frac{1}{4}, \\ \text{Pr}[\neg X \wedge Y] = \frac{1}{8} \end{array}\right\}\right]$$

$$\left\{\{X \rightarrow \{a_2, a_4\}, Y \rightarrow \{a_3, a_4\}, \Omega \rightarrow \{a_1, a_2, a_3, a_4\}\}, \left\{a_1 \rightarrow \frac{11}{24}, a_2 \rightarrow \frac{1}{4}, a_3 \rightarrow \frac{1}{8}, a_4 \rightarrow \frac{1}{6}\right\}\right\}$$

**TruthTable [MODEL1]**

X	Y	var	Pr
T	T	$a_4$	$\frac{1}{6}$
T	F	$a_2$	$\frac{1}{4}$
F	T	$a_3$	$\frac{1}{8}$
F	F	$a_1$	$\frac{11}{24}$

■ **Using PrSAT to translate probabilistic expressions into algebraic ones**

**AlgebraicForm**[ $\text{Pr}[\neg X \vee Y] < \text{Pr}[Y | X]$ , {X, Y}]

$$1 - a_2 < \frac{a_4}{a_2 + a_4}$$

**DenialOfTheorem** = **AlgebraicForm**[ $\text{Pr}[\neg X \vee Y] < \text{Pr}[Y | X]$ , {X, Y}]

$$1 - a_2 < \frac{a_4}{a_2 + a_4}$$

We can then use *Mathematica's* decision procedure for real algebra (**FindInstance**) to *refute* this claim, assuming that the underlying variables are constrained so as to constitute a *probability model*:

**FindInstance**[**DenialOfTheorem** &&  $a_1 + a_2 + a_3 + a_4 == 1$  &&  
 $0 \leq a_1 \leq 1$  &&  $0 \leq a_2 \leq 1$  &&  $0 \leq a_3 \leq 1$  &&  $0 \leq a_4 \leq 1$ , { $a_1, a_2, a_3, a_4$ }]  
 {}

■ **Using PrSAT to find a model on which  $\text{Pr}(X | Y \wedge Z) \neq \text{Pr}(X | Y \vee Z)$**

**MODEL2** = **PrSAT**[ {  
 $\text{Pr}[X | Y \wedge Z] \neq \text{Pr}[X | Y \vee Z]$   
 },  
**Probabilities** → **Regular**, **BypassSearch** → **True**]

{X → { $a_2, a_5, a_6, a_8$ }, Y → { $a_3, a_5, a_7, a_8$ },  
 Z → { $a_4, a_6, a_7, a_8$ },  $\Omega$  → { $a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8$ }},  
 $\left\{ a_1 \rightarrow \frac{1}{16}, a_2 \rightarrow \frac{1}{16}, a_3 \rightarrow \frac{1}{4}, a_4 \rightarrow \frac{5}{16}, a_5 \rightarrow \frac{1}{32}, a_6 \rightarrow \frac{1}{8}, a_7 \rightarrow \frac{1}{8}, a_8 \rightarrow \frac{1}{32} \right\}$ }

**TruthTable [MODEL2]**

X	Y	Z	var	Pr
T	T	T	$a_8$	$\frac{1}{32}$
T	T	F	$a_5$	$\frac{1}{32}$
T	F	T	$a_6$	$\frac{1}{8}$
T	F	F	$a_2$	$\frac{1}{16}$
F	T	T	$a_7$	$\frac{1}{8}$
F	T	F	$a_3$	$\frac{1}{4}$
F	F	T	$a_4$	$\frac{5}{16}$
F	F	F	$a_1$	$\frac{1}{16}$

**EvaluateProbability**[{ $\text{Pr}[X | Y \wedge Z]$ ,  $\text{Pr}[X | Y \vee Z]$ }, **MODEL2**]

$$\left\{ \frac{1}{5}, \frac{3}{14} \right\}$$

■ Using PrSAT to show the ordinal *non-equivalence* of  $\{d, r, \ell, s\}$

$$\begin{aligned}
 d[H_-, E_-] &:= \Pr[H | E] - \Pr[H]; \\
 r[H_-, E_-] &:= \frac{\Pr[H | E] - \Pr[H]}{\Pr[H | E] + \Pr[H]}; \\
 \ell[H_-, E_-] &:= \frac{\Pr[E | H] - \Pr[E | \neg H]}{\Pr[E | H] + \Pr[E | \neg H]}; \\
 s[H_-, E_-] &:= \Pr[H | E] - \Pr[H | \neg E]; \\
 J[H_-, E_-] &:= \frac{1}{\Pr[\neg H | E]} - \frac{1}{\Pr[\neg H]};
 \end{aligned}$$

A model showing that  $d$  and  $r$  are not ordinally equivalent:

```

MODEL3 = PrSAT[{
   $d[H_1, E_1] \geq d[H_2, E_2]$ ,
   $r[H_1, E_1] < r[H_2, E_2]$ 
},
Probabilities  $\rightarrow$  Regular]

{
   $E_1 \rightarrow \{a_2, a_6, a_7, a_8, a_{12}, a_{13}, a_{14}, a_{16}\}$ ,  $E_2 \rightarrow \{a_3, a_6, a_9, a_{10}, a_{12}, a_{13}, a_{15}, a_{16}\}$ ,
   $H_1 \rightarrow \{a_4, a_7, a_9, a_{11}, a_{12}, a_{14}, a_{15}, a_{16}\}$ ,  $H_2 \rightarrow \{a_5, a_8, a_{10}, a_{11}, a_{13}, a_{14}, a_{15}, a_{16}\}$ ,
   $\Omega \rightarrow \{a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8, a_9, a_{10}, a_{11}, a_{12}, a_{13}, a_{14}, a_{15}, a_{16}\}$ ,
   $a_1 \rightarrow \frac{45\,122\,521\,985\,018\,694\,529}{395\,712\,164\,128\,624\,041\,600}$ ,  $a_2 \rightarrow \frac{1}{979}$ ,  $a_3 \rightarrow \frac{4}{27}$ ,  $a_4 \rightarrow \frac{6}{19}$ ,  $a_5 \rightarrow \frac{1}{999}$ ,
   $a_6 \rightarrow \frac{1}{985}$ ,  $a_7 \rightarrow \frac{1}{998}$ ,  $a_8 \rightarrow \frac{2}{45}$ ,  $a_9 \rightarrow \frac{1}{116}$ ,  $a_{10} \rightarrow \frac{28}{141}$ ,  $a_{11} \rightarrow \frac{1}{384}$ ,
   $a_{12} \rightarrow \frac{11}{100}$ ,  $a_{13} \rightarrow \frac{1}{447}$ ,  $a_{14} \rightarrow \frac{1}{165}$ ,  $a_{15} \rightarrow \frac{2}{45}$ ,  $a_{16} \rightarrow \frac{1}{999}$ 
}

```



**TruthTable [MODEL3]**

E <sub>1</sub>	E <sub>2</sub>	H <sub>1</sub>	H <sub>2</sub>	var	Pr
T	T	T	T	a <sub>16</sub>	$\frac{1}{999}$
T	T	T	F	a <sub>12</sub>	$\frac{11}{100}$
T	T	F	T	a <sub>13</sub>	$\frac{1}{447}$
T	T	F	F	a <sub>6</sub>	$\frac{1}{985}$
T	F	T	T	a <sub>14</sub>	$\frac{1}{165}$
T	F	T	F	a <sub>7</sub>	$\frac{1}{998}$
T	F	F	T	a <sub>8</sub>	$\frac{2}{45}$
T	F	F	F	a <sub>2</sub>	$\frac{1}{979}$
F	T	T	T	a <sub>15</sub>	$\frac{2}{45}$
F	T	T	F	a <sub>9</sub>	$\frac{1}{116}$
F	T	F	T	a <sub>10</sub>	$\frac{28}{141}$
F	T	F	F	a <sub>3</sub>	$\frac{4}{27}$
F	F	T	T	a <sub>11</sub>	$\frac{1}{384}$
F	F	T	F	a <sub>4</sub>	$\frac{6}{19}$
F	F	F	T	a <sub>5</sub>	$\frac{1}{999}$
F	F	F	F	a <sub>1</sub>	$\frac{45\ 122\ 521\ 985\ 018\ 694\ 529}{395\ 712\ 164\ 128\ 624\ 041\ 600}$

**EvaluateProbability**[[{d[H<sub>1</sub>, E<sub>1</sub>], d[H<sub>2</sub>, E<sub>2</sub>]},  
 {r[H<sub>1</sub>, E<sub>1</sub>], r[H<sub>2</sub>, E<sub>2</sub>]}, MODEL3] // N  
 {{0.21837, 0.178694}, {0.182368, 0.229258}}

A model showing that  $d$  and  $\ell$  are not ordinally equivalent:

**MODEL4 = PrSAT**[[  
 d[H<sub>1</sub>, E<sub>1</sub>] ≥ d[H<sub>2</sub>, E<sub>2</sub>],  
 ℓ[H<sub>1</sub>, E<sub>1</sub>] < ℓ[H<sub>2</sub>, E<sub>2</sub>]  
 ],  
**Probabilities → Regular**]

{E<sub>1</sub> → {a<sub>2</sub>, a<sub>6</sub>, a<sub>7</sub>, a<sub>8</sub>, a<sub>12</sub>, a<sub>13</sub>, a<sub>14</sub>, a<sub>16</sub>}, E<sub>2</sub> → {a<sub>3</sub>, a<sub>6</sub>, a<sub>9</sub>, a<sub>10</sub>, a<sub>12</sub>, a<sub>13</sub>, a<sub>15</sub>, a<sub>16</sub>},  
 H<sub>1</sub> → {a<sub>4</sub>, a<sub>7</sub>, a<sub>9</sub>, a<sub>11</sub>, a<sub>12</sub>, a<sub>14</sub>, a<sub>15</sub>, a<sub>16</sub>}, H<sub>2</sub> → {a<sub>5</sub>, a<sub>8</sub>, a<sub>10</sub>, a<sub>11</sub>, a<sub>13</sub>, a<sub>14</sub>, a<sub>15</sub>, a<sub>16</sub>},  
 Ω → {a<sub>1</sub>, a<sub>2</sub>, a<sub>3</sub>, a<sub>4</sub>, a<sub>5</sub>, a<sub>6</sub>, a<sub>7</sub>, a<sub>8</sub>, a<sub>9</sub>, a<sub>10</sub>, a<sub>11</sub>, a<sub>12</sub>, a<sub>13</sub>, a<sub>14</sub>, a<sub>15</sub>, a<sub>16</sub>}},  
 {a<sub>1</sub> →  $\frac{1\ 525\ 917\ 434\ 359}{4\ 455\ 027\ 404\ 640}$ , a<sub>2</sub> →  $\frac{1}{113}$ , a<sub>3</sub> →  $\frac{1}{21}$ , a<sub>4</sub> →  $\frac{1}{129}$ , a<sub>5</sub> →  $\frac{1}{132}$ , a<sub>6</sub> →  $\frac{1}{38}$ , a<sub>7</sub> →  $\frac{1}{30}$ ,  
 a<sub>8</sub> →  $\frac{1}{32}$ , a<sub>9</sub> →  $\frac{1}{28}$ , a<sub>10</sub> →  $\frac{1}{40}$ , a<sub>11</sub> →  $\frac{1}{138}$ , a<sub>12</sub> →  $\frac{1}{20}$ , a<sub>13</sub> →  $\frac{1}{9}$ , a<sub>14</sub> →  $\frac{1}{47}$ , a<sub>15</sub> →  $\frac{1}{496}$ , a<sub>16</sub> →  $\frac{8}{33}$ }}

I leave the remaining pairs as exercises.

### ■ Selection of Properties (1)-(8)

Verification that  $r$  satisfies commutativity (4):

```
PrSAT[{
  r[H, E] ≠ r[E, H]
}, BypassSearch → True]
{}
```

Model showing that  $s$  violates (4):

```
MODEL5 = PrSAT[{
  Pr[H | E] > Pr[H],
  s[H, E] ≠ s[E, H]
}, Probabilities → Regular]
{{E → {a2, a4}, H → {a3, a4}, Ω → {a1, a2, a3, a4}}, {a1 →  $\frac{16511}{27972}$ , a2 →  $\frac{1}{4}$ , a3 →  $\frac{1}{999}$ , a4 →  $\frac{10}{63}$ }}
```

TruthTable[MODEL5]

E	H	var	Pr
T	T	a4	$\frac{10}{63}$
T	F	a2	$\frac{1}{4}$
F	T	a3	$\frac{1}{999}$
F	F	a1	$\frac{16511}{27972}$

```
EvaluateProbability[{s[H, E], s[E, H]}, MODEL5] // N
```

```
{0.386657, 0.696209}
```

Model showing that  $\ell$  violates the so-called "Law of Likelihood" (2):

```
MODEL6 = PrSAT[{
  Pr[E | H1] > Pr[E | H2],
  ℓ[H1, E] < ℓ[H2, E]
}, Probabilities → Regular]
{{E → {a2, a5, a6, a8}, H1 → {a3, a5, a7, a8},
  H2 → {a4, a6, a7, a8}, Ω → {a1, a2, a3, a4, a5, a6, a7, a8}},
 {a1 →  $\frac{71843009866}{96210749943}$ , a2 →  $\frac{3}{43}$ , a3 →  $\frac{10}{99}$ , a4 →  $\frac{1}{177}$ , a5 →  $\frac{1}{189}$ , a6 →  $\frac{1}{629}$ , a7 →  $\frac{2}{29}$ , a8 →  $\frac{1}{999}$ }}
```

Model showing that  $s$  violates (3) — people were very surprised when I reported this one:

```
MODEL7 = PrSAT[{
  Pr[H | E1] > Pr[H | E2],
  s[H, E1] < s[H, E2]
}, Probabilities → Regular]
{{E1 → {a2, a5, a6, a8}, E2 → {a3, a5, a7, a8},
  H → {a4, a6, a7, a8}, Ω → {a1, a2, a3, a4, a5, a6, a7, a8}},
 {a1 →  $\frac{211701818}{1118425347}$ , a2 →  $\frac{2}{23}$ , a3 →  $\frac{4}{27}$ , a4 →  $\frac{3}{41}$ , a5 →  $\frac{1}{31}$ , a6 →  $\frac{5}{39}$ , a7 →  $\frac{16}{69}$ , a8 →  $\frac{12}{109}$ }}
```

```
EvaluateProbability[{Pr[H | E1], Pr[H | E2], s[H, E1], s[H, E2]}, MODEL7] // N
```

```
{0.666543, 0.654647, 0.191741, 0.233022}
```

Here is a bizarre measure  $\gamma$ , which satisfies  $(\mathcal{R})$ , but violates (8). This was discovered by Crupi *et al.* See their paper: "Towards a Grammar of Bayesian Confirmation", which can be downloaded from my Mathcamp webpage <http://fitelson.org/mathcamp/>

$$\gamma[H_, E_] := \text{Pr}[H | E]^2 - \text{Pr}[H]^2;$$

This model shows that  $\gamma$  violates (8):

```
MODEL8 = PrSAT[{
  Pr[E | H1] > Pr[E | H2],
  Pr[E | ~H1] == Pr[E | ~H2],
  \gamma[H1, E] < \gamma[H2, E]
}, SearchAttempts -> 10, Probabilities -> Regular]
{E -> {a2, a5, a6, a8}, H1 -> {a3, a5, a7, a8}, H2 -> {a4, a6, a7, a8},
 \Omega -> {a1, a2, a3, a4, a5, a6, a7, a8}}, {a1 -> \frac{50\,999\,038\,403}{426\,712\,555\,305}, a2 -> \frac{14}{37},
 a3 -> \frac{118\,434\,037}{126\,860\,489\,415}, a4 -> \frac{1}{27}, a5 -> \frac{5}{33}, a6 -> \frac{9}{29}, a7 -> \frac{1}{999}, a8 -> \frac{1}{785}}}
```

```
PrSAT[{
  Pr[E | H1] > Pr[E | H2],
  Pr[E | ~H1] < Pr[E | ~H2],
  J[H1, E] < J[H2, E],
  Pr[H1] == 2 Pr[H2]
}, SearchAttempts -> 10, Probabilities -> Regular]
{E -> {a2, a5, a6, a8}, H1 -> {a3, a5, a7, a8},
 H2 -> {a4, a6, a7, a8}, \Omega -> {a1, a2, a3, a4, a5, a6, a7, a8}},
 {a1 -> \frac{28\,502}{177\,859}, a2 -> \frac{4}{37}, a3 -> \frac{280\,153}{865\,260}, a4 -> \frac{7}{46}, a5 -> \frac{13}{90}, a6 -> \frac{1}{19}, a7 -> \frac{1}{22}, a8 -> \frac{1}{76}}}
```

We can verify that  $\gamma$  satisfies  $(\mathcal{R})$ , as follows:

```
PrSAT[{
  Pr[H | E] > Pr[H] && \gamma[H, E] \le 0 ||
  Pr[H | E] == Pr[H] && \gamma[H, E] \ne 0 || Pr[H | E] < Pr[H] && \gamma[H, E] \ge 0
}, BypassSearch -> True]
{}
```

### ■ My Hint about exercise (ii)

$$\text{FullSimplify}\left[\frac{x-y}{x+y} == \text{Tanh}\left[\frac{1}{2} \text{Log}\left[\frac{x}{y}\right]\right]\right]$$

True

Both **tanh** and **log** are increasing functions. And, *increasing functions preserve order*.

### ■ Best Probability Bounds

**PrSAT** has a built-in function **PrRange** for computing best possible bounds on a probability expression, subject to (arbitrary) *numerical* constraints. Here are some examples:

*Modus ponens*:

$$\text{In[63]:= PrRange}\left[\text{Pr}[B], \text{Pr}[B | A] == \frac{1}{2} \ \&\& \ \text{Pr}[A] == \frac{1}{3}\right]$$

$$\text{Out[63]= } \left\{ \frac{1}{6}, \frac{5}{6} \right\}$$

Constraints don't have to be equalities:

$$\text{In[72]:= PrRange}\left[\text{Pr}[B], \text{Pr}[B | A] < \frac{1}{4} \ \&\& \ \text{Pr}[A] > \frac{1}{3}\right]$$

$$\text{Out[72]= } \left\{ 0, \frac{3}{4} \right\}$$

*Modus tollens:*

$$\text{In[75]:= PrRange}\left[\text{Pr}[\neg A], \text{Pr}[B | A] == \frac{1}{4} \ \&\& \ \text{Pr}[\neg B] == \frac{1}{3}\right]$$

$$\text{Out[75]= } \left\{ \frac{5}{9}, 1 \right\}$$

Denying the antecedent:

$$\text{In[76]:= PrRange}\left[\text{Pr}[\neg B], \text{Pr}[B | A] == \frac{1}{4} \ \&\& \ \text{Pr}[\neg A] == \frac{1}{3}\right]$$

$$\text{Out[76]= } \left\{ \frac{1}{2}, \frac{5}{6} \right\}$$

We can also solve analytically for general symbolic best possible bounds. Here are the bounds for *modus ponens*:

$$\text{In[77]:= MPCons = AlgebraicForm}\left[\text{Pr}[B | A] == x \ \&\& \ \text{Pr}[A] == y, \{A, B\}\right]$$

$$\text{Out[77]= } \frac{a_4}{a_2 + a_4} == x \ \&\& \ a_2 + a_4 == y$$

$$\text{In[78]:= PrB = AlgebraicForm}\left[\text{Pr}[B], \{A, B\}\right]$$

$$\text{Out[78]= } a_3 + a_4$$

$$\text{In[91]:= BackCons = } 0 < a_2 < 1 \ \&\& \ 0 < a_3 < 1 \ \&\& \ 0 < a_4 < 1 \ \&\& \ a_2 + a_3 + a_4 \leq 1;$$

$$\text{In[95]:= MPMax = Maximize}\left[\{\text{PrB}, \text{MPCons} \ \&\& \ \text{BackCons}\}, \{a_2, a_3, a_4\}, \text{Reals}\right][[1]]$$

$$\text{Out[95]= } \begin{cases} 1 - y + x y & 0 < y < 1 \ \&\& \ 0 < x < 1 \\ -\infty & \text{True} \end{cases}$$

$$\text{In[96]:= FullSimplify}\left[\text{MPMax}, 0 < y < 1 \ \&\& \ 0 < x < 1\right]$$

$$\text{Out[96]= } 1 + (-1 + x) y$$

$$\text{In[97]:= MPMin = Minimize}\left[\{\text{PrB}, \text{MPCons} \ \&\& \ \text{BackCons}\}, \{a_2, a_3, a_4\}, \text{Reals}\right][[1]]$$

$$\text{Out[97]= } \begin{cases} x y & 0 < y < 1 \ \&\& \ 0 < x < 1 \\ \infty & \text{True} \end{cases}$$

$$\text{In[98]:= FullSimplify}\left[\text{MPMin}, 0 < y < 1 \ \&\& \ 0 < x < 1\right]$$

$$\text{Out[98]= } x y$$

And, here are the bounds for *modus tollens*:

$$\text{In[99]:= MTCons = AlgebraicForm}\left[\text{Pr}[B | A] == x \ \&\& \ \text{Pr}[\neg B] == y, \{A, B\}\right]$$

$$\text{Out[99]= } \frac{a_4}{a_2 + a_4} == x \ \&\& \ 1 - a_3 - a_4 == y$$

In[101]:= **PrNotA = AlgebraicForm[Pr[- A], {A, B}]**

Out[101]=  $1 - a_2 - a_4$

In[102]:= **MTMax = Maximize[{PrNotA, MTCons && BackCons}, {a<sub>2</sub>, a<sub>3</sub>, a<sub>4</sub>}, Reals][[1]]**

Out[102]=  $\begin{cases} 1 & 0 < y < 1 \ \&\& \ 0 < x < 1 \\ -\infty & \text{True} \end{cases}$

In[103]:= **FullSimplify[MTMax, 0 < y < 1 && 0 < x < 1]**

Out[103]= 1

In[104]:= **MTMin = Minimize[{PrNotA, MTCons && BackCons}, {a<sub>2</sub>, a<sub>3</sub>, a<sub>4</sub>}, Reals][[1]]**

Out[104]=  $\begin{cases} \frac{-1+x+y}{-1+x} & (0 < y < 1 \ \&\& \ x = 1 - y) \ || \ (0 < y < 1 \ \&\& \ 0 < x < 1 - y) \\ \frac{-1+x+y}{x} & 0 < y < 1 \ \&\& \ 1 - y < x < 1 \\ \infty & \text{True} \end{cases}$

In[105]:= **FullSimplify[MTMin, 0 < y < 1 && 0 < x < 1]**

Out[105]=  $\begin{cases} 1 + \frac{y}{-1+x} & x + y \leq 1 \\ \frac{-1+x+y}{x} & \text{True} \end{cases}$

In[106]:= **FullSimplify[MTMin, 0 < y < 1 && 0 < x < 1 && x + y ≤ 1]**

Out[106]=  $1 + \frac{y}{-1+x}$

In[107]:= **FullSimplify[MTMin, 0 < y < 1 && 0 < x < 1 && x + y > 1]**

Out[107]=  $\frac{-1+x+y}{x}$