New Foundations for Comparative Probability

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- At the highest level of abstraction, our framework [13] results in epistemic coherence requirements for sets of judgments $\mathfrak{J}$ (of type J) over finite Boolean algebras $\mathcal{B}$.
- Applying our framework involves **The Three Steps**:
  - **Step 1**: Define the *vindicated (viz., perfectly accurate) judgment set* (of type J), at world w. Call this set $\mathring{\mathbf{J}}_w$.
    - Think of $\mathring{\mathbf{J}}_w$ as the judgments (of type J) that "the *omniscient/ideal* agent" would make (at world w).
  - **Step 2**: Define a notion of "distance between $\mathfrak{J}$ and $\mathring{\mathbf{J}}_w$". That is, define a measure of *distance from vindication*: $\mathfrak{D}(\mathfrak{J}, \mathring{\mathbf{J}}_w)$.
    - Think of $\mathfrak{D}$ as measuring how far one's judgment set $\mathfrak{J}$ is (in w) from the vindicated or ideal set of judgments $\mathring{\mathbf{J}}_w$ (in w).
  - **Step 3**: Adopt a *fundamental epistemic principle*, which *uses* $\mathfrak{D}(\mathfrak{J}, \mathring{\mathbf{J}}_w)$ to ground a formal coherence requirement for $\mathfrak{J}$.
    - Think of the fundamental epistemic principle as articulating an *evaluative connection* between $\mathfrak{D}$ and $\mathfrak{J}$-coherence.
- Today: *comparative confidence*. (See [10, 1, 13] for others.)

- Let $\ulcorner p \succeq q \urcorner$ be the binary relation (relative to an agent S at time t, *viz.*, over a finite Boolean Algebra $\mathcal{B}$) $\ulcorner S$ is at least as confident in the truth of p as she is in the truth of q (at t)$\urcorner$.
- There is widespread ([4], [21], [28], [11]), though not perfect ([12]), agreement that these "intuitive" axioms govern $\succeq$.

(A1) $(p \succeq q) \vee (q \succeq p)$.

(A2) If $p \succeq q$ and $q \succeq r$, then $p \succeq r$.

(A3) $\top \succ \bot$. [*i.e.*, $(\top \succeq \bot)$ & $(\bot \not\succeq \top)$]

(A4) $p \succeq \bot$.

(A5) If $\langle p, q \rangle$ and $\langle p, r \rangle$ are mutually exclusive, then:

$$q \succeq r \iff (p \vee q) \succeq (p \vee r).$$

☞ *No direct justification has been offered for **any** of these "intuitive" axioms — as epistemic coherence requirements for $\succeq$.*

- Today: we explain how to give direct justifications of this kind.
- But, first, a bit of background on comparative probability.

- Once upon a time, *comparative* confidence judgments were thought to be more "secure" or "basic" or "fundamental" than *numerical* confidence/credence judgments [20].
- In his watershed essay, de Finetti [4] begins his story about the foundation of subjective probability theory, as follows:

> Let us consider a well-defined event and suppose that we do not know in advance whether it will occur or not; the doubt about its occurrence to which we are subject lends itself to comparison, and, consequently, to gradation. If we acknowledge only, first, that one uncertain event can only appear to us (a) equally probable, (b) more probable, or (c) less probable than another; second, that an uncertain event always seems to us more probable than an impossible event and less probable than a necessary event; and finally, third, that when we judge an event E more probable than an event E', which is itself judged more probable than an event E'', the event E can only appear more probable than E'' (transitive property), it will suffice to add to these three evidently trivial axioms a fourth, itself of a purely qualitative nature, in order to construct rigorously the whole theory of probability.

- Notice how de Finetti takes the *comparative* relations ($\succ, \sim$) to be (in some sense) "prior" to numerical credence (b). He speaks of "comparison, *and consequently*, gradation."

- First, de Finetti describes the ordering [(A1)/(A2)] and normalization [(A3)/(A4)] axioms as "self-evident".
- Second, de Finetti suggests that the "additivity" axiom (A5), together with the "self-evident" axioms (A1)–(A4), *suffices* to ensure *numerical probabilistic representatability* of ⪰.
- That is, de Finetti suggests that the following claim is true:

  (†) If the relation ⪰ satisfies (A1)–(A5), then there exists a *numerical probability function* Pr such that:

  $$p \succeq q \textit{ if and only if } \Pr(p) \geq \Pr(q).$$
- de Finetti [5] reports that there are no counterexamples to (†) involving algebras $\mathcal{B}_m$ containing $m \leq 4$ states. [This is non-trivial to do by hand, but easy with today's computers.]
- Interestingly, (†) *does* have counterexamples when $m \geq 5$. This was discovered several years later by Kraft *et. al.* [22].
- Kraft *et. al.* gave necessary and sufficient conditions for Pr-representability. Dana Scott gave a "cleaner" condition.

- While Scott's Axiom is "cleaner" than Kraft *et. al.*'s, it is still quite complex. Here, I will just state Scott's Axiom and say a few things about it and the de Finetti axioms (A1)–(A5).

(SA) Let $\mathbf{X}, \mathbf{Y} \in \prod_n \mathcal{B}$ be (arbitrary) sequences of propositions (from $\mathcal{B}$), each having length $n > 0$. Let $\mathbf{X} = \langle x_1, \ldots, x_n \rangle$ and $\mathbf{Y} = \langle y_1, \ldots, y_n \rangle$. *If* conditions (i) and (ii) are satisfied:

  (i) **X** and **Y** *have the same number of truths in every state of* $\mathcal{B}$.

  (ii) For all $i \in (1, n]$, $x_i \succeq y_i$.

  then, condition (iii) must also hold

  (iii) $y_1 \succeq x_1$.

- Scott [29] proves {(A1), (A3), (A4), (SA)} are necessary and sufficient for numerical probabilistic representability of ⪰.
- Let (SA$_n$) be the n-instance of the infinite schema (SA).
- Trivially, (A1) entails (SA$_1$), *i.e.*, totality entails reflexivity.
- It is well known that (SA$_2$) ⇒ (A5) and (SA$_3$) ⇒ (A2). (See Extras.)
- **Q**: What needs to be *super-added* to (A1)–(A5) to ensure (†)?

- **Newsflash**: (A5) ⇒ (SA$_2$) and (A2) & (A5) ⇒ (SA$_3$). ∴ (A5) and (SA$_2$) are *equivalent*, as are (A2) & (A5) and (SA$_2$) & (SA$_3$)! (See Extras.)
- The Kraft *et. al.* counterexample to (†) involves (SA$_4$).
- ∴ **A**: The universal claim "$(\forall n \geq 4)(\text{SA}_n)$" is *exactly* what needs to be *super-added* to (A1)–(A5), in order to ensure (†).
- **Fun Fact**: Let $(\text{SA}_n^m) \stackrel{\text{def}}{=}$ the $\langle m, n \rangle$-instance of (SA), where m is the # of states in $\mathcal{B}$. The Kraft *et. al.* counterexample to (†) resides at (SA_4^5). And, this is *smallest in both dimensions*.
- Various complaints about (SA) have been made. Fine [11] and others have complained that (SA)'s condition (i) is not a "purely Boolean" condition (it "essentially involves *counting*").
- **Not so.** Condition (i) of (SA$_n$) is equivalent to the claim that a specific (antecedently constructible) Boolean formula (with $2n$ variables) is tautological, *i.e.*, that two specific multisets of sets of states of $\mathcal{B}$ are identical [7]. ∴ (SA)'s condition (i) is expressible *via* pure Boolean equations. (See Extras.)

- Let **C** be the set of all of S's ⪰-judgments, over all pairs in $\mathcal{B} \times \mathcal{B}$. Step 1 defines "*the vindicated set at* w" ($\mathring{\mathbf{C}}_w$). And, Step 2 defines "distance between **C** and $\mathring{\mathbf{C}}_w$" [$\delta(\mathbf{C}, \mathring{\mathbf{C}}_w)$].
- Our first attempt(s) to complete Steps 1 and 2 for ⪰ failed.
  - Our mistake was trying to *locally score* each pairwise judgment in **C** *as if it were akin to a full belief* [24, 10].
- Rather than explain our initial (failed) attempt, I'll describe our *new way* (which *works* & fits better into our framework).
- **Step 1**: here is a plausible constraint on ⪰-vindication.

($\mathcal{V}_\succ$) Vindication requires being *strictly more confident* in (all) truths than (all) falsehoods (*i.e.*, if $v_w(p) > v_w(q)$, then $p \succ q$ should be included in the w-vindicated relation: $\mathring{\mathbf{C}}_w$).

- This implies *one condition* for inclusion in the set $\mathring{\mathbf{C}}_w$. But, ($\mathcal{V}_\succ$) *leaves open* various comparisons at various worlds.
- Specifically, ($\mathcal{V}_\succ$) is *silent* on comparisons between pairs of propositions that *have the same truth-value at* w.

- One is tempted to find a (single) constraint on these remaining pairs of propositions on which ($\mathcal{V}_{\succ}$) is silent.
- And, the *natural* proposal along these lines is:

($\mathcal{V}_{\sim}$) If $v_w(p) = v_w(q)$, then $p \sim q$ should be included in $\mathring{\mathbf{C}}_w$.

- If we were to add ($\mathcal{V}_{\sim}$) to our basic principle ($\mathcal{V}_{\succ}$), then we would end-up with the following *definition* of $\mathring{\mathbf{C}}_w$.
  - $\mathring{\mathbf{C}}_w$ is the set containing $p \succ q$ ($p \sim q$) iff p is true at w and q is false at w (p and q have the same truth-value at w).
- In other words, on this proposal, $\mathring{\mathbf{C}}_w$ would correspond to *the ordering* $\succeq_w$ *that is* (i) indifferent between all truths in w, (ii) indifferent between all falsehoods in w, and (iii) ranks all truths in w *strictly above* all falsehoods in w.
- This does not seem like a crazy (initial) proposal. Arguably, an omniscient agent would have this $\succeq$-ordering.
  - Moreover, *if* $\mathring{\mathbf{C}}_w$ is generated by a (vindicated) *numerical* credal set $\mathring{\mathbf{b}}_w$, then $\mathring{\mathbf{C}}_w$ will, of course, satisfy ($\mathcal{V}_{\sim}$) and ($\mathcal{V}_{\succ}$).

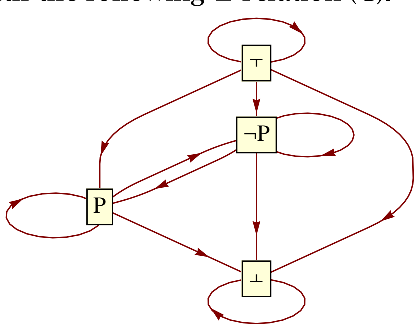
- This simple definition of $\mathring{\mathbf{C}}_w$ yields *almost* the same requirements as a more nuanced approach that does *not* assume ($\mathcal{V}_{\sim}$), but only ($\mathcal{V}_{\succ}$) + *reflexivity* of $\mathring{\mathbf{C}}_w$ (see Extras).
- **Step 2**: a binary relation $\succeq$ on $\mathcal{B}$ (containing n propositions) can be represented by its $n \times n$ *binary incidence matrix*.
- So, a natural way to think of distance between binary relations on $\mathcal{B}$ is as distance between incidence matrices. But, this is just *distance between binary vectors of length* n^2.
- In the case of full belief [10, 1], we use *Hamming distance* between binary vectors. We make the same choice here.
- This corresponds to the most oft-used measure of distance between binary relations: the *Kemeny distance* [19, 6].
  - Kemeny [19] used this measure to perform *aggregation* of ordinal (preference) rankings. His method avoids Arrow's Theorem, and leads to an optimal voting rule [30].
- Next: an illustration of the basic concepts used so far.

- Suppose we adopt $\succeq_w$ as the definition of *the vindicated comparative confidence relation at* w; and, we measure distance between relations using Kemeny distance. That is:

$\delta(\mathbf{C}, \mathring{\mathbf{C}}_w) \overset{\text{def}}{=}$ Hamming distance between the (flattened) incidence matrices of S's $\succeq$-relation and the vindicated relation $\succeq_w$.

- Consider a toy agent S with the following $\succeq$ relation (**C**).

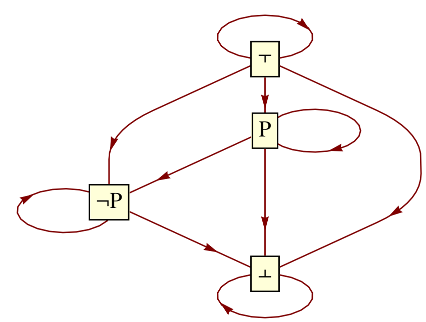
C	$\top$	P	$\neg P$	$\bot$
$\top$	1	1	1	1
P	0	1	1	1
$\neg P$	0	1	1	1
$\bot$	0	0	0	1



- Suppose $P \overset{\text{def}}{=}$ a fair coin lands heads; and, S knows ***only*** that the coin is fair (*i.e.*, S has *no other* P*-relevant evidence*).

- Intuitively, this agent's relation *shouldn't be ruled-out as incoherent* (it seems to accord with evidential requiements).
- Happily, if we adopt Kemeny distance (δ) as our distance measure, then **C** will *not* be *weakly* δ*-dominated by* ***any*** $\mathbf{C}'$.
- Of course, I won't show this *by hand*. It is easily verified by computer. However, in order to illustrate how δ works (and what $\mathring{\mathbf{C}}_w / \succeq_w$ look like), consider the following *specific* $\mathbf{C}'$:

$\mathbf{C}'$	$\top$	P	$\neg P$	$\bot$
$\top$	1	1	1	1
P	0	1	1	1
$\neg P$	0	1	1	1
$\bot$	0	0	0	1



- Here is what $\mathring{\mathbf{C}}_w$ looks like for w_1 (P true) and w_2 (P false):

$\mathring{\mathbf{C}}_{w_1}$	$\top$	P	$\neg P$	$\bot$
$\top$	1	1	1	1
P	1	1	1	1
$\neg P$	0	0	1	1
$\bot$	0	0	1	1

$\mathring{\mathbf{C}}_{w_2}$	$\top$	P	$\neg P$	$\bot$
$\top$	1	1	1	1
P	0	1	0	1
$\neg P$	1	1	1	1
$\bot$	0	1	0	1

- And, again, here is what $\mathbf{C}$ and $\mathbf{C}'$ look like:

$\mathbf{C}$	$\top$	P	$\neg P$	$\bot$
$\top$	1	1	1	1
P	0	1	1	1
$\neg P$	0	1	1	1
$\bot$	0	0	0	1

$\mathbf{C}'$	$\top$	P	$\neg P$	$\bot$
$\top$	1	1	1	1
P	0	1	1	1
$\neg P$	0	0	1	1
$\bot$	0	0	0	1

- These δ's reveal that $\mathbf{C}'$ does *not* (even weakly) dominate $\mathbf{C}$:
  - $\delta(\mathbf{C}, \mathring{\mathbf{C}}_{w_1}) = 3 > 2 = \delta(\mathbf{C}', \mathring{\mathbf{C}}_{w_1})$.
  - $\delta(\mathbf{C}, \mathring{\mathbf{C}}_{w_2}) = 3 < 4 = \delta(\mathbf{C}', \mathring{\mathbf{C}}_{w_2})$.

- **Step 3**: following Joyce [16, 17], we adopt the following FEP.

  Weak Accuracy-Dominance Avoidance (WADA).

  There does *not* exist an alternative relation $\mathbf{C}'$ such that:
  (i) $(\forall w)[\delta(\mathbf{C}', \mathring{\mathbf{C}}_w) \le \delta(\mathbf{C}, \mathring{\mathbf{C}}_w)]$, and
  (ii) $(\exists w)[\delta(\mathbf{C}', \mathring{\mathbf{C}}_w) < \delta(\mathbf{C}, \mathring{\mathbf{C}}_w)]$.

☞ (WADA) yields direct justifications of (almost) all of (A1)–(A5). The "almost" involves axioms (A1) and (A2).
  - We *can* give an accuracy-dominance justification of the *totality* axiom (A1), but one might reasonably doubt that (A1) is a requirement of rationality [15]. (See Extras.)
  - All violations of (A2) must be accuracy-dominated, *except intransitivities of indifference*. This is interesting, since transitivity of $\sim$ is controversial [12, 23]. (See Extras.)
  - That is, (WADA) yields a direct justification for the claim that coherence requires $\langle \sim, \succ \rangle$ to be a *semi-order* [23].
- We'll bracket these subtleties of grounding order structure.

- It can be shown (see [14] and the Extras for details) that:

  Theorem. Provided $\mathbf{C}$ is $\sim$-transitive, if $\mathbf{C}$ exhibits any violation (N) of any of the "intuitive" axioms (A1)–(A5), then there exists a (reflexive) relation $\mathbf{C}'$ that (a) weakly δ-dominates $\mathbf{C}$, (b) does not itself exhibit violation (N), and (c) is itself *non*-δ-dominated by any relation defined over just the propositions implicated in the violation (N) in question. [Think of $\mathbf{C}'$ as a "local fix" of the violation (N).]
- We are cautiously optimistic about the following conjecture:

  Conjecture. Provided that $\mathbf{C}$ is $\sim$-transitive, $\mathbf{C}$ is *not* weakly δ-dominated by *any* (reflexive) relation $\mathbf{C}'$ ***iff*** $\mathbf{C}$ is representable by some numerical probability function.
- One of the reasons we're optimistic about our Conjecture is:

  Theorem. The (SA_4^5)-counterexample of Kraft *et.al.* [22] is also δ-dominated (in the sense of the above Theorem).

☞ If our conjecture turns out to be true, then this would provide a neat foundation for comparative probabilism.

- Some important (additonal) open questions remain.
- How robust are the present results, across variations in the underlying measure of distance from vindication δ?
  - Most (but not all) of the naïve, counting measures of distance between binary vectors ([2], [6, §17.3], [8]) will yield the same δ-dominance norms in our present framework.
  - All (known) axiomatically derived measures of distance between *pre-orders* [3] behave similarly to Kemeny distance (for present purposes), when (A1) is weakened to reflexivity.
- How can we respond to the comparative analogue of the objection Kenny and I raise [9] regarding possible conflicts between "evidential" *vs* "alethic" norms for credences?
  - Maybe we can adopt the strategies of Jim [18] and/or Richard [26]? For full belief, we have a good response [10].
- A related question is: what are the analogues of "expected utility" norms in this context? In the cases of full belief and #-credence, there are straightforward stories ([10], [27]).

- Here, I will prove that (SA_2) entails (A5).
- let $\mathbf{X} = \langle p \vee r, q\rangle$ and $\mathbf{Y} = \langle p \vee q, r\rangle$, where $\langle p, q\rangle$ are mutually exclusive and $\langle p, r\rangle$ are mutually exclusive.
- That is, $x_1 = p \vee r$, $y_1 = p \vee q$, $x_2 = q$, and $y_2 = r$.
- Now, suppose (SA). Then, the (⇒) direction of (A5) follows.
- To see why, assume the left hand side of (A5). That is, suppose that $q \succeq r$, *i.e.*, that $x_2 \succeq y_2$. In the case at hand, this is equivalent to condition (ii) in the antecedent of (SA).
- Thus, in order to establish additivity (A5), all we need to do is show that $(p \vee q) \succeq (p \vee r)$, *i.e.*, that (iii) $y_1 \succeq x_1$.
- This will follow from (SA), provided that we can show condition (i) of (SA) must also be true in this case.
- Indeed, (i) must be true in this case, and this can most easily be seen *via* the following *schematic truth-table*.

	p	q	r	$\mathfrak{s}_i \models p \vee r$?	$\mathfrak{s}_i \models q$?	$\mathfrak{s}_i \models p \vee q$?	$\mathfrak{s}_i \models r$?
$\mathfrak{s}_1$	**T**	**T**	**T**	—	—	—	—
$\mathfrak{s}_2$	**T**	**T**	**F**	—	—	—	—
$\mathfrak{s}_3$	**T**	**F**	**T**	—	—	—	—
$\mathfrak{s}_4$	**T**	**F**	**F**	YES	NO	YES	NO
$\mathfrak{s}_5$	**F**	**T**	**T**	YES	YES	YES	YES
$\mathfrak{s}_6$	**F**	**T**	**F**	NO	YES	YES	NO
$\mathfrak{s}_7$	**F**	**F**	**T**	YES	NO	NO	YES
$\mathfrak{s}_8$	**F**	**F**	**F**	NO	NO	NO	NO

- Because $\langle p, q\rangle$ and $\langle p, r\rangle$ are mutually exclusive, the (families of) state descriptions $\mathfrak{s}_1$–$\mathfrak{s}_3$ are *impossible*. So, we can ignore those rows of the schematic truth-table.
- Now, in oder to show that (i) holds in this case, we just need to show that each of the five (*possible* families of) state descriptions $\mathfrak{s}_4$–$\mathfrak{s}_8$ satisfies condition (i) of (SA).
- This is easily verified by inspection of the table, since each of these rows contains the same number of "YES"s in both pairs of columns on the right. □ The (⇐) proof is similar.

- Next, we'll prove *transitivity* of ⪰ [(A2)] from (SA_3).
- For this proof, we'll need to exploit the fact that (SA) quantifies over (finite) *sequences* of propositions. Let:
  - $\mathbf{X} = \langle x_1, x_2, x_3\rangle \stackrel{\text{def}}{=} \langle r, p, q\rangle$.
  - $\mathbf{Y} = \langle y_1, y_2, y_3\rangle \stackrel{\text{def}}{=} \langle p, q, r\rangle$.

1. (SA_3)
   - Assumption [for ⇒I: (SA_3) ⇒ (A2)].
2. (i) of (SA_3).
   - **X** and **Y** *contain the same number of truths in all worlds*, since they involve the same (multiset of) propositions.
3. $p \succeq r$ & $q \succeq r$. [*i.e.*, (ii) of (SA_3): $x_2 \succeq y_2$ & $x_3 \succeq y_3$]
   - Assumption [for ⇒I: ($p \succeq r$ & $q \succeq r$) ⇒ $p \succeq r$].
4. $p \succeq r$. [*i.e.*, (iii) of (SA_3): $y_1 \succeq x_1$] By 1–3 (logic).
5. ($p \succeq r$ & $q \succeq r$) ⇒ $p \succeq r$. [*i.e.*, (A2)] By 3–4 (⇒I).
6. (SA_3) ⇒ (A2) By 1–5 (⇒I). □

- Let's think about (A1) — *totality*. (A1) requires that S be *opinionated* — that S makes *exactly one* of the *definite* judgments $p \succ q$, $q \succ p$, or $p \sim q$, for each $\langle p, q\rangle \in \mathcal{B} \times \mathcal{B}$.
- It would be surprising if (A1) came out as a requirement of rationality [15] *via* accuracy-domination. But, that's what our simple (total) definition of $\mathring{\mathbf{C}}_w$ implies. Next, a proof ...
  - The more complex definition of $\mathring{\mathbf{C}}_w$ does not presuppose totality. It only presupposes reflexivity and ($\mathcal{V}_\succ$). This leads to complications, since we don't get a unique $\mathring{\mathbf{C}}_w$ in each world. One can deal with this, *via* aggregation/consensus *à la* Kemeny [19]. And, then, totality does *not* follow [14].
- Suppose that S *suspends judgment* regarding $\langle p, q\rangle$, which we will write as $p \mid q$. We can represent this judgment, *via* (note: we'll *assume* ***reflexivity*** of ⪰ — think *logical omniscience*):

$$p \mid q := \begin{array}{c|cc} & p & q \\ \hline p & 1 & 0 \\ q & 0 & 1 \end{array}$$

- Now, we can show that $\mathbf{C}$ $(p \mid q)$ ***is*** δ-dominated by a $\mathbf{C}'$ $(p \sim q)$ that is *opinionated* regarding $\langle p, q\rangle$. That is:

  Theorem. $\exists$ a reflexive relation $\mathbf{C}'$ that: (a) agrees with S's $\mathbf{C}$ on all comparisons *other than* $\langle p, q\rangle$, (b) is *opinionated* regarding $\langle p, q\rangle$, (c) weakly δ-*dominates* $\mathbf{C}$, and (d) is *itself not weakly* δ*-dominated by any reflexive relation over* $\langle p, q\rangle$.

- This can be shown *by exhaustion*. There are only *three* (3) *possible* relations satisfying both (a) and (b) of the Theorem.

$p \succ q :=$

	p	q
p	1	1
q	0	1

$q \succ p :=$

	p	q
p	1	0
q	1	1

$p \sim q :=$

	p	q
p	1	1
q	1	1

- Let $\mathbf{C}_1$ ($\mathbf{C}_2$, $\mathbf{C}_3$) be the relation that (a) agrees with $\mathbf{C}$ on all pairs *other than* $\langle p, q\rangle$, and (b) is s.t. $p \succ q$ ($q \succ p$, $p \sim q$).
- There are four salient possible worlds (w_i) here, which are represented (along with the adjacency matrices of each of the four relations in question) on the following truth-table.

w_i	p	q		$\mathring{\mathbf{C}}_{w_i}$: p	$\mathring{\mathbf{C}}_{w_i}$: q	$\mathbf{C}$: p	$\mathbf{C}$: q	$\mathbf{C}_1$: p	$\mathbf{C}_1$: q	$\mathbf{C}_2$: p	$\mathbf{C}_2$: q	$\mathbf{C}_3$: p	$\mathbf{C}_3$: q
w_1	T	T	p	1	1	1	0	1	1	1	0	1	1
			q	1	1	0	1	0	1	1	1	1	1
w_2	T	F	p	1	1	1	0	1	1	1	0	1	1
			q	0	1	0	1	0	1	1	1	1	1
w_3	F	T	p	1	0	1	0	1	1	1	0	1	1
			q	1	1	0	1	0	1	1	1	1	1
w_4	F	F	p	1	1	1	0	1	1	1	0	1	1
			q	1	1	0	1	0	1	1	1	1	1

- Now, we can calculate all δ-values [focus on $\delta(\mathbf{C}_3, \mathring{\mathbf{C}}_{w_i})$ here]:

w_i	$\delta(\mathbf{C}, \mathring{\mathbf{C}}_{w_i})$	$\delta(\mathbf{C}_1, \mathring{\mathbf{C}}_{w_i})$	$\delta(\mathbf{C}_2, \mathring{\mathbf{C}}_{w_i})$	$\delta(\mathbf{C}_3, \mathring{\mathbf{C}}_{w_i})$
w_1	2	1	1	0
w_2	1	0	2	1
w_3	1	2	0	1
w_4	2	1	1	0

- Assuming (for now) that we have *established totality* (A1), there are only ***three*** *possible violations* of axiom (A2):

(NA2.1) For *some* p, q, r: $p \succ q$, $q \succ r$ *and* $r \succ p$.

(NA2.2) For *some* p, q, r: $p \succ q$, $q \sim r$ *and* $r \succ p$.

(NA2.3) For *some* p, q, r: $p \sim q$, $q \sim r$ *and* $r \succ p$.

- Establishing *full* transitivity would require the following:

(TA2S) *Each* (total) relation $\mathbf{C}$ exhibiting *any* of (NA2.i) is δ-dominated by *some* (total) relation $\mathbf{C}'$, which:

  (a) agrees with $\mathbf{C}$ on all comparisons *not involving* p, q, r.

  (b) does *not* exhibit (NA2.i).

  And, if a (total) $\mathbf{C}$ *satisfies* axiom (A2) wrt p, q, r, then it is *not* (thereby) δ-dominated by *any* (total) $\mathbf{C}'$ on p, q, r.

Theorem. *If* we *assume transitivity of* $\sim$, then (TA2S) holds. To be precise, *if* we replace occurrences 1 and 3 of "(total)" in (TA2S) with "(total ***and*** $\sim$***-transitive***)", then (TA2S) holds. [☞ Instances of (NA2.3) are *not necessarily* δ-dominated!]

- I will not sketch the proof of this Theorem. But, I will briefly explain how we have established it (it's a computer proof).
  - There are 27 total relations involving p, q, r (and 8 salient possible worlds). For each total relation $\mathbf{C}$ exhibiting either (NA2.1) or (NA2.2), we check that a $\mathbf{C}'$ satisfying Theorem exists [for *some* (NA2.3)'s, we check $\nexists$ *any* δ-dominating $\mathbf{C}'$].
- There seem to be *counterexamples* to $\sim$-transitivity. Here's an example involving (NA2.3), which is in the spirit of [23].[1]

  You observe a bank robbery. You get a good look at the robber (r_0), who has a full head of hair. The police create n perfect duplicates of the robber (r_i). They remove i hairs from the head of r_i, and make a *line-up*: $r_1, r_2, r_3, r_4, \ldots$. They show *one pair at-a-time*: $\langle r_1, r_2\rangle$, $\langle r_2, r_3\rangle$, $\langle r_3, r_4\rangle, \ldots$. They ask you, for each pair $\langle r_i, r_{i+1}\rangle$, whether you judge $p_i \succ p_{i+1}$, $p_{i+1} \succ p_i$, or $p_i \sim p_{i+1}$, where $p_i \overset{\text{def}}{=} r_i$ is the robber — that is, $p_i \overset{\text{def}}{=} r_i = r_0$. You *violate* transitivity of $\sim$.

[1] It is important to keep in mind here that *we may not assume* that S has an *underlying numerical credence function* b such that $p \sim q$ *iff* $b(p) = b(q)$.

- It is easy to prove the following theorem:

(TA3S) **Theorem**. *Each* (total) relation $\mathbf{C}$ exhibiting *either* of (NA3.i) is (*strictly!*) δ-dominated by *some* (total) relation $\mathbf{C}'$, which:

(a) agrees with $\mathbf{C}$ on all comparisons *not involving* $\top, \bot$.

(b) does *not* exhibit (NA3.i).

(c) is *not* δ-dominated by *any* (total) relation involving $\top, \bot$.

- What makes this proof easy is that *the vindicated relation involving only* $\top, \bot$ *is the same in every possible world.*

☞ Therefore, we can let $\mathbf{C}'$ be $\mathring{\mathbf{C}}_w \stackrel{\text{def}}{=} \succeq_w = \begin{array}{c|c|c} & \top & \bot \\ \hline \top & 1 & 1 \\ \bot & 0 & 1 \end{array}$.

(NA3.1) $\bot \succ \top$. $\quad \mathbf{C}_1 = \begin{array}{c|c|c} & \top & \bot \\ \hline \top & 1 & 0 \\ \bot & 1 & 1 \end{array}$. $\quad$ So, $\delta(\mathbf{C}_1, \mathring{\mathbf{C}}_w) = 2 > 0$.

(NA3.2) $\top \sim \bot$. $\quad \mathbf{C}_2 = \begin{array}{c|c|c} & \top & \bot \\ \hline \top & 1 & 1 \\ \bot & 1 & 1 \end{array}$. $\quad$ So, $\delta(\mathbf{C}_2, \mathring{\mathbf{C}}_w) = 1 > 0$. $\quad\square$

- The argument for (A4) is almost as easy. Assuming (A1) has been proven (for now), there is *only one* way to violate (A4).

(NA4) For *some* p, $\bot \succ p$. $\quad \mathbf{C} = \begin{array}{c|c|c} & p & \bot \\ \hline p & 1 & 0 \\ \bot & 1 & 1 \end{array}$

- Any $\mathbf{C}$ exhibiting (NA4) is δ-dominated by $\mathbf{C}'$ ($p \succ \bot$):

w_i	p	$\mathring{\mathbf{C}}_{w_i}$	$\mathbf{C}$	$\mathbf{C}'$	$\delta(\mathbf{C}, \mathring{\mathbf{C}}_{w_i})$	$\delta(\mathbf{C}', \mathring{\mathbf{C}}_{w_i})$
w_1	T	$\begin{array}{c\|c\|c} & p & \bot \\ \hline p & 1 & 1 \\ \bot & 0 & 1 \end{array}$	$\begin{array}{c\|c\|c} & p & \bot \\ \hline p & 1 & 0 \\ \bot & 1 & 1 \end{array}$	$\begin{array}{c\|c\|c} & p & \bot \\ \hline p & 1 & 1 \\ \bot & 0 & 1 \end{array}$	2	0
w_2	F	$\begin{array}{c\|c\|c} & p & \bot \\ \hline p & 1 & 1 \\ \bot & 1 & 1 \end{array}$	$\begin{array}{c\|c\|c} & p & \bot \\ \hline p & 1 & 0 \\ \bot & 1 & 1 \end{array}$	$\begin{array}{c\|c\|c} & p & \bot \\ \hline p & 1 & 1 \\ \bot & 0 & 1 \end{array}$	1	1

- Finally, it is easy to verify that $\mathbf{C}'$ will *not* be (weakly) δ-dominated by *any* total relation (involving $p, \bot$). $\quad\square$

- That brings us to *additivity* (A5). This is the most complex of the "intuitive" $\succeq$-axioms to establish *via* δ-dominance.
- Assuming (A1) has been proven (for now), there are *only four* (really, only *three*) ways to violate (A5).

(NA5.1) For some p, q, r such that $\langle p, q \rangle$ and $\langle p, r \rangle$ are mutually exclusive pairs: $q \succ r$ and $(p \vee r) \succ (p \vee q)$.

(NA5.2) For some p, q, r such that $\langle p, q \rangle$ and $\langle p, r \rangle$ are mutually exclusive pairs: $q \sim r$ and $(p \vee r) \succ (p \vee q)$.

(NA5.3) For some p, q, r such that $\langle p, q \rangle$ and $\langle p, r \rangle$ are mutually exclusive pairs: $(p \vee q) \succ (p \vee r)$ and $r \succ q$.

(NA5.4) For some p, q, r such that $\langle p, q \rangle$ and $\langle p, r \rangle$ are mutually exclusive pairs: $(p \vee q) \sim (p \vee r)$ and $r \succ q$.

- Note: (NA5.1) & (NA5.3) are *equivalent*. So, there are *only 3*.
- In order to establish (A5), we'll need to assume that S satisfies *both* (A1) *and* (A2) — full transitivity of $\succeq$. In other words, we will now assume that S's $\succeq$ is a *weak total order*.

- With this background in place, we can prove the following:

(TA5S) **Theorem**. *Each weak total order* $\mathbf{C}$ exhibiting *any* of (NA5.i) is (weakly) δ-dominated by *some* (total) relation $\mathbf{C}'$, which:

(a) agrees with $\mathbf{C}$ on all comparisons *not involving* the four propositions $q, r, p \vee q, p \vee r$.

(b) does *not* exhibit (NA5.i).

Furthermore, if a weak total order $\mathbf{C}$ *satisfies* axiom (A5) with respect to $q, r, p \vee q, p \vee r$, then it is *not* (thereby) δ-dominated by *any* (total) relation $\mathbf{C}'$ on $q, r, p \vee q, p \vee r$.

- I will not attempt to even sketch this proof (it's an even more complex computer proof). But, here's what it requires.
  - There are 75 weak total orders (and 729 total relations) over $q, r, p \vee q, p \vee r$. And, there are 5 salient possible worlds (3/8 of the worlds are ruled-out by the precondition of A5).
  - For each weak total order $\mathbf{C}$ exhibiting each (NA5.i), we need to verify that $\exists \mathbf{C}'$ satisfying (TA5S). And, for each weak total order *satisfying* (A5), we need to show $\nexists$ such a $\mathbf{C}'$.

- I mentioned that (SA$_n$)'s condition (i) is equivalent (assuming $\mathcal{B}$ is generated by a sentential language $\mathscr{L}$) to the tautologousness of a Boolean $\mathscr{L}$-formula with $2n$ atoms.
- Let's look at the (SA$_2$) case. When $n = 2$, (SA$_n$)'s condition (i) asserts that $\mathbf{X} = \langle x_1, x_2\rangle$ and $\mathbf{Y} = \langle y_1, y_2\rangle$ have the same number of truths in every state of $\mathcal{B}$. This means:
  (1) $x_1 \,\&\, x_2 \dashv\vDash y_1 \,\&\, y_2$,
  (2) $(x_1 \,\&\, \neg x_2) \vee (\neg x_1 \,\&\, x_2) \dashv\vDash (y_1 \,\&\, \neg y_2) \vee (\neg y_1 \,\&\, y_2)$, and
  (3) $\neg x_1 \,\&\, \neg x_2 \dashv\vDash \neg y_1 \,\&\, \neg y_2$.
- But, the joint truth of (1)–(3) is equivalent to the logical truth (tautologousness) of the following conjunction:

$$x_1 \,\&\, x_2 \equiv y_1 \,\&\, y_2$$
$$\&$$
$$(x_1 \,\&\, \neg x_2) \vee (\neg x_1 \,\&\, x_2) \equiv (y_1 \,\&\, \neg y_2) \vee (\neg y_1 \,\&\, y_2)$$
$$\&$$
$$\neg x_1 \,\&\, \neg x_2 \equiv \neg y_1 \,\&\, \neg y_2$$

- This allows us to encode (SA$_2$)'s (i) as a purely Boolean formula (*viz.*, as 3 Boolean equations, given a background equational axiomatization of Boolean algebra [25]).
- Once we've done that, we can then use a (first-order) theorem prover (*e.g.*, `prover9`) to show that (A5) entails (SA$_2$). This is because the rest of the conditions in (A5) and (SA$_2$) are just first-order conditions on a binary relation.
- If anyone is interested, I have a simple `prover9` input file that yields a (non-trivial!) proof that (A5) entails (SA$_2$). [Together with the argument above, this shows (A5) ⇔ (SA$_2$).]
- The same trick can be used to encode the claim that axioms (A2) & (A5) *jointly entail* (SA$_3$). That is a *highly* non-trivial proof! And, it, together with the proofs above that the Scott Axiom instances (SA$_2$) & (SA$_3$) jointly entail (A2) & (A5), establishes the *equivalence* of (SA$_2$) & (SA$_3$) and (A2) & (A5). [I have a `prover9` input file for this too, if want it. . .]

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